REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 69, No. 1, 2026, Pages 155–160
Published online: December 19, 2025
https://doi.org/10.33044/revuma.4820

A CONNECTION BETWEEN HYPERREALS AND TOPOLOGICAL FILTERS

MOHAMED BENSLIMANE

Abstract. The aim of this paper is to show that the ultrapower ${}^*\mathbb{R}$ of the real line $\mathbb{R}$ with respect to a selective ultrafilter on the natural numbers (Choquet's absolute ultrafilter) can be naturally embedded in the prime spectrum of the usual topology on $\mathbb{R}$, viewed as a distributive lattice. Moreover, the topology induced on ${}^*\mathbb{R} \setminus \mathbb{R}$ through this embedding is separated (Hausdorff).

1. Introduction

Following the research devoted to the study of the hyperreals and of topological filters, it appears that these two notions are independent. The present paper aims to build a bridge between them.

Let τ denote the usual topology on $\mathbb{R}$. Recall that a nonempty collection θ of elements of τ is said to be a *prime filter* of τ if θ satisfies the following four conditions:

(i) $\emptyset \notin \theta$.
(ii) If $A, B \in \theta$, then $A \cap B \in \theta$.
(iii) If $A \in \theta$ and $A \subseteq B \in \tau$, then $B \in \theta$.
(iv) If $A, B \in \tau$ with $A \cup B \in \theta$, then $A \in \theta$ or $B \in \theta$.

We will use the symbol $\widetilde{\mathbb{R}}$ to denote the set of all prime filters in the bounded distributive lattice $(\tau, \cup, \cap, \emptyset, \mathbb{R})$. Note that the space $\widetilde{\mathbb{R}}$ is homeomorphic to the space of prime ideals of τ, usually called the *prime spectrum* of the lattice τ (see [6]). This clearly implies that $\widetilde{\mathbb{R}}$ is endowed with the well-known topology generated by the basis consisting precisely of all subsets $\widetilde{\omega} = \{\theta \in \widetilde{\mathbb{R}} \mid \omega \in \theta\}$ of $\widetilde{\mathbb{R}}$, where ω is an open set of $\mathbb{R}$. Some authors called this topology the *Zariski topology*.

In this paper, we discover an important connection between hyperreals and topological filters (Theorem 2.6). We refer the interested reader to [5] for definitions and basic facts about hyperreals. We conclude the paper by proving that ${}^*\mathbb{R} \setminus \mathbb{R}$ endowed with the induced topology by the space $\widetilde{\mathbb{R}}$ is a separated topological space (Theorem 2.7).

2020 *Mathematics Subject Classification.* Primary 54J05; Secondary 54A20.

Key words and phrases. (absolute) ultrafilter, hyperreal, prime spectrum, topological filter, ultrapower.

Notation. In this paper we will use the following notation. The letters $\mathbb{N}$ and $\mathbb{R}$ are used for the sets of non-negative integers and the field of real numbers, respectively. For given subsets A and B of a set X, $A \setminus B = \{a \in A \mid a \notin B\}$, and we write $A^c = X \setminus A$. We use $|X|$ to denote the cardinality of a set X. Given a nontrivial ultrafilter U on $\mathbb{N}$, we let ${}^*\mathbb{R}$ denote the set of hyperreals modulo U. We refer to [4, 5, 6] for all the undefined terminology in this paper.

2. Hyperreals versus topological filters

Let $a \in \mathbb{R}$. For simplicity of notation, we write

$$\omega(a) = \{W \mid W \text{ is an open subset of } \mathbb{R} \text{ with } a \in W\},$$
$$\omega(\infty) = \{K^c = \mathbb{R} \setminus K \mid K \text{ is a compact subset of } \mathbb{R}\}.$$

Lemma 2.1. *For each $\theta \in \widetilde{\mathbb{R}}$, there exists a unique element $a \in \mathbb{R} \cup \{\infty\}$ such that $\omega(a) \subseteq \theta$.*

Proof. Let $\theta \in \widetilde{\mathbb{R}}$.

Case 1: Suppose that $\bigcap_{A\in\theta} \overline{A} = \emptyset$. Let K be a compact subset of $\mathbb{R}$. Then $\bigcap_{A\in\theta}(\overline{A} \cap K) = \emptyset$. Therefore there exist finitely many elements $A_1, \ldots, A_n$ of θ such that $(\bigcap_{i=1}^{n} A_i) \cap K = \emptyset$. Put $A = \bigcap_{i=1}^{n} A_i$. Then $A \in \theta$ and $A \cap K = \emptyset$. This implies that $A \subseteq K^c$ and hence $K^c \in \theta$. It follows that $\omega(\infty) \subseteq \theta$.

Case 2: Assume that $\bigcap_{A\in\theta} \overline{A} \neq \emptyset$ and take $a \in \bigcap_{A\in\theta} \overline{A}$. Let $V \in \omega(a)$. So there exists $W \in \omega(a)$ such that $W \subseteq \overline{W} \subseteq V$. Since $W \cap (\overline{W})^c = \emptyset$, we deduce that $(\overline{W})^c \notin \theta$. Note that $\mathbb{R} = V \cup (\overline{W})^c$. Then $V \in \theta$. Consequently, $\omega(a) \subseteq \theta$. □

For any $a \in \mathbb{R}$, we consider the following sets:

$$\omega^+(a) = \{W \text{ open subset of } \mathbb{R} \mid]a, a+\varepsilon[\subseteq W \text{ for some } \varepsilon > 0\},$$
$$\omega^-(a) = \{W \text{ open subset of } \mathbb{R} \mid]a-\varepsilon, a[\subseteq W \text{ for some } \varepsilon > 0\},$$
$$\omega^+(\infty) = \{W \text{ open subset of } \mathbb{R} \mid]M, +\infty[\subseteq W \text{ for some } M > 0\},$$
$$\omega^-(\infty) = \{W \text{ open subset of } \mathbb{R} \mid]-\infty, -M[\subseteq W \text{ for some } M > 0\}.$$

Proposition 2.2. *For each $\theta \in \widetilde{\mathbb{R}} \setminus \mathbb{R}$, there exists a unique $a \in \mathbb{R} \cup \{\infty\}$ such that either $\omega^+(a) \subseteq \theta$ or $\omega^-(a) \subseteq \theta$.*

Proof. By Lemma 2.1, there exists a unique $a \in \mathbb{R} \cup \{\infty\}$ such that $\omega(a) \subseteq \theta$. Since θ is not principal, there exists $A \in \theta \setminus \omega(a)$.

Case 1: a is a real number. Since $a \notin A$, we have

$$A \cap]a-1, a+1[= (A \cap]a-1, a[) \cup (A \cap]a, a+1[) \in \theta.$$

But θ is a prime filter, so $A \cap]a-1, a[\in \theta$ or $A \cap]a, a+1[\in \theta$. Without loss of generality we can assume that $A \cap]a-1, a[\in \theta$. This implies that $]a-1, a[\in \theta$. To show that $\omega^-(a) \subseteq \theta$, take $W \in \omega^-(a)$. Then $]a-\varepsilon, a[\subseteq W$ for some $\varepsilon > 0$. If $\varepsilon \geq 1$, then $]a-1, a[\subseteq]a-\varepsilon, a[$ and hence $]a-\varepsilon, a[\in \theta$. If $\varepsilon < 1$, then $]a-\varepsilon, a[=]a-1, a[\cap]a-\varepsilon, a+\varepsilon[\in \theta$. It follows that $W \in \theta$, and so $\omega^-(a) \subseteq \theta$.

Case 2: $a = \infty$. The proof of this case runs as before by using the fact that

$$A \cap [-1, 1]^c = (A \cap]-\infty, -1[) \cup (A \cap]1, +\infty[) \in \theta. \qquad \square$$

Choquet introduced the notion of absolute ultrafilters in 1968 [2]. A nontrivial ultrafilter U on $\mathbb{N}$ is said to be *absolute* if, for every application $f : \mathbb{N} \to \mathbb{N}$ such that $\widetilde{f}(U)$ is not trivial, we have $\widetilde{f}(U) \cong U$. Ultrafilters of this type are usually called *selective* or *Ramsey* ultrafilters (see [1]). Choquet showed that the Continuum Hypothesis guarantees the existence of such an ultrafilter and established the following result (see [2]).

Theorem 2.3. *Let U be a nontrivial ultrafilter. Then the following conditions are equivalent:*

(i) *U is an absolute ultrafilter;*
(ii) *For every partition $(Q_n)_{n\geq 0}$ of $\mathbb{N}$ with $Q_n \notin U$ for all $n \geq 0$, there exists $A \in U$ such that $|Q_n \cap A| \leq 1$ for all $n \geq 0$.*

The next result is taken from [3].

Theorem 2.4. *Let U be an ultrafilter on a set E and let $f : E \to E$ be an application. Then there exists $A \in U$ such that exactly one of the following two conditions is satisfied:*

(i) *$f(x) = x$ for every $x \in A$, or*
(ii) *$A \cap f(A) = \emptyset$.*

As a consequence of the preceding theorem, we obtain the following useful corollary.

Corollary 2.5. *Let U be an ultrafilter on a set E and let $P \in U$. Assume that there exists an application $f : P \to E$ such that*

$$\forall S \in U,\ S \subseteq P \Rightarrow f(S) \in U.$$

Then there exists $A \in U$ such that $A \subseteq P$ and $f(x) = x$ for every $x \in A$.

Proof. Let $g : E \to E$ be the extension of f defined by $g(x) = x$ for every $x \in E \backslash P$. From Theorem 2.4, it follows that there exists $Q \in U$ such that $Q \cap g(Q) = \emptyset$ or $g(x) = x$ for every $x \in Q$. Hence the proof is done by setting $A = Q \cap P$ and using the fact that $A \cap f(A) \neq \emptyset$. $\square$

In the remainder of this section we assume U to be an absolute ultrafilter on $\mathbb{N}$. Given a real sequence $x = (x_n)_{n\in\mathbb{N}}$, we set

$$U(x) = \{W \subseteq \mathbb{R} \mid W \text{ is an open subset of } \mathbb{R} \text{ with } \{n \in \mathbb{N} \mid x_n \in W\} \in U\}.$$

It is easily seen that $U(x) \in \widetilde{\mathbb{R}}$.

Now we are ready to prove the main results of this paper. The first one, which is maybe unexpected, connects hyperreals to topological filters.

Theorem 2.6. *Let U be an absolute ultrafilter on $\mathbb{N}$ and let x and y be two real sequences. Then $U(x) = U(y)$ if and only if $\overline{x} = \overline{y}$.*

Proof. The sufficiency is clear. Conversely, suppose that $U(x) = U(y)$.

Case 1: Assume that $U(x) \in \mathbb{R}$. Then $U(x) = \omega(a)$ for some real number a. We claim that $\overline{x} = \overline{a}$. Suppose the contrary and let $P = \{n \geq 0 \mid x_n \neq a\}$. It is clear that $P \in U$. Let W be an open subset of $\mathbb{R}$ such that $x_n \in W$ for every $n \in P$ but $a \notin W$. Then $W \in U(x) \setminus \omega(a)$, a contradiction. It follows that $\overline{x} = \overline{y} = \overline{a}$.

Case 2: Now assume that $U(x) \notin \mathbb{R}$. By Proposition 2.2, there exists a unique $a \in \mathbb{R} \cup \{\infty\}$ such that either $\omega^+(a) \subseteq U(x)$ or $\omega^-(a) \subseteq U(x)$. For simplicity of notation, if $a \in \mathbb{R}$ set

$$\begin{aligned} I_k &= \left]a + \tfrac{1}{k+1}, a + \tfrac{1}{k}\right], \\ Q_k &= \{n \geq 0 \mid x_n \in I_k\} \text{ for any positive integer } k, \\ Q &= \{n \geq 0 \mid x_n \in \left]a, a+1\right]\}, \\ Q_0 &= Q^c. \end{aligned}$$

If $a = \infty$, set

$$\begin{aligned} I_k &= \left]k, k+1\right] \\ Q_k &= \{n \geq 0 \mid x_n \in I_k\} \text{ for any positive integer } k, \\ Q &= \{n \geq 0 \mid x_n \in \left]1, +\infty\right[\} \\ Q_0 &= Q^c. \end{aligned}$$

Note that $Q \in U$ and $Q_n \notin U$ for every $n \geq 0$. Moreover, $(Q_n)_{n \geq 0}$ is a partition of $\mathbb{N}$ satisfying the conditions of Theorem 2.3. The sequence y can be handled in much the same way, and then there exists $A \in U$ such that the following two conditions hold:

(1) $\forall k \in A, \exists \alpha(k) \geq 1, \exists \beta(k) \geq 1, x_k \in I_{\alpha(k)}$ and $y_k \in I_{\beta(k)}$.
(2) For every $n \geq 1$, I_n contains at most one element of x and at most one element of y.

For ease of notation, set $B(z, r) = \left]z - r, z + r\right[$ for any $z \in \mathbb{R}$ and any positive real r. Let $P = \{i \in A \mid \exists j = \varphi(i) \in A, x_i = y_j\}$, $E = \{x_i \mid i \in A\}$, $F = \{y_j \mid j \in A\}$ and $G = E \cup F$. Taking into account the distribution of the points of G on the intervals I_n $(n \geq 1)$, we see that the mapping $\varphi : P \to \mathbb{N}$ is well defined. Let $R = \varphi(P)$ and consider the following neighbourhoods of the points of G.

If $i \in P$ and $j = \varphi(i) \in R$, $\exists \varepsilon_i = \mu_j > 0$, $B(x_i, \varepsilon_i) \cap G = \{x_i\}$.

If $i \notin P$, $\exists \varepsilon_i > 0$, $B(x_i, \varepsilon_i) \cap G = \{x_i\}$.

Similarly, if $j \notin R$, $\exists \mu_j > 0$, $B(y_j, \mu_j) \cap G = \{y_j\}$.

For $M \subseteq A$, set $V_M = \bigcup_{i \in M} B(x_i, \varepsilon_i)$ and $W_M = \bigcup_{j \in M} B(y_j, \mu_j)$. Note that

$$M \in U \Leftrightarrow V_M \in U(x) \Leftrightarrow W_M \in U(y).$$

Since $U(x) = U(y)$, we must have $P \in U$ and for every $S \subseteq P$, $S \in U$ implies $\varphi(S) \in U$. In fact,

$$P \notin U \Rightarrow T = P^c \cap A \in U \Rightarrow V_T \in U(x) \setminus U(y) \text{ and } W_T \in U(y) \setminus U(x),$$

and

$$S \in U \Rightarrow V_S = W_{\varphi(S)} \in U(x) = U(y) \Rightarrow \varphi(S) \in U.$$

Now, using Corollary 2.5, it follows that there exists $H \in U$ such that $\varphi(i) = i$ for every $i \in H$. This clearly forces $\overline{x} = \overline{y}$. □

Recall that the set of hyperreals (modulo U) is denoted by ${}^*\mathbb{R}$. Then the application $h : {}^*\mathbb{R} \to \widetilde{\mathbb{R}}$ defined by $h(\overline{x}) = U(x)$ is injective. Hence the set ${}^*\mathbb{R}$ can be identified to a subset of $\widetilde{\mathbb{R}}$ and then it can be endowed with the induced topology.

Theorem 2.7. *The set ${}^*\mathbb{R} \setminus \mathbb{R}$ endowed with the induced topology by the space $\widetilde{\mathbb{R}}$ is a separated topological space.*

Proof. Let $\overline{x}, \overline{y} \in {}^*\mathbb{R} \setminus \mathbb{R}$ with $\overline{x} \neq \overline{y}$. Then by Proposition 2.2, there exist $a, b \in \mathbb{R} \cup \{\infty\}$ and $\alpha, \beta \in \{+, -\}$ such that $\omega^\alpha(a) \subseteq U(x)$ and $\omega^\beta(b) \subseteq U(y)$. The proof falls naturally into three parts: (i) $a \neq b$, (ii) $a = b$ and $\alpha \neq \beta$, and (iii) $a = b$ and $\alpha = \beta$.

For (i) and (ii), it is clear that there exist two disjoint open sets V and W such that $V \in U(x)$ and $W \in U(y)$. Now assume that $a = b$ and $\alpha = \beta = +$. We will use the same notation of the proof of Theorem 2.6. Note that U is an absolute ultrafilter. Applying Theorem 2.3, we conclude that there exists $A \in U$ satisfying the following two conditions:

(1) every interval I_n contains at most an element x_i and at most an element y_j with $i, j \in A$, and
(2) for each $k, l \in A$, $\exists m, n \geq 0$, $x_k \in I_n$ and $y_l \in I_m$.

Let $P = \{i \in A, \exists j = \varphi(i) \in A,\ x_i = y_j\}$ and set $R = \varphi(P) = \{j \in A, \exists i \in A,\ x_i = y_j\}$. It is easily seen that $\varphi : P \to R$ is a bijection.

Case 1: Assume that $P, Q \in U$. Since $\overline{x} \neq \overline{y}$, it follows from Corollary 2.5 that there exists $T \in U$ with $T \subseteq P$ and $\varphi(T) \notin U$. Therefore $S = R \cap (\varphi(T))^c \in U$. In this case, V_T and W_S are two disjoint open sets such that $V_T \in U(x)$ and $W_S \in U(y)$.

Case 2: Assume now that $P \notin U$. Then $T = A \cap P^c \in U$. Therefore V_T and W_A are two disjoint open sets such that $V_T \in U(x)$ and $W_A \in U(y)$. This completes the proof. □

Remark 2.8. Note that ${}^*\mathbb{R}$ endowed with the induced topology by the space $\widetilde{\mathbb{R}}$ need not be a separated topological space. To see this, consider the real sequence x defined by $x_n = 1/n$ for every $n \geq 1$. Let U be a free ultrafilter on $\mathbb{N}$. We have $\omega(0) \subseteq U(x)$ and $\omega(0) \neq U(x)$. It is clear that 0 and $\overline{x}$ are not separated.

Acknowledgements

The author wishes to thank the referee for carefully reading the manuscript and for making suggestions that improved the presentation of the paper. He would like to express his gratitude to Professor Labib Haddad of Clermont-Ferrand University for his kind help and many stimulating discussions during the preparation of this work. His feedback led to substantial improvements on the quality of the paper.

He would also like to thank him for providing several interesting papers related to the study. Finally, the author wishes to thank Professor Rachid Tribak for translating an earlier version of this paper from French into English, for preparing the manuscript in LaTeX, and for improving its presentation.

References

[1] D. Booth, Ultrafilters on a countable set, *Ann. Math. Logic* **2** no. 1 (1970), 1–24. DOI MR Zbl

[2] G. Choquet, Deux classes remarquables d'ultrafiltres sur N, *Bull. Sci. Math. (2)* **92** (1968), 143–153. MR Zbl

[3] G. Choquet, Une propriété des germes suivant un ultrafiltre, in *Séminaire Choquet, 10e année (1970/71), Initiation à l'analyse, Fasc. 2*, Secrétariat Math., Paris, 1971, p. Exp. no. C1. MR Zbl Available at https://www.numdam.org/item/SC_1970-1971__10_2_A9_0/.

[4] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, Cambridge Math. Textbooks, Cambridge University Press, Cambridge, 1990. MR Zbl

[5] R. Goldblatt, *Lectures on the hyperreals: An introduction to nonstandard analysis*, Grad. Texts in Math. 188, Springer, New York, 1998. DOI MR Zbl

[6] P. T. Johnstone, *Stone spaces*, Cambridge Stud. Adv. Math. 3, Cambridge University Press, Cambridge, 1982. MR Zbl

Mohamed Benslimane
Department of Mathematics, Faculty of Sciences, Abdelmalek Essaâdi University, B.P. 2121 Tetouan, Morocco
med.bens@gmail.com

Received: May 27, 2024
Accepted: July 7, 2025
Early view: July 9, 2025