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ON THE SUM OF THE EIGENVALUES OF THE DISTANCE LAPLACIAN MATRIX OF GRAPHS WITH DIAMETER THREE AND FOUR

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Abstract. We study an inequality proposed by Zhou et al. (2025) relating the distance Laplacian eigenvalues of a connected graph to its Wiener index. For a connected graph G on n vertices, let $U_r(G)$ denote the sum of the r largest distance Laplacian eigenvalues, and let $W(G)$ be the Wiener index of G. Zhou et al. conjectured that, for all $r = 2, \dots, n$, one has $U_r(G) \leq W(G) + \binom{r+2}{3}$. We prove this inequality for several families of graphs. In particular, for $n \geq 95$ we verify it for all graphs in Γ_n, that is, graphs of order n and diameter 3 that contain a spanning tree of diameter 3; as a consequence, the conjecture holds for all trees of diameter 3. Moreover, we show that if G has maximum degree $n-2$, then the inequality holds for all $1 \leq r \leq n$. Finally, we prove that sun graphs and partial sun-type graphs of diameter 4 also satisfy the inequality for all $1 \leq r \leq n$.

1. Introduction

We consider a simple connected graph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The adjacency matrix $A(G) = (a_{ij})$ of G is a $(0,1)$-matrix of order n, where the (i,j)-entry is equal to 1 if v_i is adjacent to v_j, and 0 otherwise.

Let $\mathcal{D}'(G) = \operatorname{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees, where $d_i = d_G(v_i)$ for $i = 1, 2, \dots, n$. The Laplacian matrix of G is defined as $L(G) = \mathcal{D}'(G) - A(G)$, and the signless Laplacian matrix as $Q(G) = \mathcal{D}'(G) + A(G)$. The spectra of these matrices are known as the Laplacian spectrum and the signless Laplacian spectrum of G, respectively. These matrices are real, symmetric, and positive semi-definite. We write the Laplacian spectrum of G as $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$, and the signless Laplacian spectrum as $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$.

In a graph G, the distance $d_G(u,v)$ (or d_{uv} for short) between the vertices u and v is the length of a shortest path connecting them. The diameter of G is the maximum

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distance between any two vertices in G. The transmission $\mathrm{Tr}_G(v)$ of a vertex v is the sum of the distances from v to all other vertices in G, i.e., $\mathrm{Tr}_G(v) = \sum_{u\in V(G)} d_{uv}$. A graph G is said to be r-transmission regular if $\mathrm{Tr}_G(v) = r$ for each $v \in V(G)$. The Wiener index (or transmission) of a graph G, denoted by $W(G)$, is the sum of the distances between all unordered pairs of vertices in G, which can be expressed as $W(G) = \frac{1}{2}\sum_{v\in V(G)} \mathrm{Tr}_G(v)$. The distance matrix of G, denoted by $D(G)$, is defined as $D(G) = (d_{uv})_{u,v\in V(G)}$. The quantity $\mathrm{Tr}_i = \mathrm{Tr}_G(v_i)$ is referred to as the transmission degree of the vertex v_i, and the sequence $\{\mathrm{Tr}_1, \mathrm{Tr}_2, \dots, \mathrm{Tr}_n\}$ is the transmission degree sequence of the graph G.

Let $\mathrm{Tr}(G) = \mathrm{diag}(\mathrm{Tr}_1, \mathrm{Tr}_2, \dots, \mathrm{Tr}_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [2] defined the distance Laplacian matrix of G as $D^L(G) = \mathrm{Tr}(G) - D(G)$, and the distance signless Laplacian matrix as $D^Q(G) = \mathrm{Tr}(G) + D(G)$. If G is connected, then D(G) is symmetric, non-negative and irreducible. Some recent works include [12, 14, 15]. Let $S_k(G) = \sum_{i=1}^{k} \mu_i$, for $1 \le k \le n$, be the sum of the k-largest Laplacian eigenvalues of G. For a graph with n vertices and degree sequence $\{d_v \mid v \in V(G)\}$, Grone and Merris [11] conjectured that $S_k(G) = \sum_{i=1}^{k} \mu_i(G) \le \sum_{i=1}^{k} |\{v \in V(G) \mid d_v \ge i\}|$ for $k = 1, 2, \dots, n$. This was proved by Bai [4]. As a variation of the Grone–Merris theorem, Brouwer [5] conjectured that for a graph G with n vertices and m edges, $S_k(G) = \sum_{i=1}^{k} \mu_i \le m + \binom{k+1}{2}$ for $k = 1, 2, \dots, n$. For the progress on this conjecture, we refer to [6, 10, 9, 8, 7, 13].

Analogous to Brouwer's conjecture, Alhevaz et al. [1] conjectured the following for the sum of the k-largest distance signless Laplacian eigenvalues $U_k(G)$.

Problem 1. If G is a connected graph of order n that is not a path, then $U_k(G) \le W(G) + \binom{k+2}{3}$ for any $k = 2, \dots, n$.

They proved that this inequality is true for graphs of diameter one and two.

For $1 \le r \le n-1$, let U_r be the sum of the r largest distance Laplacian eigenvalues of a graph G, i.e., $U_r = \sum_{i=1}^{r} \partial_i^L(G)$, where $\partial_i^L(G)$, $i = 1, 2, \dots, n$, are the distance Laplacian eigenvalues of G. Similar to Alhevaz et al. [1], the following problem was proposed by Zhou et al. [17].

Problem 2. Determine which connected graphs of order n, and any $r = 2, \dots, n$, satisfy

$$U_r(G) \le W(G) + \binom{r+2}{3}. \tag{A}$$

Zhou et al. [17] confirmed the validity of inequality (A) for graphs with diameter one and graphs of diameter two with given maximum degree for all r.

Definition 1.1. If $d(v) = 1$, then v is called a *pendant vertex*. A tree containing exactly two non-pendant vertices is called a *double-star*. A double-star with degree sequence $(k_1 + 1, k_2 + 1, 1, 1, \dots, 1)$ is denoted by S_{k_1,k_2}.

The following observations will be used in what follows.

Lemma 1.2 ([3]). *Let G be a graph on n vertices and $N(v)$ represent the set of all vertices adjacent to vertex v. If $S = \{v_1, v_2, \dots, v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$, then $\partial = \mathrm{Tr}(v_i) = \mathrm{Tr}(v_j)$ for all $i, j \in \{1, 2, \dots, p\}$, and $\partial + 2$ is an eigenvalue of $D^L(G)$ with multiplicity at least $p - 1$.*

Lemma 1.3 ([2]). *Let G be a connected graph on n vertices and $m \geq n$ edges. Consider the connected graph G' obtained from G by the deletion of an edge. Let $\partial_1^L(G) \geq \partial_2^L(G) \geq \cdots \geq \partial_n^L(G)$ and $\partial_1^L(G') \geq \partial_2^L(G') \geq \cdots \geq \partial_n^L(G')$ be the distance Laplacian eigenvalues of G and G', respectively. Then $\partial_i^L(G') \geq \partial_i^L(G)$ holds for all $1 \leq i \leq n$.*

The paper is organized as follows. In Section 2, we prove that inequality (A) holds for all graphs in the family Γ_n with $n \geq 95$ and for any graph with a maximum degree $n - 2$ and $n \geq 95$. In Section 3, we establish that inequality (A) is true for all sun-type and partial sun-type graphs.

2. Sum of the distance Laplacian eigenvalues of graphs with diameter three

Graphs with diameter three can be divided into two categories: those that have a spanning tree of diameter at most three, and those whose spanning tree has diameter greater than three. We use the notation Γ_n for the first category. It is important to note that every graph in Γ_n contains the double star S_{k_1,k_2}, where $k_1 + k_2 + 2 = n$, as a spanning subgraph. The graph H given in Figure 1 is the largest possible graph in Γ_n in terms of edge count. The graph H is formed by connecting an isolated vertex u to a complete graph K_{k_1}, and another isolated vertex v to a complete graph K_{k_2}. Finally, the two complete graphs K_{k_1} and K_{k_2} are joined together. This construction yields the largest graph by edge count in the family Γ_n. In this section, we prove that inequality (A) is true for all graphs in Γ_n for which $n \geq 95$. As a consequence of this, we finally show that inequality (A) is true for any graph G with maximum degree $\Delta = n - 2$.

In the following lemma, we first obtain the distance Laplacian spectrum of S_{k_1,k_2}.

Lemma 2.1. *The distance Laplacian spectrum of $S_{k,n-k-2}$ is $\{(3n-k-3)^{k-1}, (2n+k-1)^{n-k-3}\}$ together with the roots of the polynomial*

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

Proof. Suppose that $V_1 = \{u, u_1, u_2, \dots, u_k\}$ and $V_2 = \{v, v_1, v_2, \dots, v_{n-k-2}\}$ are the sets of vertices of $K_{1,k}$ and $K_{1,n-k-2}$, respectively. We assume that the double star graph $S_{k,n-k-2}$ is obtained by joining the central vertices u and v of the star graphs $K_{1,k}$ and $K_{1,n-k-2}$. Then the vertex set of $S_{k,n-k-2}$ is $V(S_{k,n-k-2}) = V_1 \cup V_2$. The sets of vertices $\{u_1, u_2, \dots, u_k\}$ and $\{v_1, v_2, \dots, v_{n-k-2}\}$ are clearly independent, with the transmission values $\mathrm{Tr}(u_i) = 3n - k - 5$ and $\mathrm{Tr}(v_i) = 2n + k - 3$. By Lemma 1.2, the distance Laplacian eigenvalues $3n - k - 3$ and $2n - k - 1$ of $S_{k,n-k-2}$ have multiplicities of at least $k - 1$ and $n - k - 3$, respectively.

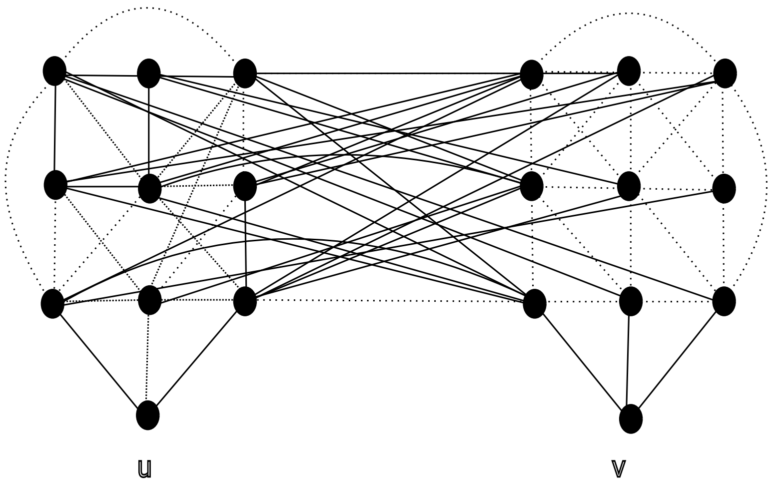

Figure 1. Graph H

The remaining are the eigenvalues of the matrix B, given below:

$$B = \begin{pmatrix} 2n-k-3 & -k & -1 & -2n+2k+4 \\ -1 & 3n-3k-3 & -2 & -3n+3k+6 \\ -1 & -2k & n+k-1 & -n+2+k \\ -2 & -3k & -1 & 3k+3 \end{pmatrix}.$$

The characteristic polynomial of the matrix B is given by

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

Clearly, 0 is one of the zeros of the polynomial $p(\lambda)$ and the sum of the remaining three zeros is $6n-4$. This completes the proof. □

Theorem 2.2. *Let G be a graph in the class Γ_n with $n \geq 95$, containing $S_{k,n-k-2}$ as a spanning subgraph. Then, for all $1 \leq r \leq k-1$ and $k \geq 2$, inequality* (A) *holds for G.*

Proof. The graph H has the smallest Wiener index among all possible connected graphs in Γ_n, and its Wiener index is $\frac{n^2+n}{2}$. Moreover, for any graph G' of diameter 3, $\partial_i^L(S_{k,n-k-2}) \geq \partial_i^L(G')$ for all $i = 1, 2, \ldots, n$. Therefore, in view of Lemma 1.3 and the fact that the Wiener index of any graph in Γ_n is at least $\frac{n^2+n}{2}$, the following inequality will establish the result:

$$U_r(S_{k,n-k-2}) \leq \frac{n^2+n}{2} + \binom{r+2}{3}.$$

From Lemma 2.1, the distance Laplacian spectrum of $S_{k,n-k-2}$ is $\{(3n-k-3)^{k-2}, (2n+k-1)^{n-k-2}\}$ together with the roots of the polynomial

$$p(\lambda) = \lambda(\lambda^3 - (6n-4)\lambda^2 + (11n^2 - 13n - k^2 + kn - 2k + 3)\lambda + (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)).$$

Since $p(3n-k) < 0$ and $p(6n-4) > 0$, it follows that the spectral radius is one of the zeros of the polynomial $p(\lambda)$ and lies in the interval $(3n-k, 6n-4)$. To establish the result, we consider the following two cases based on the multiplicity of the second largest distance Laplacian eigenvalue $\partial_2^L(S_{k,n-k-2})$.

Case (i). $n \geq 2k+2$ and $\partial_2^L = 3n-k-3$ with multiplicity at least $k-1$.

For $n \geq 2k+2$, we have $3n-k-3 \geq 2n+k-1$ and $3n-k-3$ is a decreasing function of k, so we can write $(3n-k-3)(r-1) \leq (3n-4)(r-1)$. As $\partial_1^L(S_{k,n-k-2}) < 6n-4$, we prove that, for any r satisfying $1 \leq r \leq k-1$, the inequality

$$(6n-4) + (3n-4)(r-1) \leq \frac{n^2+n}{2} + \binom{r+2}{3}$$

holds, which can be further simplified to $r^3 + 3r^2 + 26r - 18nr + 3n^2 - 15n \geq 0$. Since $r \geq 1$, in order to prove $r^3 + 3r^2 + 26r - 18nr + 3n^2 - 15n \geq 0$ we need to prove that $r^3 + 3r^2 - 18nr + 3n^2 - 15n \geq 0$. Let $f(r) = r^3 + 3r^2 - 18nr + 3n^2 - 15n$, $r \in [1, k-2]$. The function $f(r)$ is decreasing in $[1, -1+\sqrt{1+6n}]$ and increasing in $[-1+\sqrt{1+6n}, k-2]$. Therefore, to conclude the proof it is necessary to prove that $f(-1+\sqrt{1+6n}) \geq 0$, that is,

$$\begin{aligned} f(-1+\sqrt{1+6n}) &= (-1+\sqrt{1+6n})^3 + 3(-1+\sqrt{1+6n})^2 \\ &\quad - 18n(-1+\sqrt{1+6n}) + 3n^2 - 15n \\ &= (-1 + (1+6n)\sqrt{1+6n} + 3\sqrt{1+6n} - 3(1+6n)) \\ &\quad + 6 + 18n - 6\sqrt{1+6n}) + 18n - 18n(\sqrt{1+6n}) + 3n^2 - 15n \\ &\geq 0, \end{aligned}$$

which is true for $n \geq 95$.

Case (ii). $n < 2k+2$ and $\partial_2^L = 2n+k-1$ with multiplicity at least $n-k-2$. For $n < 2k+1$, we have $3n-k-3 < 2n+k-1$ and $2n+k-1$ is an increasing function of k, so we can write $(2n+k-1)(r-1) \leq (3n-4)(r-1)$. Now, proceeding in the same way as in Case (i), we can establish the result. □

Theorem 2.3. *Inequality* (A) *is true for all connected graphs in* Γ_n *for which* $n \geq 95$.

Proof. Assume that G is any graph in Γ_n and $S_{k,n-k-2}$ is the spanning tree of G. If $r \leq k-2$, then the result is true by Theorem 2.2. We need to prove this for $r > k-2$. The graph H has the smallest Wiener index among all possible connected graphs in Γ_n, and its Wiener index is $\frac{n^2+n}{2}$. Moreover, for any graph G' in Γ_n, $\partial_i^L(S_{k,n-k-2}) \geq \partial_i^L(G')$ for all $i = 1, 2, \ldots, n$. Therefore, in view of Lemma 1.3 and the fact that the Wiener index of any graph in Γ_n at least $\frac{n^2+n}{2}$, the following inequality will establish the result:

$$U_r(S_{k,n-k-2}) \leq \frac{n^2+n}{2} + \binom{r+2}{3}.$$

From Lemma 2.1, the distance Laplacian spectrum of $S_{k,n-k-2}$ is $\{(3n-k-3)^{k-2}, (2n+k-1)^{n-k-2}\}$ together with the roots of the polynomial

$$\begin{aligned} p(\lambda) = \lambda(\lambda^3 &- (6n-4)\lambda^2 + (11n^2-13n-k^2+kn-2k+3)\lambda \\ &+ (3k^2n - 3kn^2 + 6kn - 6n^3 + 9n^2 - 3n)). \end{aligned}$$

We consider the following cases.

Case (i). $n \geq 2k+2$ and assume that the multiplicity of $\partial^L_{n-4} = 2n+k-1$ is s.

Now, for $n \geq 2k+2$, we have $3n-k-3 \geq 2n+k-1$ and $3n-k-3$ is a decreasing function of k and $2n+k-1$ is an increasing function of k, so we can write

$$(3n-k-3)(r-t) + (2n+k-1)t \leq (3n-4)(r-t) + \left(\frac{5n}{2} - 2\right)t,$$

where $1 \leq t \leq s$.

For any r satisfying $1 \leq r \leq n-4$, we prove that the inequality

$$(3n-4)(r-t) + \left(\frac{5n}{2} - 2\right)t \leq \frac{n^2+n}{2} + \binom{r+2}{3}$$

holds. Since $(3n-4)(r-t) + (\frac{5n}{2}-2)t \leq (3n-4)(r)$, we show that $(3n-4)(r) \leq \frac{n^2+n}{2} + \binom{r+2}{3}$, which can be further simplified as

$$r^3 + 3r^2 + 6r - 18nr + 3n^2 + 3n \geq 0.$$

Since $r \geq 1$, in order to prove that $r^3+3r^2+6r-18nr+3n^2+3n \geq 0$ we need to prove that $r^3-18nr+3n^2+3n+9 \geq 0$. Let $f(r) = r^3-18nr+3n^2+3n+9$, $r \in [1, n-4]$. The function $f(r)$ is decreasing in $[1, \sqrt{6n}]$ and increasing in $[\sqrt{6n}, n-4]$. Hence, to conclude the proof, it is necessary to prove that $f(\sqrt{6n}) \geq 0$, that is,

$$\begin{aligned} f(\sqrt{6n}) &= (\sqrt{6n})^3 - 18n(\sqrt{6n}) + 3n^2 + 3n + 9 \\ &= (-12n)\sqrt{6n} + 3n^2 + 3n + 9 \\ &\geq 0, \end{aligned}$$

which is true for $n \geq 95$.

Case (ii). $n < 2k+1$ and $\partial^L_2 = 2n+k-2$ with multiplicity at least $n-k-2$. For $n < 2k+1$, we have $3n-k-3 < 2n+k-2$ and $2n+k-2$ is an increasing function of k, so we can write $(2n+k-2)r \leq (3n-3)r$. Now, proceeding similarly as in Case (i), we can establish the result. □

Zhou et al. [17] proved the following result for graphs with diameter 2 and maximum degree $n-2$.

Theorem 2.4 ([17])**.** *Let G be a connected graph of order n and size m having maximum degree Δ.*

(1) *If $\Delta = n-1$, then Problem 2 is true for all $n \geq 21$.*
(2) *If $\Delta = n-2$, then Problem 2 is true for all $n \geq 26$.*

Now, we generalize this to all graphs with maximum degree $n-2$, without any constraints on the diameter.

Theorem 2.5. *Let G be a connected graph of order n and size m with maximum degree $\Delta = n-2$. Then inequality* (A) *is true for all $n \geq 95$.*

Proof. Any graph G with maximum degree $n-2$ has diameter either 2 or 3. If G has diameter 2, the result holds by Theorem 2.4. Therefore, we only need to prove the case where G has a maximum degree $n-2$ and diameter 3.

Now, if G has diameter 3 and maximum degree $n-2$, then G belongs to the family Γ_n. Consequently, in this case, the result follows from Theorem 2.3. □

3. Sum of distance Laplacian eigenvalues of sun-type graphs

Definition 3.1. The *sun graph* $S_n(a,a)$ is a tree of order $n = 2a+1$ containing pendent vertices, each attached to a vertex of degree 2, and a vertex of degree a. The sun graph of order n can be obtained from the star graph $K_{1,n-1}$ by deleting $\frac{n-1}{2}$ pendent vertices and inserting a new vertex on each of the remaining $\frac{n-1}{2}$ edges of $K_{1,n-1}$. We define a *partial sun graph* $S_n(a,k)$ as a tree of order $n = a+k+1$ containing k pendent vertices ($1 \leq k \leq a-1$), each attached to a vertex of degree 2, and $a-k$ pendent vertices, each attached to a vertex of degree a.

Theorem 3.2 ([16])**.** *For $a \geq 2$, let $S_n(a,a)$ be the sun graph of order $n = 2a+1$. Then the distance Laplacian spectrum of $S_n(a,a)$ is*

$$\Big\{0, \tfrac{1}{2}(12a-1 \pm \sqrt{4a^2+4a+17})^{a-1}, \tfrac{1}{2}(9a-1 \pm \sqrt{9a^2-22a+17})\Big\}.$$

Now, we prove that inequality (A) holds for both the sun graph $S_n(a,a)$ and the partial sun graph $S_n(a,k)$, each of diameter 4.

Theorem 3.3. *If $a \geq 12$, then inequality* (A) *is true for $S_n(a,a)$.*

Proof. We first find the Wiener index of $S_n(a,a)$. Since $S_n(a,a)$ has a vertices of transmission $7a-4$, a vertices of transmission $5a-3$, and one vertex of transmission $3a$, the Wiener index of $S_n(a,a)$ is $\frac{12a^2-4a}{2}$. Thus, we need to prove the inequality

$$U_r(S_n(a,a)) \leq \frac{12a^2-4a}{2} + \binom{r+2}{3}. \tag{3.1}$$

Since $\partial_1^L(S_n(a,a)) = \frac{1}{2}(12a-1+\sqrt{4a^2+4a+17})$ is of multiplicity $a-1$, in order to establish (3.1) we show that the following inequality is true for all r satisfying $1 \leq r \leq n$:

$$\frac{1}{2}(12a-1+\sqrt{4a^2+4a+17})r \leq \frac{12a^2-4a}{2} + \binom{r+2}{3}.$$

As $\sqrt{4a^2+4a+17} \leq 3a$, we show that $\frac{1}{2}(15a)r \leq \frac{12a^2-4a}{2} + \binom{r+2}{3}$, which can be further simplified to

$$r^3 + 3r^2 - 45ar + 2r + 36a^2 - 12a \geq 0.$$

Again, since $r \geq 1$, in order to prove that $r^3 + 3r^2 - 45ar + 2r + 36a^2 - 12a \geq 0$ we need to prove that $r^3 - 45ar + 5 + 36a^2 - 12a \geq 0$.

Let $f(r) = r^3 - 45ar + 5 + 36a^2 - 12a$, $r \in [1, n]$. The function $f(r)$ is decreasing in $[1, \sqrt{15a}]$ and increasing in $[\sqrt{15a}, n]$. Thus, to conclude the proof, it is necessary to prove that $f(\sqrt{15a}) \geq 0$, that is,

$$\begin{aligned} f(\sqrt{15a}) &= (\sqrt{15a})^3 - 45a(\sqrt{15a}) + 5 + 36a^2 - 12a \\ &= -30a\sqrt{15a} + 36a^2 - 12a \\ &\geq 0 \end{aligned}$$

if $a \geq 12$. □

Theorem 3.4 ([16]). *For $a \geq 2$ and $1 \leq k \leq a-1$, let $S_n(a,k)$ be the partial sun graph of order $n = a+k+1$. Then the distance Laplacian spectrum of $S_n(a,k)$ consists of the simple eigenvalue 0; the eigenvalues $\frac{1}{2}(5a+7k-1 \pm \sqrt{(a+k)^2+2(a+k)+17})$, each of multiplicity $k-1$; the eigenvalue $2a+3k+1$, of multiplicity $a-k-1$; and the zeros of the polynomial*

$$\begin{aligned} &x^3 - (6a+8k)x^2 + (-5+a+11a^2+4k+31ak+21k^2)x + 4 \\ &\qquad + 7a - 3a^2 - 6a^3 + 12k - 10ak - 27a^2k - 10k^2 - 39ak^2 - 18k^3. \end{aligned}$$

Theorem 3.5. *For $a \geq 450$, inequality* (A) *is true for the partial sun graph $S_n(a,k)$.*

Proof. The Wiener index of $S_n(a,k)$ is $\frac{6ak+4k^2+2a^2-4k}{2}$. Therefore, we need to prove the inequality

$$U_r(S_n(a,k)) \leq \frac{6ak+4k^2+2a^2-4k}{2} + \binom{r+2}{3}.$$

Since $\frac{1}{2}(5a+7k-1+\sqrt{(a+k)^2+2(a+k)+17})$ is the spectral radius of $S_n(a,k)$ with multiplicity $k-1$, we prove that

$$\left(\frac{1}{2}(5a+7k-1+\sqrt{(a+k)^2+2(a+k)+17})\right)r \leq \frac{6ak+4k^2+2a^2-4k}{2} + \binom{r+2}{3}$$

for any r, $1 \leq r \leq n-1$. Since $k \leq a-1$ and $a+k \geq 4$, we have $\frac{1}{2}(5a+7k-1+\sqrt{(a+k)^2+2(a+k)+17}) \leq 8a$. So to establish the above inequality, we prove that

$$(8a)r \leq \frac{6ak+4k^2+2a^2-4k}{2} + \binom{r+2}{3}.$$

This can be further simplified as

$$r^3 + 3r^2 + 2r - 48ar + 18ak + 12k^2 + 6a^2 - 12k \geq 0.$$

Since $r \geq 1$, to prove that

$$r^3 + 3r^2 + 2r - 48ar + 18ak + 12k^2 + 6a^2 - 12k \geq 0$$

we can instead show that

$$r^3 - 48ar + 18ak + 12k^2 + 6a^2 - 12k \geq 0.$$

Let $f(r) = r^3 - 48ar + 18ak + 12k^2 + 6a^2 - 12k$. The function $f(r)$ is decreasing in $[1, 4\sqrt{a}]$ and increasing in $[4\sqrt{a}, n]$. Thus, to conclude the proof, it is necessary to prove that $f(4\sqrt{a}) \geq 0$, that is,

$$\begin{aligned} f(4\sqrt{a}) &= (4\sqrt{a})^3 - 48a(4\sqrt{a}) + 18ak + 12k^2 + 6a^2 - 12k \\ &= -128a\sqrt{a} + 18ak + 12k^2 + 6a^2 - 12k \\ &\geq 0 \end{aligned}$$

if $a \geq 450$. □

Conclusion. We have extended the study of Zhou et al. [17] by determining the class of connected graphs of diameter 3 and 4 that satisfy the inequality $U_r(G) \leq W(G) + \binom{r+2}{3}$ for any $r = 2, \ldots, n$, where $U_r(G)$ represents the sum of the r largest distance Laplacian eigenvalues of G. Our findings demonstrate that this upper bound holds for certain classes of graphs with diameters beyond 2, expanding the understanding of the spectral properties of graphs with larger diameters. Further research may continue to explore the validity of this bound for all graphs of diameter 3, 4, and beyond, potentially providing deeper insights into the relationships between graph structure and spectral bounds for distance Laplacian eigenvalues.

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