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LEFT AND RIGHT W-WEIGHTED G-DRAZIN INVERSES AND NEW MATRIX PARTIAL ORDERS

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Abstract. This paper investigates a way to define left and right versions of the class of G-Drazin inverses for complex rectangular matrices. More precisely, the concepts of W-weighted left and right G-Drazin inverses are introduced and characterized by means of a simultaneous core-nilpotent decomposition as well as by a certain system of matrix equations. Then new partial orders associated with these weighted generalized inverses are presented and studied.

1. Introduction

The set of complex rectangular matrices of size $m \times n$ will be denoted by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, the symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\operatorname{rank}(A)$, and $\operatorname{Ind}(A)$ denote the column space, the null space, the rank, and the index $(m = n)$ of A, respectively.

By considering a fixed nonzero complex matrix W of size $n \times m$ and an arbitrary complex matrix A of size $m \times n$, Cline and Greville [3] showed the existence and uniqueness of a solution X to the system of equations

$$XWAWX = X, \quad AWX = XWA, \quad XW(AW)^{k+1} = (AW)^k$$

for some positive integer k.

This solution is called the W-weighted Drazin inverse of A and is written as $A^{d,W}$. When $m = n$ and $W = I_n$, we recover the classical Drazin inverse, that is, $A^{d,I_n} = A^d$.

It is well known that $A^{d,W} = A[(WA)^d]^2 = [(AW)^d]^2A$, $WA^{d,W} = (WA)^d$, and $A^{d,W}W = (AW)^d$.

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Using the W-weighted Drazin inverse, the weighted Drazin pre-order [7] was defined as follows:

$$A \preceq^{d,W} B \quad \Leftrightarrow \quad AWA^{d,W} = BWA^{d,W} \ \text{ and } \ A^{d,W}WA = A^{d,W}WB.$$

Note that $\preceq^{d,W}$ is an extension to the rectangular case of the pre-order $\preceq^{d}$ defined and studied in depth by Mitra, Bhimasankaram, and Malik [9]. More interesting results related to pre-orders introduced by the W-weighted Drazin inverse can be found in [8, 10, 14].

Recently, Wang and Liu [17] studied the class of G-Drazin inverses of a square matrix. Recall that X is an inner inverse of $A \in \mathbb{C}^{m\times n}$ if it satisfies $AXA = A$ [1].

Definition 1.1 ([17])**.** Let $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{Ind}(A)$. Then a matrix X is called a *G-Drazin inverse* of A if X is an inner inverse of A such that $XA^{k+1} = A^k$ and $A^{k+1}X = A^k$.

The authors observed that, in general, the G-Drazin inverse is not unique. Thus, the class of G-Drazin inverses of A is denoted by $A\{GD\}$. Clearly, $A\{GD\} \subseteq A\{1\} = \{A^- \in \mathbb{C}^{n\times m} : AA^-A = A\}$. Using the G-Drazin inverses, the authors also introduced a new partial order on the set of square matrices called the G-Drazin partial order.

Definition 1.2 ([17])**.** Let $A, B \in \mathbb{C}^{n\times n}$. The matrix A is said to be below the matrix B under the G-Drazin partial order if there exist $X_1, X_2 \in A\{GD\}$ such that $X_1A = X_1B$ and $AX_2 = BX_2$, and we denote this by $A \leq^{GD} B$.

The notions of left and right invertibility as extensions of invertibility motivated many authors to investigate similar problems for various kinds of generalized invertibility. As weaker versions of the G-Drazin inverse, the concepts of left and right G-Drazin inverses were proposed in [16].

Definition 1.3 ([16])**.** Let $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{Ind}(A)$. Then a matrix X is called a *left* (resp. *right*) *G-Drazin inverse* of A if X is an inner inverse of A such that $XA^{k+1} = A^k$ (resp. $A^{k+1}X = A^k$).

The sets of all left (or right) G-Drazin inverses of A will be denoted by $A\{l, GD\}$ (or $A\{r, GD\}$). Replacing $X_1, X_2 \in A\{GD\}$ in Definition 1.2 with $X_1, X_2 \in A\{l, GD\}$ (or $X_1, X_2 \in A\{r, GD\}$), a left (or right) G-Drazin partial order was defined in [16] on the set of square matrices. More details related to the G-Drazin inverses can be found in [5, 6, 12, 13].

Later, Coll et al. [4] extended the G-Drazin inverses to the rectangular case using an appropriate weighted W in Definition 1.1. More precisely, a matrix $X \in \mathbb{C}^{m\times n}$ is a W-weighted G-Drazin inverse of $A \in \mathbb{C}^{m\times n}$ if it satisfies the three equations

$$WAWXWAW = WAW, \quad (AW)^{k+1}XW = (AW)^k, \quad WX(WA)^{k+1} = (WA)^k, \tag{1.1}$$

where $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. The set of W-weighted G-Drazin inverses of A is denoted by $A\{W\text{-}GD\}$. Moreover, the following binary relation on $\mathbb{C}^{m\times n}$ was considered in order to extend Definition 1.2:

$$A \preceq_W^{GD} B \quad \Leftrightarrow \quad WAWX_1 = WBWX_1 \ \text{ and } \ X_2WAW = X_2WBW \tag{1.2}$$

for some $X_1, X_2 \in A\{GD - W\}$.

The relation $\preceq_W^{GD}$ is a pre-order on $\mathbb{C}^{m\times n}$; however, it is not a partial order because it is not antisymmetric. The concept of the W-weighted G-Drazin inverse was studied for operators between two Banach spaces in [11], and the definition of the strong W-weighted G-Drazin inverse was given in [15]. In particular, a matrix $X \in \mathbb{C}^{m\times n}$ is a strong W-weighted G-Drazin inverse of $A \in \mathbb{C}^{m\times n}$ if it is a solution to the system of two equations

$$AWXWA = A, \quad (WA)^d WAWXW = WXW(AW)^d AW.$$

The definitions of left and right G-Drazin inverses, as well as the W-weighted G-Drazin inverse and the strong W-weighted G-Drazin inverse, motivated us to consider weighted versions of left and right G-Drazin inverses. The main goal of this article is to present a variant of the system (1.1) and therefore an alternative binary relation to the one given in (1.2) in order to obtain a weighted partial order on the set of complex rectangular matrices. Precisely, as solutions of corresponding systems of two equations, we introduce the W-weighted left (resp. right) G-Drazin inverse for rectangular matrices. Also, the concept of the W-weighted left-right G-Drazin inverse is given as a both left and right G-Drazin inverse. Since left (or right) G-Drazin inverses and G-Drazin inverses are particular cases of our new inverses, we define three new wider classes of already existing generalized inverses. Especially, these new inverses can be applied to propose three new kinds of partial orders on the set of rectangular matrices. As we previously mentioned, based on weighted generalized inverses many pre-orders were defined, but we present partial orders using the W-weighted left (or right) G-Drazin inverse and the W-weighted left-right G-Drazin inverse.

The structure of this article is as follows. In Section 2, we introduce and characterize three new types of generalized inverses for rectangular matrices: the W-weighted left (resp. right) G-Drazin inverses and the W-weighted left-right G-Drazin inverses. Using a simultaneous decomposition, we obtain a canonical form for these generalized inverses. In Section 3, parameterized representations of W-weighted left (resp. right) G-Drazin inverses are obtained. Section 4 is devoted to the study of new binary relations involving the W-weighted left (resp. right) G-Drazin inverses which result in partial orders on the set of rectangular matrices.

Throughout this paper, we fix a nonzero matrix $W \in \mathbb{C}^{n\times m}$ and use it as a weight.

2. Weighted left and right G-Drazin inverses

We start with the following definitions.

Definition 2.1. Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. A matrix $X \in \mathbb{C}^{m\times n}$ is

(i) a *W-weighted left G-Drazin inverse* of A if the following equalities hold:

$$AWXWA = A, \quad XW(AW)^{k+1} = (AW)^k;$$

(ii) a *W-weighted right G-Drazin inverse* of A if the following equalities hold:

$$AWXWA = A, \quad (WA)^{k+1}WX = (WA)^k;$$

(iii) a *W-weighted left-right G-Drazin inverse* if X is both W-weighted left and right G-Drazin inverse of A.

We denote by $A\{l, W\text{-}GD\}$, $A\{r, W\text{-}GD\}$, and $A\{l, r, W\text{-}GD\}$ the sets of all W-weighted left, right, and left-right G-Drazin inverses of A, respectively.

Remark 2.2. When $m = n$ and $W = I_n$, Definition 2.1 recovers the concepts of left and right G-Drazin inverses [16, Definition 3.1] and G-Drazin inverses [17, Definition 3.1].

Necessary and sufficient conditions for a matrix to be a W-weighted left G-Drazin inverse are investigated below.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the following conditions are equivalent:*

(a) $X \in A\{l, W\text{-}GD\}$;
(b) $AWXWA = A$ *and* $XWAW(AW)^d = (AW)^d$;
(c) $AWXWA = A$ *and* $XWAW(AW)^dAW = (AW)^dAW$;
(d) $AWXWA = A$ *and* $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(XWAW)$.

Proof. (a) ⇒ (b) It is sufficient to prove the second condition. In fact, from

$$XW(AW)^{k+1} = (AW)^k,$$

we have

$$XW(AW)^{k+1}[(AW)^d]^{k+1} = (AW)^k[(AW)^d]^{k+1}.$$

As $AW(AW)^d$ is idempotent, we get

$$XWAW(AW)^d = (AW)^d.$$

(b) ⇒ (c) Trivial.

(c) ⇒ (a) Multiplying $XWAW(AW)^dAW = (AW)^dAW$ from the right side by $(AW)^d(AW)^{k+1}$, we obtain $XW(AW)^{k+1} = (AW)^k$. Since $AWXWA = A$, we deduce that $X \in A\{l, W\text{-}GD\}$.

(b) ⇔ (d) Note that $AWXWA = A$ implies that $XWAW$ is idempotent. Thus, $XWAW(AW)^d = (AW)^d$ is equivalent to

$$\mathcal{R}((AW)^k) = \mathcal{R}((AW)^d) \subseteq \mathcal{N}(I_m - XWAW) = \mathcal{R}(XWAW). \qquad \square$$

In a manner similar to the above theorem, we can characterize the W-weighted right G-Drazin inverses.

Theorem 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the following conditions are equivalent:*

(a) $X \in A\{r, W\text{-}GD\}$;
(b) $AWXWA = A$ *and* $(WA)^dWAWX = (WA)^d$;
(c) $AWXWA = A$ *and* $WA(WA)^dWAWX = WA(WA)^d$;
(d) $AWXWA = A$ *and* $\mathcal{N}(WAWX) \subseteq \mathcal{N}((WA)^k)$.

As a consequence of Theorem 2.3 and Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the following conditions are equivalent:*

(a) $X \in A\{l, r, W\text{-}GD\}$;
(b) $AWXWA = A$, $XWAW(AW)^d = (AW)^d$, *and* $(WA)^d WAWX = (WA)^d$;
(c) $AWXWA = A$, $XWAW(AW)^d AW = (AW)^d AW$, *and* $WA(WA)^d WAWX = WA(WA)^d$;
(d) $AWXWA = A$, $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(XWAW)$, *and* $\mathcal{N}(WAWX) \subseteq \mathcal{R}((WA)^k)$.

In order to obtain more properties and representations of $A^{d,W}$, the following simultaneous decomposition of A and W (called weighted core-nilpotent decomposition) was introduced by Wei [18]:

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} Q^{-1} \quad \text{and} \quad W = Q \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} P^{-1}, \tag{2.1}$$

where $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, $A_1, W_1 \in \mathbb{C}^{t\times t}$ are nonsingular matrices, $A_2 \in \mathbb{C}^{(m-t)\times(n-t)}$ and $W_2 \in \mathbb{C}^{(n-t)\times(m-t)}$ are rectangular matrices such that A_2W_2 and W_2A_2 are nilpotent of indices $\mathrm{Ind}(AW)$ and $\mathrm{Ind}(WA)$, respectively. In particular, if $m = n$ and $AW = WA$, then $Q = P$. In this case, if $W = I_n$, then $W_1 = I_t$ and $W_2 = I_{n-t}$.

Now, we provide a characterization of a W-weighted left G-Drazin inverse by means of the weighted core-nilpotent decomposition.

Theorem 2.6. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. If A and W are written as in* (2.1), *then $X \in A\{l, W\text{-}GD\}$ if and only if*

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} Q^{-1}, \tag{2.2}$$

where $X_3W_2A_2 = 0$ and $W_2X_2W_2 \in A_2\{1\}$.

Proof. Consider $X \in A\{l, W\text{-}GD\}$ written as

$$X = P \begin{bmatrix} X_1 & X_3 \\ X_4 & X_2 \end{bmatrix} Q^{-1}, \tag{2.3}$$

according to the sizes of the blocks in A. From (2.1) and (2.3), we have

$$AW = P \begin{bmatrix} A_1W_1 & 0 \\ 0 & A_2W_2 \end{bmatrix} P^{-1}, \quad XW = P \begin{bmatrix} X_1W_1 & X_3W_2 \\ X_4W_1 & X_2W_2 \end{bmatrix} P^{-1}. \tag{2.4}$$

As A_2W_2 is nilpotent of index k, from (2.4) we get

$$(AW)^k = P \begin{bmatrix} (A_1W_1)^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1},$$

and therefore

$$XW(AW)^{k+1} = (AW)^k \;\Leftrightarrow\; X_4W_1(A_1W_1)^{k+1} = 0 \;\Leftrightarrow\; X_4 = 0. \tag{2.5}$$

Now, from (2.4) and (2.5), we obtain

$$AWXWA = A \quad \Leftrightarrow \quad X_1 = (W_1A_1W_1)^{-1},\ X_3W_2A_2 = 0,\ A_2W_2X_2W_2A_2 = A_2. \tag{2.6}$$

From (2.5) and (2.6), it follows that (2.2) holds. □

Similarly, we can establish the following result concerning W-weighted right G-Drazin inverses.

Theorem 2.7. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. If A and W are written as in* (2.1)*, then $X \in A\{r, W\text{-}GD\}$ if and only if*

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ X_4 & X_2 \end{bmatrix} Q^{-1},$$

where $A_2W_2X_4 = 0$ and $W_2X_2W_2 \in A_2\{1\}$.

As a consequence of Theorem 2.6 and Theorem 2.7, we get a canonical form for the W-weighted left-right G-Drazin inverses.

Corollary 2.8. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. If A and W are written as in* (2.1)*, then $X \in A\{l, r, W\text{-}GD\}$ if and only if*

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & X_2 \end{bmatrix} Q^{-1},$$

where $W_2X_2W_2 \in A_2\{1\}$.

It is clear that if $m = n$ and $W = I_n$ in (2.1), we recover the classical core-nilpotent decomposition of a matrix $A \in \mathbb{C}^{n\times n}$ of index k,

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P^{-1}, \tag{2.7}$$

where $t = \mathrm{rank}(A^k)$, $A_1 \in \mathbb{C}^{t\times t}$ is nonsingular, and $A_2 \in \mathbb{C}^{(n-t)\times(n-t)}$ is nilpotent of index k.

Corollary 2.9. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$*. If A is written as in* (2.7)*, then the following hold:*

(a) *$X \in A\{l, GD\}$ if and only if $X = P \begin{bmatrix} A_1^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} P^{-1}$, where $X_3A_2 = 0$ and $X_2 \in A_2\{1\}$;*

(b) *$X \in A\{r, GD\}$ if and only if $X = P \begin{bmatrix} A_1^{-1} & 0 \\ X_4 & X_2 \end{bmatrix} P^{-1}$, where $A_2X_4 = 0$ and $X_2 \in A_2\{1\}$;*

(c) *$X \in A\{l, r, GD\}$ if and only if $X = P \begin{bmatrix} A_1^{-1} & 0 \\ 0 & X_2 \end{bmatrix} P^{-1}$, where $X_2 \in A_2\{1\}$.*

Remark 2.10. Notice that if $m = n$ and $W = I_n$, the set of all G-Drazin inverses of A coincides with the set of all left-right G-Drazin inverses of A, that is, $A\{GD\} = A\{l, r, GD\}$.

3. Parametrizing W-weighted left and right G-Drazin inverses

The following two lemmas are due to Penrose.

Lemma 3.1 ([1]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{m\times q}$, $A^- \in A\{1\}$, and $B^- \in B\{1\}$. Then the equation $AXB = C$ is consistent (in X) if and only if $AA^-CB^-B = C$, in which case the general solution is*

$$X = A^-CB^- + Y - A^-AYBB^-,$$

where Y is an arbitrary matrix.

Lemma 3.2 ([1]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $D \in \mathbb{C}^{m\times p}$, and $E \in \mathbb{C}^{n\times q}$. The matrix equations*

$$AX = B \quad \text{and} \quad XD = E \tag{3.1}$$

have a common solution if and only if each equation separately has a solution and $AE = BD$. In this case, the general solution of (3.1) *is*

$$X = A^-B + ED^- - A^-AED^- + (I_n - A^-A)Y(I_p - DD^-),$$

for arbitrary $A^- \in A\{1\}$, $D^- \in D\{1\}$, and $Y \in \mathbb{C}^{n\times p}$.

Now, we provide a parameterized representation of the W-weighted left G-Drazin inverses.

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the general solution of the system*

$$AWXWA = A \quad \text{and} \quad XW(AW)^{k+1} = (AW)^k, \tag{3.2}$$

is given by

$$\begin{aligned} X = {} & (I_m - YWAW)(AW)^dW^- + Y \\ & + (AW)^-(I_m - AWYW)(I_m - AW(AW)^d)A(WA)^-, \end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{m\times n}$ and for fixed but arbitrary $W^- \in W\{1\}$, $(AW)^- \in (AW)\{1\}$, and $(WA)^- \in (WA)\{1\}$.

Proof. From Lemma 3.1, the equation $XW(AW)^{k+1} = (AW)^k$ has a solution if and only if

$$(AW)^k(W(AW)^{k+1})^-W(AW)^{k+1} = (AW)^k \tag{3.3}$$

for some $(W(AW)^{k+1})^- \in W(AW)^{k+1}\{1\}$. Note that the equation (3.3) holds for $(W(AW)^{k+1})^- = [(AW)^d]^{k+1}W^-$. In fact, we know that $W(AW)^d = (WA)^dW$, $(WA)^{k+1}W = W(AW)^{k+1}$, and $BB^d = B^m(B^d)^m$ for any square matrix B with

$m \in \mathbb{N}$. Consequently, by taking $B = AW$ and $m = k+1$ we obtain

$$\begin{aligned} W(AW)^{k+1}[(AW)^d]^{k+1}W^-W(AW)^{k+1} &= WAW(AW)^dW^-W(AW)^{k+1} \\ &= WA(WA)^d[WW^-W](AW)^{k+1} \\ &= WA(WA)^dW(AW)^{k+1} \\ &= WA(WA)^d(WA)^{k+1}W \\ &= (WA)^{k+1}W \\ &= W(AW)^{k+1}. \end{aligned}$$

Thus, the general solution of the second equation in (3.2) is given by

$$X = (AW)^k(W(AW)^{k+1})^- + Z - ZW(AW)^{k+1}(W(AW)^{k+1})^-$$

for an arbitrary $Z \in \mathbb{C}^{m\times n}$, which is equivalent to

$$\begin{aligned} X &= (AW)^k[(AW)^d]^{k+1}W^- + Z - ZW(AW)^{k+1}[(AW)^d]^{k+1}W^- \\ &= (AW)^dW^- + Z - ZWAW(AW)^dW^- \end{aligned} \tag{3.4}$$

for an arbitrary $Z \in \mathbb{C}^{m\times n}$.

Now, substituting (3.4) in the first equation of (3.2), we have

$$AW(AW)^dW^-WA + AWZWA - AWZWAW(AW)^dW^-WA = A,$$

or equivalently

$$AWZW(I_m - AW(AW)^d)A = (I_m - AW(AW)^d)A. \tag{3.5}$$

Note that the above equation is consistent (in Z). In fact, as $AWXWA = A$, we have that $\operatorname{rank}(AW) = \operatorname{rank}(A) = \operatorname{rank}(WA)$, whence $AW(AW)^-A = A$ and $A(WA)^-WA = A$. Moreover, as $(WA)^- \in [W(I_m - AW(AW)^d)A]\{1\}$, we obtain

$$\begin{aligned} &AW(AW)^-(I_m - AW(AW)^d)A(WA)^-W(I_m - AW(AW)^d)A \\ &\quad = [AW(AW)^-A(I_m - W(AW)^dA)][(WA)^-WA(I_m - W(AW)^dA)] \\ &\quad = [A(I_m - W(AW)^dA)]\ [(WA)^-WA(I_m - W(AW)^dA)] \\ &\quad = [(I_m - AW(AW)^d)][A(WA)^-WA](I_m - W(AW)^dA) \\ &\quad = [(I_m - AW(AW)^d)]A(I_m - W(AW)^dA) \\ &\quad = (I_m - AW(AW)^d)(I_m - AW(AW)^d)A \\ &\quad = (I_m - AW(AW)^d)A. \end{aligned}$$

Thus, the general solution of (3.5) is given by

$$Z = (AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^-, \tag{3.6}$$

where $P := I_m - AW(AW)^d$.

Now, note that as $A(WA)^-WA = A$ we have

$$\begin{aligned}
PA(WA)^-WAW(AW)^d &= (I_m - AW(AW)^d)[A(WA)^-WA]W(AW)^d \\
&= (I_m - AW(AW)^d)AW(AW)^d \\
&= 0.
\end{aligned}$$

Thus, substituting (3.6) in (3.4), we obtain

$$\begin{aligned}
X &= (AW)^dW^- + (AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^- \\
&\quad - [(AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^-]WAW(AW)^dW^- \\
&= (AW)^dW^- + (AW)^-PA(WA)^- + Y - (AW)^-AWYWPA(WA)^- \\
&\quad - YWAW(AW)^dW^- \\
&= (I_m - YWAW)(AW)^dW^- + Y + (AW)^-(I_m - AWYW)PA(WA)^- \\
&= (I_m - YWAW)(AW)^dW^- + Y \\
&\quad + (AW)^-(I_m - AWYW)(I_m - AW(AW)^d)A(WA)^-. \qquad \square
\end{aligned}$$

Parameterized expressions for the W-weighted right G-Drazin inverses are also given below.

Theorem 3.4. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the general solution of the system*

$$AWXWA = A \quad \textit{and} \quad (WA)^{k+1}WX = (WA)^k,$$

is given by

$$\begin{aligned}
X &= W^-(WA)^d(I_n - WAWY) + Y \\
&\quad + (AW)^-A(I_n - WA(WA)^d)(I_n - WYWA)(WA)^-,
\end{aligned}$$

for arbitrary $Y \in \mathbb{C}^{m\times n}$ and for fixed but arbitrary $W^- \in W\{1\}$, $(AW)^- \in (AW)\{1\}$, and $(WA)^- \in (WA)\{1\}$.

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Then the general solution of the system*

$$AWXWA = A, \quad XW(AW)^{k+1} = (AW)^k, \quad \textit{and} \quad (WA)^{k+1}WX = (WA)^k \tag{3.7}$$

is given by

$$\begin{aligned}
X &= (AW)^dW^- \\
&\quad - (AW)^-A(I_n - WA(WA)^d)WYW(I_m - AW(AW)^d)A(WA)^- \\
&\quad + [W^-(WA)^d + (AW)^-A(I_n - WA(WA)^d) \\
&\quad + (I_m - W^-(WA)^dWAW)Y](I_n - WAW(AW)^dW^-),
\end{aligned} \tag{3.8}$$

for arbitrary $Y \in \mathbb{C}^{m\times n}$ and for fixed but arbitrary $W^- \in W\{1\}$, $(AW)^- \in (AW)\{1\}$, and $(WA)^- \in (WA)\{1\}$.

Proof. As in the proofs of Theorems 3.3 and 3.4, each equation $XW(AW)^{k+1} = (AW)^k$ and $(WA)^{k+1}WX = (WA)^k$ separately has a solution. Also, we know that $[(AW)^d]^{k+1}W^- \in W(AW)^{k+1}\{1\}$ and $W^-[(WA)^d]^{k+1} \in (WA)^{k+1}W\{1\}$. Thus, from Lemma 3.2, the general solution of the last two equations in (3.7) is given by

$$\begin{aligned} X &= W^-(WA)^d + (AW)^dW^- - W^-(WA)^dWAW(AW)^dW^- \\ &\quad + (I_m - W^-(WA)^dWAW)Z(I_n - WAW(AW)^dW^-) \end{aligned} \tag{3.9}$$

for arbitrary Z.

Substituting (3.9) in $AWXWA = A$ we get

$$A(I_n - WA(WA)^d)WZW(I_m - AW(AW)^d)A = A(I_n - WA(WA)^d). \tag{3.10}$$

Since $(AW)^-$ and $(WA)^-$ are inner inverses of $A(I_n - WA(WA)^d)W$ and $W(I_m - AW(AW)^d)A$, respectively, the equation (3.10) is consistent (in Z), and therefore from Lemma 3.1 we obtain

$$\begin{aligned} Z &= (AW)^-A(I_n - WA(WA)^d)(WA)^- + Y \\ &\quad - (AW)^-A(I_n - WA(WA)^d)WYW(I_m - AW(AW)^d)A(WA)^-. \end{aligned}$$

Now, replacing this expression of Z in (3.9) and using the facts that

$$(I_m - AW(AW)^d)A(WA)^-WAW(AW)^d = 0$$

and

$$(WA)^dWAW(AW)^-A(I_n - WA(WA)^d) = 0,$$

we obtain

$$\begin{aligned} X &= (AW)^dW^- + W^-(WA)^d(I_n - WAW(AW)^dW^-) \\ &\quad + (AW)^-A(I_n - WA(WA)^d)(I_n - WAW(AW)^dW^-) \\ &\quad + (I_m - W^-(WA)^dWAW)Y(I_n - WAW(AW)^dW^-) \\ &\quad - (AW)^-A(I_n - WA(WA)^d)WYW(I_m - AW(AW)^d)A(WA)^-, \end{aligned}$$

which implies (3.8). □

When $m = n$ and $W = I_n$, we get some useful parameterized representations of the different lateral G-Drazin inverses.

Corollary 3.6. *Let $A \in \mathbb{C}^{n\times n}$ and $A^- \in A\{1\}$. Then*

$$\begin{aligned} A\{l, GD\} &= \{Y + (I_n - YA)A^d + A^-(I_n - AY)(I_n - AA^d)AA^- : Y \in \mathbb{C}^{n\times n}\}; \\ A\{r, GD\} &= \{Y + A^d(I_n - AY) + A^-A(I_n - AA^d)(I_n - YA)A^- : Y \in \mathbb{C}^{n\times n}\}; \\ A\{l, r, GD\} &= \{A^d + A^-A(I_n - AA^d)[I_n - Y(I_n - AA^d)AA^-] \\ &\quad + (I_n - AA^d)Y(I_n - AA^d) : Y \in \mathbb{C}^{n\times n}\}. \end{aligned}$$

4. Weighted left and right G-Drazin partial orders

It is well known that the binary relation $\preceq^d$ (called the Drazin pre-order) is only a pre-order on $\mathbb{C}^{n\times n}$, because the nilpotent part of matrices is not considered. To extend the Drazin pre-order to the rectangular case, the order $\preceq^{d,W}$ [7], and more recently the order $\preceq_W^{GD}$ [4], were introduced and characterized. However, none of these weighted matrix orders has the antisymmetric property, and each therefore yields only pre-orders on $\mathbb{C}^{m\times n}$. The aim of this section is to give new binary relations on the set $\mathbb{C}^{m\times n}$ by using W-weighted left (resp. right) G-Drazin inverses, resulting in matrix partial orders.

Definition 4.1. Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$. Then we say that

(i) A is below B under the W-weighted left G-Drazin order (denoted by $A \leq_W^{l,GD} B$) if there exist $X_1, X_2 \in A\{l, W\text{-}GD\}$ such that

$$AWX_1 = BWX_1, \quad X_2WA = X_2WB;$$

(ii) A is below B under the W-weighted right G-Drazin order (denoted by $A \leq_W^{r,GD} B$) if there exist $X_1, X_2 \in A\{r, W\text{-}GD\}$ such that

$$AWX_1 = BWX_1, \quad X_2WA = X_2WB;$$

(iii) A is below B under the W-weighted left-right G-Drazin order (denoted by $A \leq_W^{l,r,GD} B$) if there exist $X_1, X_2 \in A\{l, r, W\text{-}GD\}$ such that

$$AWX_1 = BWX_1, \quad X_2WA = X_2WB.$$

Remark 4.2. When $m = n$ and $W_n = I_n$ in the above definition, we recover [16, Definition 7.1] and [17, Definition 3.1].

Our first result gives some useful characterizations of the binary relation $\leq_W^{l,GD}$.

Theorem 4.3. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (2.1)*, then the following conditions are equivalent:*

(a) $A \leq_W^{l,GD} B$*;*
(b) *there exists $X \in A\{l, W\text{-}GD\}$ such that $AWX = BWX$ and $XWA = XWB$;*
(c) *there exists $X \in A\{l, W\text{-}GD\}$ such that*

$$\begin{aligned} X &= P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} Q^{-1}, \\ B &= P \begin{bmatrix} A_1 & -A_1W_1X_3W_2B_2 \\ 0 & B_2 \end{bmatrix} Q^{-1}, \end{aligned} \tag{4.1}$$

where $X_3W_2A_2 = 0$, $W_2X_2W_2 \in A_2\{1\}$, $A_2W_2X_2 = B_2W_2X_2$, and $X_2W_2A_2 = X_2W_2B_2$;
(d) *there exists $X \in A\{l, W\text{-}GD\}$ such that $AWXWB = BWXWA = A$.*

Proof. $(a) \Rightarrow (b)$ If $A \leq_W^{l,GD} B$, there exist $X_1, X_2 \in A\{l, W\text{-}GD\}$ such that $AWX_1 = BWX_1$ and $X_2WA = X_2WB$. Now, we consider the matrix $X := X_1WAWX_2$. First, we note that $X \in A\{l, W\text{-}GD\}$. In fact,

$$AWXWA = (AWX_1WA)WX_2WA = AWX_2WA = A$$

and

$$\begin{aligned} XW(AW)^{k+1} &= X_1WAW(X_2W(AW)^{k+1}) = X_1WAW(AW)^k \\ &= X_1W(AW)^{k+1} = (AW)^k. \end{aligned}$$

Moreover,

$$AWX = (AWX_1)WAWX_2 = BWX_1WAWX_2 = BWX$$

and

$$XWA = X_1WAW(X_2WA) = X_1WAWX_2WB = XWB.$$

$(b) \Rightarrow (c)$ Let $X \in A\{l, W\text{-}GD\}$ be such that $AWX = BWX$ and $XWA = XWB$. Thus, Theorem 2.6 implies

$$X = P \begin{bmatrix} (W_1A_1W_1)^{-1} & X_3 \\ 0 & X_2 \end{bmatrix} Q^{-1}, \tag{4.2}$$

where $X_3W_2A_2 = 0$ and $W_2X_2W_2 \in A_2\{1\}$. Let us consider the partition of $B = P \left[\begin{smallmatrix} B_1 & B_3 \\ B_4 & B_2 \end{smallmatrix}\right] Q^{-1}$. Hence,

$$\begin{aligned} AWX &= P \begin{bmatrix} W_1^{-1} & A_1W_1X_3 \\ 0 & A_2W_2X_2 \end{bmatrix} Q^{-1}, \\ BWX &= P \begin{bmatrix} B_1(W_1A_1)^{-1} & B_1W_1X_3 + B_3W_2X_2 \\ B_4(W_1A_1)^{-1} & B_4W_1X_3 + B_2W_2X_2 \end{bmatrix} Q^{-1}, \\ XWA &= P \begin{bmatrix} W_1^{-1} & 0 \\ 0 & X_2W_2A_2 \end{bmatrix} Q^{-1}, \\ XWB &= P \begin{bmatrix} (A_1W_1)^{-1}B_1 & (A_1W_1)^{-1}B_3 + X_3W_2B_2 \\ 0 & X_2W_2B_2 \end{bmatrix} Q^{-1}. \end{aligned}$$

Therefore,

$$AWX = BWX \quad \Leftrightarrow \quad A_1 = B_1,\ B_4 = 0,\ B_3W_2X_2 = 0,\ A_2W_2X_2 = B_2W_2X_2. \tag{4.3}$$

In consequence,

$$XWA = XWB \quad \Leftrightarrow \quad B_3 + A_1W_1X_3W_2B_2 = 0,\ X_2W_2A_2 = X_2W_2B_2. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we obtain the desired implication.

$(c) \Rightarrow (d)$ This implication follows from direct calculations.

$(d) \Rightarrow (a)$ Let $X \in A\{l, W\text{-}GD\}$ be such that $AWXWB = BWXWA = A$. Taking $Y = XWAWX$ we have

$$AWYWA = AW(XWAWX)WA = (AWXWA)WXWA = AWXWA = A$$

and

$$YW(AW)^{k+1} = XWAW[XW(AW)^{k+1}] = XW(AW)^{k+1} = (AW)^k.$$

Thus, $Y \in A\{l, W\text{-}GD\}$. Moreover, as $AWXWA = A$ and $BWXWA = A$, we obtain

$$AWY = (AWXWA)WX = AWX = BW(XWAWX) = BWY.$$

Similarly, from $AWXWB = A$, we get $YWA = YWB$. Thus, $A \leq_W^{l,GD} B$. □

Remark 4.4. By the proof of Theorem 4.3, we deduce that

$$A\{l, W\text{-}GD\} \cdot WAW \cdot A\{l, W\text{-}GD\} \subseteq A\{l, W\text{-}GD\}.$$

Corollary 4.5. *Let $X, B \in \mathbb{C}^{m\times n}$ be of the form* (4.1). *Then Z is a W-weighted left G-Drazin inverse of B if and only if*

$$Z = P\begin{bmatrix}(W_1A_1W_1)^{-1} & Z_3\\ 0 & Z_2\end{bmatrix}Q^{-1}, \tag{4.5}$$

where $Z_3W_2B_2 = X_3W_2B_2$ and $Z_2 \in B_2\{l, W_2\text{-}GD\}$.

Proof. Let $Z \in B\{W\text{-}GD\}$ and consider

$$Z = P\begin{bmatrix}Z_1 & Z_3\\ Z_4 & Z_2\end{bmatrix}Q^{-1}, \tag{4.6}$$

according to the sizes of the blocks in B. From (4.1) and (4.6), we have

$$\begin{aligned} BW &= P\begin{bmatrix}A_1W_1 & -A_1W_1X_3W_2B_2W_2\\ 0 & B_2W_2\end{bmatrix}P^{-1},\\ ZWBW &= P\begin{bmatrix}Z_1W_1A_1W_1 & M\\ Z_4W_1A_1W_1 & N\end{bmatrix}P^{-1}, \end{aligned} \tag{4.7}$$

where $S = -A_1W_1X_3W_2B_2W_2$, $M = Z_1W_1S + Z_3W_2B_2W_2$, and $N = Z_4W_1S + Z_2W_2B_2W_2$.

As BW is an upper-triangular block matrix with its block A_1W_1 nonsingular, from [2, Theorem 7.7.2], we get

$$(BW)^d = P\begin{bmatrix}(A_1W_1)^{-1} & R\\ 0 & (B_2W_2)^d\end{bmatrix}P^{-1} \quad \text{for some matrix } R. \tag{4.8}$$

Now, from (4.7) and (4.8), we obtain

$$ZWBW(BW)^d = P\begin{bmatrix}Z_1W_1 & Z_1W_1A_1W_1R + M(B_2W_2)^d\\ Z_4W_1 & Z_4W_1A_1W_1R + N(B_2W_2)^d\end{bmatrix}P^{-1}. \tag{4.9}$$

Thus, from (4.8) and (4.9), direct calculations imply that $ZWBW(BW)^d = (BW)^d$ holds if and only if

$$\begin{aligned} Z_1 &= (W_1A_1W_1)^{-1},\\ Z_4 &= 0,\\ Z_3W_2B_2W_2 &= X_3W_2B_2W_2,\\ Z_2W_2B_2W_2(B_2W_2)^d &= (B_2W_2)^d. \end{aligned} \tag{4.10}$$

From (4.10), we obtain

$$\begin{aligned} &BWZWB \\ &= P\begin{bmatrix} A_1 & A_1W_1Z_3W_2B_2 - A_1W_1X_3W_2B_2W_2Z_2W_2B_2 - A_1W_1X_3W_2B_2 \\ 0 & B_2W_2Z_2W_2B_2 \end{bmatrix} Q^{-1}, \end{aligned}$$

whence

$$BWZWB = B \quad \Leftrightarrow \quad B_2W_2Z_2W_2B_2 = B_2,\ Z_3W_2B_2 = X_3W_2B_2. \tag{4.11}$$

Thus, (4.10) and (4.11) imply (4.5). □

In a manner similar to Theorem 4.3, we derive characterizations of the W-weighted right G-Drazin order.

Theorem 4.6. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (2.1)*, then the following conditions are equivalent:*

(a) $A \leq_W^{r,GD} B$;
(b) *there exists $X \in A\{r, W\text{-}GD\}$ such that $AWX = BWX$ and $XWA = XWB$;*
(c) *there exists $X \in \mathbb{C}^{m\times n}$ such that*

$$X = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ X_4 & X_2 \end{bmatrix} Q^{-1} \quad \text{and} \quad B = P\begin{bmatrix} A_1 & 0 \\ -B_2W_2X_4A_1W_1 & B_2 \end{bmatrix} Q^{-1}, \tag{4.12}$$

where $A_2W_2X_4 = 0$, $W_2X_2W_2 \in A_2\{1\}$, $A_2W_2X_2 = B_2W_2X_2$, and $X_2W_2A_2 = X_2W_2B_2$;
(d) *there exists $X \in A\{r, W\text{-}GD\}$ such that $AWXWB = BWXWA = A$.*

Corollary 4.7. *Let $X, B \in \mathbb{C}^{m\times n}$ be of the form* (4.12)*. Then Z is a W-weighted right G-Drazin inverse of B if and only if*

$$Z = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ Z_4 & Z_2 \end{bmatrix} Q^{-1},$$

where $B_2W_2Z_4 = B_2W_2X_4$ and $Z_2 \in B_2\{r, W_2\text{-}GD\}$.

Theorem 4.8. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$. If A and W are written as in* (2.1)*, then the following conditions are equivalent:*

(a) $A \leq_W^{l,r,GD} B$;
(b) *there exists $X \in A\{l, r, W\text{-}GD\}$ such that $AWX = BWX$ and $XWA = XWB$;*
(c) *there exists $X \in \mathbb{C}^{m\times n}$ such that*

$$X = P\begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & X_2 \end{bmatrix} Q^{-1} \quad \text{and} \quad B = P\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix} Q^{-1}, \tag{4.13}$$

where $W_2X_2W_2 \in A_2\{1\}$, $A_2W_2X_2 = B_2W_2X_2$, and $X_2W_2A_2 = X_2W_2B_2$;
(d) *there exists $X \in A\{l, r, W\text{-}GD\}$ such that $AWXWB = BWXWA = A$.*

Corollary 4.9. *Let $X, B \in \mathbb{C}^{m\times n}$ be of the form* (4.13)*. Then Z is a W-weighted left-right G-Drazin inverse of B if and only if*

$$Z = P \begin{bmatrix} (W_1A_1W_1)^{-1} & 0 \\ 0 & Z_2 \end{bmatrix} Q^{-1},$$

where $Z_2 \in B_2\{l, r, W_2\text{-}GD\}$.

In order to prove that the binary relations $\leq_W^{l,GD}$, $\leq_W^{r,GD}$ and $\leq_W^{l,r,GD}$ are partial orders on the set of complex rectangular matrices, we need an auxiliary lemma.

Lemma 4.10. *Let $W \in \mathbb{C}^{n\times m}$ and $A, B \in \mathbb{C}^{m\times n}$.*

(a) *If $A \leq_W^{l,GD} B$, then $B\{l, W\text{-}GD\} \subseteq A\{l, W\text{-}GD\}$.*
(b) *If $A \leq_W^{r,GD} B$, then $B\{r, W\text{-}GD\} \subseteq A\{r, W\text{-}GD\}$.*
(c) *If $A \leq_W^{l,r,GD} B$, then $B\{l, r, W\text{-}GD\} \subseteq A\{l, r, W\text{-}GD\}$.*

Proof. (a) Since $A \leq_W^{l,GD} B$, by Theorem 4.3 (b), there exists $X \in A\{l, W\text{-}GD\}$ such that $AWX = BWX$ and $XWA = XWB$. Suppose that $Y \in B\{l, W\text{-}GD\}$, i.e., $BWYWB = B$ and $YWBW(BW)^d = (BW)^d$. We will prove that $Y \in A\{l, W\text{-}GD\}$. In fact, as $A = AWXWA$ due to $X \in A\{l, W\text{-}GD\}$, we have

$$\begin{aligned} AWYWA &= (AWXWA)WYW(AWXWA) \\ &= AW(XWA)WYW(AWX)WA \\ &= AW(XWB)WYW(BWX)WA \\ &= AWXW(BWYWB)WXWA \\ &= AW(XWB)WXWA \\ &= AW(XWA)WXWA \\ &= (AWXWA)WXWA \\ &= AWXWA \\ &= A. \end{aligned}$$

From (4.1) and [2, Theorem 7.7.1], we have

$$\begin{aligned} (AW)^d &= P \begin{bmatrix} (A_1W_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \\ (BW)^d &= P \begin{bmatrix} (A_1W_1)^{-1} & R \\ 0 & (B_2W_2)^d \end{bmatrix} P^{-1} \end{aligned} \tag{4.14}$$

for some matrix R.

From (4.14), it is easy to see that $(AW)^d = (BW)^d BW(AW)^d$. Hence, using that $AWX = BWX$, $XWAW(AW)^d = (AW)^d$, and $YWBW(BW)^d = (BW)^d$:

$$\begin{aligned} YWAW(AW)^d &= YW(AWX)WAW(AW)^d = YWBW(XWAW(AW)^d) \\ &= YWBW(AW)^d = YWBW(BW)^d BW(AW)^d \\ &= (BW)^d BW(AW)^d = (AW)^d. \end{aligned}$$

A similar proof can be applied to statements (b) and (c). □

Theorem 4.11. *Let $W \in \mathbb{C}^{n\times m}$. Then the binary relation $\leq_W^{l,GD}$ is a partial order on $\mathbb{C}^{m\times n}$.*

Proof. Clearly, the binary relation $\leq_W^{l,GD}$ is reflexive. Suppose $A, B, C \in \mathbb{C}^{m\times n}$ are such that $A \leq_W^{l,GD} B$ and $B \leq_W^{l,GD} C$. By Theorem 4.3 (b) we know that there exist $X \in A\{l, W\text{-}GD\}$ and $Y \in B\{l, W\text{-}GD\}$ such that $AWX = BWX$, $XWA = XWB$, $BWY = CWY$, and $YWB = YWC$. Now, by applying Lemma 4.10, we deduce that $Y \in A\{l, W\text{-}GD\}$. Thus,

$$\begin{aligned} AWYWC &= AWYWB = AW(XWA)WYWB = AWXW(BWYWB) \\ &= AW(XWB) = AWXWA = A. \end{aligned}$$

Similarly, we obtain $CWYWA = A$. Therefore, by Theorem 4.3 (d) we obtain $A \leq_W^{l,GD} C$.

In order to prove that the binary relation is antisymmetric, we have to show that the conjunction $A \leq_W^{l,GD} B$ and $B \leq_W^{l,GD} A$ implies $A = B$. By Theorem 4.3 (b), the second of these conditions ensures that there exists $Y \in B\{l, W\text{-}GD\}$ such that $BWY = AWY$ and $YWB = YWA$. By Lemma 4.10, we know that $Y \in A\{l, W\text{-}GD\}$. In consequence, $A = (AWY)WA = BW(YWA) = BWYWB = B$. □

By a similar procedure, from Theorem 4.6, Theorem 4.8, and Lemma 4.10 we can identify two new matrix partial orders on the set of complex rectangular matrices.

Theorem 4.12. *Let $W \in \mathbb{C}^{n\times m}$. Then the binary relations $\leq_W^{r,GD}$ and $\leq_W^{l,r,GD}$ are partial orders on the set $\mathbb{C}^{m\times n}$.*

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