REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 69, No. 1, 2026, Pages 143–154
Published online: December 19, 2025
https://doi.org/10.33044/revuma.5106

ADDITIVE AND MULTIPLICATIVE RELATIONS WITH ALGEBRAIC CONJUGATES

ARTŪRAS DUBICKAS AND PAULIUS VIRBALAS

Abstract. We prove that every nontrivial additive relation between algebraic conjugates of degree d over $\mathbb{Q}$ has a corresponding multiplicative relation. The proof is constructive. The reverse statement is not true. These findings supplement the research of Smyth, Dixon, Girstmair and others. In addition, following a result of Kitaoka we show that all additive relations between four distinct algebraic conjugates of degree 4 over $\mathbb{Q}$ can be described as $\mathbb{Z}$-linear combinations of several basic nontrivial relations. On the other hand, we prove that an analogous result no longer holds for algebraic conjugates of degree 6 over $\mathbb{Q}$.

1. Introduction

Let K be a field and let α be an algebraic number of degree d with conjugates $\alpha_1, \dots, \alpha_d$ over K. The additive relation

$$k_1\alpha_1 + \dots + k_d\alpha_d = 0 \tag{1.1}$$

is called *nontrivial* if it holds for some $k_1, \dots, k_d \in K$, not all equal. Likewise, the multiplicative relation

$$\alpha_1^{k_1} \dots \alpha_d^{k_d} = 1 \tag{1.2}$$

is called *nontrivial* if it holds for some $k_1, \dots, k_d \in \mathbb{Z}$, not all equal. Note that if $K = \mathbb{Q}$, then multiplying (1.1) by an appropriate integer we can assume that $k_1, \dots, k_d \in \mathbb{Z}$.

First, note that not every nontrivial multiplicative relation has a corresponding nontrivial additive relation. That is, if (1.2) holds for some α of degree d with $(k_1, \dots, k_d) \in \mathbb{Z}^d$, then there may not exist another algebraic number of the same degree d and some $(k_1, \dots, k_d) \in \mathbb{Z}^d$ satisfying (1.1). Indeed, consider the polynomial $x^6 - 3x^3 + 1$, which is irreducible over $\mathbb{Q}$. Three of its roots lie on the circle $|z| = \sqrt[3]{(3 + \sqrt{5})/2}$. One of them, say α_1, is real, and two others, say α_2 and α_3,

2020 *Mathematics Subject Classification.* Primary 11R04; Secondary 11D09, 11R09, 11R32, 12F10.

Key words and phrases. conjugate algebraic numbers, additive relations, multiplicative relations, Galois group.

are complex conjugates. Consequently, their product $\alpha_2\alpha_3$ equals α_1^2, which implies the multiplicative relation $\alpha_1^2\alpha_2^{-1}\alpha_3^{-1} = 1$. This means that (1.2) holds for the vector

$$(k_1, k_2, k_3, k_4, k_5, k_6) = (2, -1, -1, 0, 0, 0) \tag{1.3}$$

and the sextic algebraic number α_1. In contrast, by a result of Smyth [23, Lemma 1], we know that there is no corresponding nontrivial additive relation for the vector (1.3) between distinct algebraic conjugates of a sextic algebraic number.

The first result of this paper shows that the converse is actually true if the ground field is taken to be $\mathbb{Q}$. That is, if an additive relation (1.1) holds for some α of degree d over $\mathbb{Q}$ and some $(k_1, \dots, k_d) \in \mathbb{Z}^d$, then there exists β of degree d over $\mathbb{Q}$ such that the corresponding multiplicative relation (1.2) holds with the same vector $(k_1, \dots, k_d)$.

Theorem 1.1. *Assume that there is an algebraic number α of degree d over $\mathbb{Q}$ with conjugates $\alpha_1, \dots, \alpha_d$ and some $k_1, \dots, k_d \in \mathbb{Z}$, not all equal, such that*

$$k_1\alpha_1 + \dots + k_d\alpha_d = 0.$$

Then, there is an algebraic number β of degree d over $\mathbb{Q}$ with conjugates $\beta_1, \dots, \beta_d$ such that

$$\beta_1^{k_1} \dots \beta_d^{k_d} = 1.$$

The proof of Theorem 1.1 is constructive, i.e., it shows how to construct β_i from $\alpha_1, \dots, \alpha_d$. This supplements the techniques applied by Girstmair in [13], where the link between additive and multiplicative relations was investigated by non-constructive arguments.

The research in this area originated with the works of Kurbatov [19], Girstmair [12] and Smyth [23, 24]. The latter paper of Smyth contains a similar characterization of nontrivial additive and multiplicative relations, but the conjugates of the corresponding algebraic number are allowed to be equal and there is no restriction on its degree. (Note that nontrivial has a different meaning in (1.1), (1.2) and in [24].) The problem of determining whether a particular nontrivial additive or multiplicative relation may occur is then reduced to a purely combinatorial problem; see [24, Theorem 1]. In that case, additive and multiplicative problems are in some sense equivalent [24, Corollary 1].

Since then, some general results related to (1.1) or (1.2) have been obtained in [3, 5, 7, 8, 9, 13, 17]. However, the only nontrivial relations of the form (1.1) or (1.2) that are completely understood remain the simplest ones, namely,

$$\alpha_i + \alpha_j = 0 \quad \text{and} \quad \alpha_i\alpha_j = 1,$$

where α_i, α_j denote distinct algebraic conjugates of an algebraic number α of degree $d > 2$ over $\mathbb{Q}$. Note that the relation $\alpha_i + \alpha_j = 0$ occurs if and only if the minimal polynomial f of α is of the form $f(x) = g(x^2)$ for some $g \in \mathbb{Q}[x]$, while the relation $\alpha_i\alpha_j = 1$ occurs if and only if the minimal polynomial f of α satisfies $f(x) = x^d f(1/x)$ for $x \neq 0$. Note that in both cases d must be even, but the degree of α whose three conjugates sum to zero is not necessarily divisible by 3 (see [11]).

In this direction, it is still not known what the sufficient and necessary conditions are for the relations

$$\alpha_i + \alpha_j + \alpha_k = 0, \quad \alpha_i\alpha_j\alpha_k = 1 \tag{1.4}$$

or

$$\alpha_i + \alpha_j = \alpha_k, \quad \alpha_i\alpha_j = \alpha_k \tag{1.5}$$

to hold.

Since the general solution is not attainable at the moment, the topic of nontrivial relations has been mainly investigated either by imposing restrictions on the Galois group G of the splitting field of f, where $f(x)$ is the minimal polynomial of α over $\mathbb{Q}$, or by fixing the degree d of α. Various aspects of relations (1.4) or (1.5) were investigated in [2, 4, 10, 14, 15, 16, 22, 25], while the papers [20, 21, 23] deal with slightly different types of nontrivial relations.

In this paper, we deviate from the research described earlier by seeking to obtain some structural insights about the nature of all possible nontrivial relations for a fixed degree d of α by considering all possible relations as a lattice in $\mathbb{Z}^d$. See, e.g., [26, 27] for some computational results related to the lattice in $\mathbb{Z}^d$ of multiplicative relations (1.2) for some α of degree d.

If d is a prime number, then there are no nontrivial additive relations between algebraic conjugates of degree d (see [4, 19]). For $d = 4$ we show the following result about all possible nontrivial additive relations (1.1) over $\mathbb{Q}$.

Theorem 1.2. *Assume that there is an algebraic number α of degree 4 over $\mathbb{Q}$ with conjugates $\alpha_1, \dots, \alpha_4$ and some $k_1, \dots, k_4 \in \mathbb{Z}$, not all equal, such that*

$$k_1\alpha_1 + \cdots + k_4\alpha_4 = 0.$$

Then, after re-indexing of the conjugates if necessary, we have

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0 \tag{1.6}$$

and

$$k_1 = k_2, \quad k_3 = k_4. \tag{1.7}$$

In the proof of Theorem 1.2, we obtain that for some $r \in \mathbb{Q}$ we have $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = r$. The relation (1.6), although not stated explicitly, can be derived from the work of Kitaoka [18]. Our main contribution is the proof that the relation (1.7) must also hold. Moreover, it will be shown that if $r \neq 0$, then any nontrivial additive relation with four conjugates of an algebraic number of degree 4 over $\mathbb{Q}$, up to the re-indexing of conjugates, has the form

$$\mathbb{Z}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4), \tag{1.8}$$

while for $r = 0$ it has the more general form

$$\mathbb{Z}(\alpha_1 + \alpha_2) + \mathbb{Z}(\alpha_3 + \alpha_4). \tag{1.9}$$

Hence, for $d = 4$, there is a finite list of nontrivial additive relations (one nontrivial relation $\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0$ or two such relations $\alpha_1 + \alpha_2 = 0$ and $\alpha_3 + \alpha_4 = 0$) such that any possible additive relation is a $\mathbb{Z}$-linear form of those one or two basic relations.

It seems very likely that such a result holds only for $d = 4$, and we should not expect it for other composite degrees d. We next show that it is impossible to expect a result analogous to Theorem 1.2 and (1.8), (1.9) for $d = 6$.

Theorem 1.3. *Let $m \in \mathbb{N}$ and let*

$$f_i(x_1, \dots, x_6) = \sum_{j=1}^{6} u_{ij} x_j,$$

$i = 1, \dots, m$, be a set of m linear forms such that for each i the coefficients $u_{i1}, \dots, u_{i6} \in \mathbb{Z}$ are not all equal. Then, there are infinitely many algebraic numbers α of degree 6 over $\mathbb{Q}$ whose conjugates $\alpha_1, \dots, \alpha_6$ satisfy a nontrivial additive relation of the form

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0,$$

where $k_1, k_2, k_3 \in \mathbb{N}$, but this relation is not of the form

$$\sum_{j \in S} \mathbb{Z} f_j(\alpha_1, \dots, \alpha_6),$$

where $S \subseteq \{1, \dots, m\}$ is such that $f_j(\alpha_1, \dots, \alpha_6) = 0$ for each $j \in S$.

The paper is structured as follows. In Section 2, we state some auxiliary results that will be applied later. Then, in Sections 3, 4 and 5, we present the proofs of Theorems 1.1, 1.2 and 1.3, respectively.

2. Auxiliary results

The following lemma will be the main tool in the proof of Theorem 1.1.

Lemma 2.1 ([6, Lemma 2.1]). *Let n and X be positive integers, and let $z_1, \dots, z_n$ be pairwise distinct complex numbers. Then, there is a positive integer k for which the equality*

$$(k + z_1)^{x_1} \dots (k + z_n)^{x_n} = 1$$

does not hold for $x_1, \dots, x_n \in \mathbb{Z}$ satisfying $|x_1|, \dots, |x_n| \le X$, unless we have $x_1 = \dots = x_n = 0$.

The next two lemmas will be used in the proof of Theorem 1.2.

Lemma 2.2 ([18, Proposition 3]). *Let $p \in \mathbb{Q}[x]$ be a quartic monic irreducible polynomial over $\mathbb{Q}$. If there is a nontrivial additive relation between the roots of p, then*

$$p(x) = f(g(x))$$

for quadratic polynomials g and f.

Lemma 2.3 ([25, Lemma 2.3]). *If for some non-zero rationals m_i, m_j and distinct algebraic conjugates α_i, α_j over $\mathbb{Q}$ we have*

$$m_i\alpha_i + m_j\alpha_j = 0,$$

then $m_i = m_j$ and $\alpha_i = -\alpha_j$.

The next lemma is essential for the proof of Theorem 1.3.

Lemma 2.4. *Let $\mathcal{S}$ be a finite set, and let R be a positive integer. Then, there are infinitely many triplets of positive integers (a, b, c) satisfying*

$$a^2 + b^2 - ab = c^2, \tag{2.1}$$

$$a, b, c > R, \tag{2.2}$$

and

$$\frac{a}{b} \notin \mathcal{S}. \tag{2.3}$$

Proof. Our original proof was based on considering Pell's equation $X^2-3Y^2 = 1$. It has infinitely many solutions $(X, Y) = (2x_n, 2y_n-1)$ with $(x_n, y_n) \in \mathbb{N}^2$. Then, one can take $(a, b, c) = (y_n, 2y_n - 1, x_n)$ and for infinitely many $n \in \mathbb{N}$ both conditions (2.2), (2.3) clearly hold.

Alternatively, the referee observed that setting in (2.1) $X = a/b$ and $Y = c/b$ we get the equation

$$X^2 - X + 1 - Y^2 = 0.$$

It is a non-degenerate conic with integer coefficients. Since it has a rational solution $(X, Y) = (1, 1)$, it must have infinitely many rational solutions. In fact, it is easy to see that infinitely many of them satisfy $X, Y > 0$. So, for infinitely many of them, the conditions (2.2) and (2.3) also hold for the corresponding positive integers a, b, c, where $(a, b, c) = (bX, b, bY)$ and b is the least positive integer for which $bX, bY \in \mathbb{N}$. □

3. Proof of Theorem 1.1

Assume that there exists an algebraic number α of degree d over $\mathbb{Q}$ with conjugates $\alpha_1, \dots, \alpha_d$ and some $k_1, \dots, k_d \in \mathbb{Z}$, not all equal, such that

$$k_1\alpha_1 + \cdots + k_d\alpha_d = 0. \tag{3.1}$$

Let L be the Galois closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Set $G = \mathrm{Gal}(L/\mathbb{Q})$, and put $n = [L : \mathbb{Q}] = |G|$. Let $\sigma_1, \dots, \sigma_n$ denote all distinct automorphisms of G. By the normal basis theorem (see [1]), there exists $w \in L$ such that $w_j = \sigma_j(w)$, where $j = 1, \dots, n$, form a basis of L. Hence, there are rational numbers $a_1, \dots, a_n$ for which

$$\alpha = a_1w_1 + \cdots + a_nw_n.$$

Note that replacing α by its proper integer multiple does not change the property (3.1), so we can assume that $a_1, \dots, a_d \in \mathbb{Z}$.

Next, observe that for each $\sigma \in G$, we have

$$\sigma(\alpha) = a_1\sigma(w_1) + \cdots + a_n\sigma(w_n) = a_{\tau(1)}w_1 + \cdots + a_{\tau(n)}w_n,$$

where $\tau = \sigma^{-1} \in G$ acts as a permutation of the set $\{1, \dots, n\}$. Note that the sums $a_{\tau(1)}w_1 + \cdots + a_{\tau(n)}w_n$ and $a_{\tau'(1)}w_1 + \cdots + a_{\tau'(n)}w_n$ are distinct if and only if the vectors $(a_{\tau(1)}, \dots, a_{\tau(n)})$ and $(a_{\tau'(1)}, \dots, a_{\tau'(n)})$ are distinct. Since the degree of α over $\mathbb{Q}$ is d, there are d distinct vectors among $(a_{\tau(1)}, \dots, a_{\tau(n)})$ as $\tau = \sigma^{-1}$ runs over all n distinct automorphisms $\sigma \in G$. (All other possible vectors are repetitions of those d vectors, and each distinct vector occurs exactly n/d times.)

Select d automorphisms $\tau_1, \dots, \tau_d \in G$ so that

$$\alpha_i = a_{\tau_i(1)} w_1 + \cdots + a_{\tau_i(n)} w_n,$$

where $i = 1, \dots, d$. Then, from (3.1), and the fact that $w_1, \dots, w_n$ is a basis of L, it follows that

$$k_1 a_{\tau_1(j)} + \cdots + k_d a_{\tau_d(j)} = 0 \tag{3.2}$$

for $j = 1, \dots, n$.

Now we are in the position to construct an algebraic number β of degree d with conjugates $\beta_1, \dots, \beta_d$ over $\mathbb{Q}$ that satisfy

$$\beta_1^{k_1} \dots \beta_d^{k_d} = 1$$

with the same vector $(k_1, \dots, k_d) \in \mathbb{Z}^d$ as in (3.1). Consider the number

$$\beta = (k + w_1)^{a_1} \dots (k + w_n)^{a_n}, \tag{3.3}$$

where k is a positive integer that will be chosen later. It is clear that $\beta \in L$, since $w_1, \dots, w_n \in L$ and $a_1, \dots, a_n \in \mathbb{Z}$. Applying the automorphism $\sigma_i = \tau_i^{-1}$ to (3.3) we obtain

$$\sigma_i(\beta) = (k + w_{\sigma_i(1)})^{a_1} \dots (k + w_{\sigma_i(n)})^{a_n} = (k + w_1)^{a_{\tau_i(1)}} \dots (k + w_n)^{a_{\tau_i}(n)}.$$

Set $\beta_i = \sigma_i(\beta)$, where $i = 1, \dots, d$. Then, from (3.2) it follows that

$$\beta_1^{k_1} \dots \beta_d^{k_d} = 1.$$

It remains to show that β is of degree d over $\mathbb{Q}$. Observe that the conjugates of β are all of the form

$$(k + w_1)^{a_{\tau(1)}} \dots (k + w_n)^{a_{\tau(n)}}, \tag{3.4}$$

where τ runs through all n distinct automorphisms of G corresponding to n permutations of the set $\{1, \dots, n\}$. Consequently, the degree of α equals the number of distinct products among the n conjugates (not necessarily distinct) in (3.4). Now, in Lemma 2.1, choosing

$$X = 2 \max\{|a_1|, \dots, |a_n|\}$$

and an appropriate $k \in \mathbb{N}$, we conclude that two such products are equal if and only if the vectors $(a_{\tau(1)}, \dots, a_{\tau(n)})$ and $(a_{\tau'(1)}, \dots, a_{\tau'(n)})$ are equal for some permutations τ and τ'. This implies that the degree of β is equal to the number of distinct vectors among $(a_{\tau(1)}, \dots, a_{\tau(n)})$. As it was mentioned earlier, the number of distinct vectors is equal to d. This completes the proof of the theorem. □

4. Proof of Theorem 1.2

Assume that there is an algebraic number α of degree 4 over $\mathbb{Q}$ with conjugates $\alpha_1, \dots, \alpha_4$ and some $k_1, \dots, k_4 \in \mathbb{Z}$, not all equal, such that

$$k_1 \alpha_1 + \cdots + k_4 \alpha_4 = 0. \tag{4.1}$$

From Lemma 2.2 we deduce that the minimal polynomial of α over $\mathbb{Q}$ must be of the form $f(g(x))$, where $f, g \in \mathbb{Q}[x]$ are both quadratic. Writing

$$g(x) = (x + u)^2 + u' \quad \text{and} \quad f(x) = (x + v)^2 + v'$$

with $u, u', v, v' \in \mathbb{Q}$ we deduce that

$$f(g(x)) = ((x+u)^2 + u' + v)^2 + v' = (x+u)^4 + w(x+u)^2 + s, \tag{4.2}$$

where $u, w, s \in \mathbb{Q}$.

Consider the polynomial

$$h(x) = x^4 + wx^2 + s.$$

By its construction, h is irreducible over $\mathbb{Q}$. From (4.2) it follows that $\beta = \alpha + u$ is a root of h. Moreover, it is clear that $-\beta$ is also a root of h. Let $\beta, -\beta, \beta', -\beta'$ denote all four roots of h. Then

$$h(x) = (x-\beta)(x+\beta)(x-\beta')(x+\beta') = (x^2 - \beta^2)(x^2 - \beta'^2). \tag{4.3}$$

Writing $\alpha_1 = \beta - u$, $\alpha_2 = -\beta - u$, $\alpha_3 = \beta' - u$ and $\alpha_4 = -\beta' - u$, we obtain

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = -2u \tag{4.4}$$

which implies the relation in (1.6).

Next, set $m_1 = k_1 - k_2$ and $m_2 = k_3 - k_4$. In view of (4.1) and (4.4), we get

$$0 = k_1\alpha_1 + \cdots + k_4\alpha_4 = m_1\alpha_1 - 2uk_2 + m_2\alpha_3 - 2uk_4.$$

By setting $m = 2uk_2 + 2uk_4$, we see that

$$m_1\alpha_1 + m_2\alpha_3 = m \in \mathbb{Q}. \tag{4.5}$$

To establish the relation in (1.7) it suffices to show that $m_1 = m_2 = 0$.

It is clear that exactly one of the integers m_1, m_2 cannot be zero, since α_1 and α_3 are irrational. Hence $m_1 m_2 \neq 0$. Consider an automorphism σ of the Galois group $\mathrm{Gal}(\mathbb{Q}(\alpha_1, \ldots, \alpha_4)/\mathbb{Q})$ that maps α_1 to α_2. Then, by (4.4), we get

$$\alpha_2 + \sigma(\alpha_2) = -2u.$$

It follows then that $\sigma(\alpha_2) = \alpha_1$, and consequently, $\sigma(\alpha_3) = \alpha_j$, where $j \in \{3, 4\}$. Applying the same automorphism σ to (4.5), we find that

$$m_1\alpha_2 + m_2\alpha_j = m. \tag{4.6}$$

Addition of (4.5) with (4.6) and application of (4.4) gives

$$2m = m_1(\alpha_1 + \alpha_2) + m_2(\alpha_3 + \alpha_j) = -2um_1 + m_2(\alpha_3 + \alpha_j).$$

As $m_2 \neq 0$ and $\alpha_3 \notin \mathbb{Q}$, this is impossible in the case when $j = 3$. Therefore, $j = 4$ and so, by (4.4),

$$2m = -2um_1 - 2um_2,$$

which yields

$$um_1 + um_2 = -m.$$

Now, combining this with (4.5), we find that

$$0 = m_1\alpha_1 + m_2\alpha_3 - m = m_1(\alpha_1 + u) + m_2(\alpha_3 + u) = m_1\beta + m_2\beta'.$$

By Lemma 2.3, this is only possible if $\beta = -\beta'$. However, this implies that h defined in (4.3) has a double root, and so is reducible over $\mathbb{Q}$, a contradiction. This completes the proof of the theorem. □

Thus, we have proved that if (4.1) holds, then

$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = -2u \quad \text{and} \quad k_1 = k_2,\ k_3 = k_4. \tag{4.7}$$

Consequently, we must have

$$k_1(\alpha_1 + \alpha_2) + k_3(\alpha_3 + \alpha_4) = 0. \tag{4.8}$$

Set $r = -2u$ in (4.7). If $r \neq 0$, then (4.8) forces $k_3 = -k_1$. Therefore, in this case, any nontrivial additive relation between four distinct algebraic conjugates of an algebraic number α of degree 4 over $\mathbb{Q}$ (up to re-indexing of the conjugates) has the form

$$\mathbb{Z}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4),$$

as claimed in (1.8). If $r = 0$, then it has the more general form

$$\mathbb{Z}(\alpha_1 + \alpha_2) + \mathbb{Z}(\alpha_3 + \alpha_4),$$

as claimed (1.9).

5. Proof of Theorem 1.3

Let

$$f_i(x_1, \dots, x_6) = \sum_{j=1}^{6} u_{ij} x_j,$$

$i = 1, \dots, m$, be a set of m linear forms such that for each i the coefficients $u_{i1}, \dots, u_{i6} \in \mathbb{Z}$ are not all equal. Set

$$U = \max_{1 \le i \le m,\, 1 \le j \le 6} |u_{ij}|.$$

Next, according to Lemma 2.4, fix positive integers a, b, c such that

$$a, b, c > 2U \quad \text{and} \quad a^2 + b^2 - ab = c^2. \tag{5.1}$$

We first introduce the nontrivial relation between algebraic conjugates of degree 6 that was discovered in [25]. Let L be the splitting field of the irreducible polynomial $x^6 + 3$ over $\mathbb{Q}$. It is easy to verify that $L = \mathbb{Q}(\sqrt[6]{-3}, e^{\frac{2\pi i}{6}})$ is a normal extension of $\mathbb{Q}$, so $[L : \mathbb{Q}] = 6$. The Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$ has transitivity number 6T2 (it is permutation isomorphic to the symmetric group S_3 acting regularly on its six elements). Hence, by the normal basis theorem, there exists $\beta \in L$ whose conjugates can be re-indexed so that G is generated by $(\beta_1, \beta_3, \beta_2)(\beta_4, \beta_5, \beta_6)$ and $(\beta_1, \beta_4)(\beta_2, \beta_5)(\beta_3, \beta_6)$. With this notation, let us consider the algebraic number α of degree 6 with conjugates

$$\begin{aligned}
\alpha_1 &= c\beta_1 + c\beta_2 - 2c\beta_3 - (a+b)\beta_4 + (2b-a)\beta_5 + (2a-b)\beta_6,\\
\alpha_2 &= c\beta_1 - 2c\beta_2 + c\beta_3 + (2a-b)\beta_4 - (a+b)\beta_5 + (2b-a)\beta_6,\\
\alpha_3 &= -(a+b)\beta_1 + (2b-a)\beta_2 + (2a-b)\beta_3 + c\beta_4 + c\beta_5 - 2c\beta_6,\\
\alpha_4 &= -2c\beta_1 + c\beta_2 + c\beta_3 + (2b-a)\beta_4 + (2a-b)\beta_5 - (a+b)\beta_6,\\
\alpha_5 &= (2a-b)\beta_1 - (a+b)\beta_2 + (2b-a)\beta_3 + c\beta_4 - 2c\beta_5 + c\beta_6,\\
\alpha_6 &= (2b-a)\beta_1 + (2a-b)\beta_2 - (a+b)\beta_3 - 2c\beta_4 + c\beta_5 + c\beta_6.
\end{aligned} \tag{5.2}$$

In view of (5.1) and (5.2), the conjugates of α satisfy the relation

$$a\alpha_1 + b\alpha_2 + c\alpha_3 = 0. \tag{5.3}$$

From (5.2) it is also clear that

$$\alpha_1 + \alpha_2 + \alpha_4 = 0 \tag{5.4}$$

and

$$\alpha_3 + \alpha_5 + \alpha_6 = 0. \tag{5.5}$$

Fix $i \in \{1, \dots, m\}$. In what follows, we show that

$$f_i(\alpha_1, \dots, \alpha_6) = 0$$

is true only if

$$f_i(x_1, \dots, x_6) = \mathbb{Z}(x_1 + x_2 + x_4) + \mathbb{Z}(x_3 + x_5 + x_6). \tag{5.6}$$

In other words, (5.4), (5.5) and their $\mathbb{Z}$-linear combinations are the only nontrivial additive linear relations with conjugates of α and coefficients of moduli at most U.

For a contradiction, assume that for some integers $u_1, \dots, u_6$, not all equal, satisfying $|u_i| \le U$ we have

$$u_1\alpha_1 + \dots + u_6\alpha_6 = 0.$$

By (5.4) and (5.5), we obtain

$$(u_1 - u_4)\alpha_1 + (u_2 - u_4)\alpha_2 + (u_3 - u_6)\alpha_3 + (u_5 - u_6)\alpha_5 = 0. \tag{5.7}$$

If

$$u_1 = u_2 = u_4 \quad \text{and} \quad u_3 = u_5 = u_6, \tag{5.8}$$

then (5.6) holds and we are done. In contrast, if one of the equalities in (5.8) is not true, then from (5.7) it follows that for some integers

$$k_1 = u_1 - u_4, \quad k_2 = u_2 - u_4, \quad k_3 = u_3 - u_6, \quad k_5 = u_5 - u_6,$$

not all zeros, satisfying $|k_1|, |k_2|, |k_3|, |k_4| \le 2U$, the additive relation

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_5\alpha_5 = 0$$

holds. Multiplying this equality by $c \neq 0$ and using (5.3) we get

$$(ck_1 - ak_3)\alpha_1 + (ck_2 - bk_3)\alpha_2 + ck_5\alpha_5 = 0. \tag{5.9}$$

We claim that this implies

$$ak_5 - bk_3 + ck_2 = 0. \tag{5.10}$$

Indeed, inserting the expressions for $\alpha_1, \alpha_2, \alpha_5$ from (5.2) into (5.9), and collecting the terms for β_1 and β_2, we find that

$$c(ck_1 - ak_3) + c(ck_2 - bk_3) + c(2a - b)k_5 = 0$$

and

$$c(ck_1 - ak_3) - 2c(ck_2 - bk_3) - c(a + b)k_5 = 0.$$

Subtracting the latter equality from the former, we obtain

$$3c(ck_2 - bk_3) + 3cak_5 = 3c(ck_2 - bk_3 + ak_5) = 0,$$

which implies (5.10).

We will show that (5.10) cannot hold for the triplets (a, b, c) constructed in Lemma 2.4 with $R = 2U$ and an appropriate finite set $\mathcal{S}$. Indeed, if $k_2 = 0$, then (5.10) implies that $a/b = k_3/k_5$. As $|k_3|, |k_5| \leq 2U$, the set of such quotients is finite.

If $k_2 \neq 0$, then $c = (ak_5 - bk_3)/k_2$, and therefore

$$a^2 + b^2 - ab = c^2 = \frac{a^2k_5^2 + b^2k_3^2 - 2abk_5k_3}{k_2^2}.$$

Hence, $q = a/b$ satisfies the quadratic equation

$$q^2\bigl(1 - (k_5/k_2)^2\bigr) + q(2k_5k_3/k_2^2 - 1) + 1 - (k_3/k_2)^2 = 0.$$

This means that q belongs to a finite set depending on U only, unless all three coefficients of the quadratic polynomial

$$1 - (k_5/k_2)^2, \quad 2k_5k_3/k_2^2 - 1, \quad 1 - (k_3/k_2)^2$$

are equal to zero. However, looking at the first and the last coefficients, we see that this is possible only if

$$k_5 = \pm k_2, \quad k_3 = \pm k_2.$$

But then the second coefficient equals

$$2k_5k_3/k_2^2 - 1 = \pm 2 - 1 \neq 0,$$

a contradiction.

Thus, we have shown that if

$$f_i(\alpha_1, \dots, \alpha_6) = 0$$

for some $i \in \{1, \dots, m\}$, then

$$f_i(x_1, \dots, x_6) = \mathbb{Z}(x_1 + x_2 + x_4) + \mathbb{Z}(x_3 + x_5 + x_6).$$

This implies that if some nontrivial relation between the conjugates of α is of the form

$$\sum_{j \in S} \mathbb{Z} f_j(\alpha_1, \dots, \alpha_6),$$

where $S \subseteq \{1, \dots, m\}$ and $f_j(\alpha_1, \dots, \alpha_6) = 0$ for each $j \in S$, then it can be expressed in the simpler form, namely

$$\mathbb{Z}(\alpha_1 + \alpha_2 + \alpha_4) + \mathbb{Z}(\alpha_3 + \alpha_5 + \alpha_6). \tag{5.11}$$

On the other hand, it is clear that the nontrivial relation introduced in (5.3), namely,

$$a\alpha_1 + b\alpha_2 + c\alpha_3 = 0 \tag{5.12}$$

is not of the form described in (5.11). According to Lemma 2.4, there are infinitely many nontrivial relations with conjugates of a sextic algebraic number satisfying (5.12), each of them induced by distinct $(a, b, c) \in \mathbb{N}^3$. This completes the proof because different triplets $(a, b, c) \in \mathbb{N}^3$ define different algebraic numbers α in (5.2). □

Acknowledgement

We thank the referee for careful reading and some useful comments.

References

[1] E. Artin, *Galois theory*, second ed., Dover, Mineola, NY, 1998. MR Zbl

[2] G. Baron, M. Drmota, and M. Skałba, Polynomial relations between polynomial roots, *J. Algebra* **177** no. 3 (1995), 827–846. DOI MR Zbl

[3] J. D. Dixon, Polynomials with nontrivial relations between their roots, *Acta Arith.* **82** no. 3 (1997), 293–302. DOI MR Zbl

[4] M. Drmota and M. Skałba, On multiplicative and linear independence of polynomial roots, in *Contributions to general algebra 7 (Vienna, 1990)*, Hölder-Pichler-Tempsky, Vienna, 1991, pp. 127–135. MR Zbl

[5] M. Drmota and M. Skałba, Relations between polynomial roots, *Acta Arith.* **71** no. 1 (1995), 65–77. DOI MR Zbl

[6] P. Drungilas and A. Dubickas, On degrees of three algebraic numbers with zero sum or unit product, *Colloq. Math.* **143** no. 2 (2016), 159–167. DOI MR Zbl

[7] A. Dubickas, On the degree of a linear form in conjugates of an algebraic number, *Illinois J. Math.* **46** no. 2 (2002), 571–585. MR Zbl Available at http://projecteuclid.org/euclid.ijm/1258136212.

[8] A. Dubickas, Additive relations with conjugate algebraic numbers, *Acta Arith.* **107** no. 1 (2003), 35–43. DOI MR Zbl

[9] A. Dubickas, Multiplicative relations with conjugate algebraic numbers, *Ukraïn. Mat. Zh.* **59** no. 7 (2007), 890–900, and in *Ukrainian Math. J.* **59** no. 7 (2007), 984–995. DOI MR Zbl

[10] A. Dubickas and J. Jankauskas, Simple linear relations between conjugate algebraic numbers of low degree, *J. Ramanujan Math. Soc.* **30** no. 2 (2015), 219–235. MR Zbl

[11] A. Dubickas and C. J. Smyth, Polynomials with three distinct zeros summing to zero (problem 11123), *Amer. Math. Monthly* **113** no. 10 (2006), 941–942, solution by R. Stong. DOI

[12] K. Girstmair, Linear dependence of zeros of polynomials and construction of primitive elements, *Manuscripta Math.* **39** no. 1 (1982), 81–97. DOI MR Zbl

[13] K. Girstmair, Linear relations between roots of polynomials, *Acta Arith.* **89** no. 1 (1999), 53–96. DOI MR Zbl

[14] K. Girstmair, The Galois relation $x_1 = x_2 + x_3$ and Fermat over finite fields, *Acta Arith.* **124** no. 4 (2006), 357–370. DOI MR Zbl

[15] K. Girstmair, The Galois relation $x_1 = x_2 + x_3$ for finite simple groups, *Acta Arith.* **127** no. 3 (2007), 301–303. DOI MR Zbl

[16] K. Girstmair, The Galois relations $x_1 = x_2 + x_3$ and $x_1 = x_2x_3$ for certain solvable groups, *Ann. Sci. Math. Québec* **32** no. 2 (2008), 171–174. MR Zbl

[17] W. Hardt and J. Yin, Linear relations among Galois conjugates over $\mathbb{F}_q(t)$, *Res. Number Theory* **8** no. 2 (2022), Paper No. 34. DOI MR Zbl

[18] Y. Kitaoka, Notes on the distribution of roots modulo a prime of a polynomial, *Unif. Distrib. Theory* **12** no. 2 (2017), 91–117. MR Zbl

[19] V. A. Kurbatov, Galois extensions of prime degree and their primitive elements, *Izv. Vysš. Učebn. Zaved. Matematika* no. 1(176) (1977), 61–66, English transl.: *Soviet Math. (Iz. VUZ)* **21** no. 1 (1977), 49–53. MR Zbl

[20] F. Lalande, Relations linéaires entre les racines d'un polynôme et anneaux de Schur, *Ann. Sci. Math. Québec* **27** no. 2 (2003), 169–175. MR Zbl

[21] F. Lalande, La relation linéaire $a = b+c+\cdots+t$ entre les racines d'un polynôme, *J. Théor. Nombres Bordeaux* **19** no. 2 (2007), 473–484. DOI MR Zbl

[22] F. Lalande, À propos de la relation galoisienne $x_1 = x_2 + x_3$, *J. Théor. Nombres Bordeaux* **22** no. 3 (2010), 661–673. DOI MR Zbl

[23] C. J. Smyth, Conjugate algebraic numbers on conics, *Acta Arith.* **40** no. 4 (1982), 333–346. DOI MR Zbl

[24] C. J. Smyth, Additive and multiplicative relations connecting conjugate algebraic numbers, *J. Number Theory* **23** no. 2 (1986), 243–254. DOI MR Zbl

[25] P. Virbalas, Linear relations between three conjugate algebraic numbers of low degree, *J. Korean Math. Soc.* **62** no. 2 (2025), 253–284. DOI MR Zbl

[26] T. Zheng, Characterizing triviality of the exponent lattice of a polynomial through Galois and Galois-like groups, in *Computer algebra in scientific computing*, Lecture Notes in Comput. Sci. 12291, Springer, Cham, 2020, pp. 621–641. DOI MR Zbl

[27] T. Zheng, A fast algorithm for computing multiplicative relations between the roots of a generic polynomial, *J. Symbolic Comput.* **104** (2021), 381–401. DOI MR Zbl

Artūras Dubickas✉
Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University,
Naugarduko 24, LT-03225 Vilnius, Lithuania
arturas.dubickas@mif.vu.lt

Paulius Virbalas
Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University,
Naugarduko 24, LT-03225 Vilnius, Lithuania
paulius.virbalas@mif.vu.lt

Received: January 3, 2025
Accepted: March 13, 2025
Early view: March 16, 2025