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CORE PARTIAL ORDER FOR FINITE POTENT ENDOMORPHISMS

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Abstract. The aim of this paper is to generalize the core inverse to arbitrary vector spaces using finite potent endomorphisms. As an application, the core partial order is studied in the set of finite potent endomorphisms (of index less than or equal to one), thus generalizing the theory of this order to infinite-dimensional vector spaces. Moreover, a pre-order is presented using the CN decomposition of a finite potent endomorphism. Finally, some questions concerning this pre-order are posed. Throughout the paper, some remarks are made in the framework of arbitrary Hilbert spaces using bounded finite potent endomorphisms.

1. Introduction

In this paper, the set of all $m \times n$ matrices over a field k is represented by $\mathrm{Mat}_{m\times n}(k)$. Given $A \in \mathrm{Mat}_{m\times n}(k)$, the symbols A^{-1}, A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathrm{rk}(A)$ and $P_{\mathcal{R}(A)}$ will stand for the usual inverse (when it exists), the conjugate transpose, the column space, the null space, the rank of the matrix A, and the orthogonal projection onto the column space of A, respectively. Moreover, $\mathrm{Id} \in \mathrm{Mat}_{n\times n}(k)$ will denote the identity matrix.

For an arbitrary $(n \times n)$-matrix A with entries in a field k, the index of A, $\mathrm{i}(A) \geq 0$, is the smallest non-negative integer such that $\mathrm{rk}(A^{\mathrm{i}(A)}) = \mathrm{rk}(A^{\mathrm{i}(A)+1})$. The Drazin inverse of A is the unique solution that satisfies

$$A^{\mathrm{i}(A)+1} \cdot X = A^{\mathrm{i}(A)}, \qquad X \cdot A \cdot X = X, \qquad A \cdot X = X \cdot A,$$

and it is represented by A^D. If $\mathrm{i}(A) \leq 1$, then A^D is called the group inverse of A and is denoted by $A^{\#}$.

For $A \in \mathrm{Mat}_{m\times n}(\mathbb{C})$, if $X \in \mathrm{Mat}_{n\times m}(\mathbb{C})$ satisfies

$$A \cdot X \cdot A = A, \qquad X \cdot A \cdot X = X, \qquad (A \cdot X)^* = X \cdot A, \qquad (X \cdot A)^* = X \cdot A,$$

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then X is called the Moore–Penrose inverse of A. This matrix is unique and it is represented by $A^\dagger$. For details, see [4].

In 2010, Baksalary and Trenkler [3] introduced in the notion of the core inverse of a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ with $\mathrm{i}(A) \leq 1$ as the unique matrix $X \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ satisfying the conditions

$$A \cdot X = P_{\mathcal{R}(A)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A).$$

It is denoted by $A^{\textcircled{\#}}$. Initially, it was presented as an alternative to the group inverse. Firstly, they noted that this generalized inverse could be used to solve certain linear systems and gained interest as it coincides with the Bott–Duffin inverse $P_{\mathcal{R}(A)} \cdot [(A - \mathrm{Id}) \cdot P_{\mathcal{R}(A)} + \mathrm{Id}]^{-1}$. The Bott–Duffin inverse occurs in the solutions of some constrained systems of equations arising in electrical network theory; see [4, Chapter 2, Section 10] and [6]. Ever since its appearance, the core inverse and its applications have attracted a lot of researchers in the area; see, for instance, [16], [15], [8], [9], [11], [12].

John Tate [24] introduced the notion of finite potent endomorphism. Let k be an arbitrary field and let V be an arbitrary k-vector space. Let us now consider an endomorphism φ of V. The ring of endomorphisms over the k-vector space V will be denoted by $\mathrm{End}_k(V)$. We say that φ is finite potent if $\varphi^n(V)$ is a finite-dimensional vector subspace for some n. Tate used this operator in order to give an intrinsic definition of abstract residue.

In [3], the authors studied some properties of the core inverse using the singular value decomposition, and they used this inverse to define a partial order in the class of square complex matrices of index one. In this paper, the core inverse and the core order are extended to arbitrary vector spaces, in general infinite-dimensional ones, endowed with an inner product over a field $k = \mathbb{R}$ or $k = \mathbb{C}$, using finite potent endomorphisms.

The article is organized as follows. Firstly, Section 2 is a gathering of results that will be used later. Section 3 contains some new results related to the Moore–Penrose inverse of a bounded finite potent endomorphism. Section 4 includes the generalization of the theory related to the core inverse to arbitrary vector spaces, following [3] as a guideline. However, the study here presented about the core inverse is done with a different approach to the case of matrices. In particular, we do not start by requiring our finite potent endomorphism to be of index less than or equal to one. We start by deducing that when the core inverse of our finite potent endomorphism exists, the endomorphism shall be of index less than or equal to one (Proposition 4.2). From this previous fact and the definition of the core inverse we deduce that the core inverse is also a unique finite potent endomorphism of index less than or equal to one (Corollary 4.3). Further, the well-known algebraic characterization of the core inverse, involving the group inverse and the Moore–Penrose inverse is presented, as well as a geometric characterization. Using the obtained geometric characterization, we further calculate the Moore–Penrose inverse and the core inverse of the core inverse.

As an application of the previous theory, the core partial order is also generalized to arbitrary vector spaces, namely, infinite-dimensional ones, using finite potent endomorphisms and this is included in Section 5. In the same paper in which the core inverse was introduced [3], the core partial order for complex matrices was extensively investigated. After relating the core order for finite potent endomorphisms with the space pre-order, a characterization is derived in Theorem 5.4. This is the key element to prove that the core order is indeed a partial order on the set of finite potent endomorphisms of index less than or equal to one. Finally, in Section 6 we introduce the so-called "general core order", which is a pre-order on the set of finite potent endomorphisms of arbitrary index. Moreover, some conjectures are posed concerning this pre-order, possibly relating it with the core-EP pre-order and, in general, relating the theory exposed here with the core-EP inverse. It is worth noticing two final points. Every proof and result presented can be specialized to finite square matrices over arbitrary ground fields. Finite potent endomorphisms do not form an ideal of the endomorphisms. Namely, the sum and the composition of two finite potent endomorphisms is not, in general, a finite potent endomorphism. Therefore, the generalization presented here is not merely a generalization from finite-dimensional vector spaces (finite square matrices) to infinite-dimensional vector spaces, but it also carries the additional problems arising from the impossibility of using the usual ring structure of endomorphisms to generalize the theory.

2. Preliminaries

This section is included for the sake of completeness.

2.1. **Finite potent endomorphisms.** Let k be an arbitrary field and let V be a k-vector space. Let us now consider an endomorphism φ of V. We say that φ is *finite potent* if $\varphi^n(V)$ is finite-dimensional for some n.

In 2007, M. Argerami, F. Szechtman and R. Tifenbach [1] showed that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|_{U_\varphi}}$ is nilpotent, W_φ is finite-dimensional and $\varphi_{|_{W_\varphi}} : W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism. Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST decomposition of V.

Moreover, we shall call the nilpotency order of $\varphi_{|_{U_\varphi}}$ the *index* of φ, denoted by $\mathrm{i}(\varphi)$. One has that $\mathrm{i}(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

We shall remark that the sum and the composition of finite potent endomorphisms is not necessarily a finite potent endomorphism.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector space, and finite-rank or nilpotent endomorphisms of infinite-dimensional vector spaces.

For more details on the theory of finite potent endomorphisms, the reader is referred to [18] and [19].

2.1.1. *CN decomposition of a finite potent endomorphism.* Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \operatorname{End}_k(V)$ are finite potent endomorphisms satisfying

- $\mathrm{i}(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

Also, the following hold:

$$\varphi = \varphi_1 \iff U_\varphi = \operatorname{Ker}\varphi \iff W_\varphi = \operatorname{Im}\varphi \iff \mathrm{i}(\varphi) \leq 1. \tag{2.1}$$

Moreover, if $V = W_\varphi \oplus U_\varphi$ is the AST decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi, \\ 0 & \text{if } v \in U_\varphi, \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi, \\ \varphi(v) & \text{if } v \in U_\varphi. \end{cases}$$

2.2. **Bounded finite potent endomorphisms on Hilbert spaces.** In 2021, the author of [21] studied the set of bounded finite potent endomorphisms on arbitrary Hilbert spaces. Henceforth, this set will be denoted by $B_{\mathrm{fp}}(\mathcal{H})$.

Theorem 2.1 (Characterization of bounded finite potent endomorphisms [21, Theorem 3.7])**.** *Given a Hilbert space $\mathcal{H}$ and an endomorphism $\varphi \in \operatorname{End}_{\mathbb{C}}(\mathcal{H})$, the following conditions are equivalent:*

- $\varphi \in B_{\mathrm{fp}}(\mathcal{H})$.
- *$\mathcal{H}$ admits a decomposition $\mathcal{H} = W_\varphi \oplus U_\varphi$, where W_φ and U_φ are closed φ-invariant subspaces of $\mathcal{H}$, W_φ is finite-dimensional, $\varphi_{|W_\varphi}$ is a homeomorphism of W_φ and $\varphi_{|U_\varphi}$ is a bounded nilpotent operator.*
- *φ has a decomposition $\varphi = \psi + \phi$, where ψ is a bounded finite-rank operator, ϕ is a bounded nilpotent operator and $\psi \circ \phi = 0 = \phi \circ \psi$.*

2.2.1. *The adjoint operator of a bounded finite potent endomorphism.* Let us now consider two inner product vector spaces (V, g) and $(H, \bar{g})$. If $\varphi\colon V \to H$ is a linear map, a linear operator $\varphi^*\colon H \to V$ is called the *adjoint* of φ when

$$g(\varphi^*(h), v) = \bar{g}(h, \varphi(v))$$

for all $v \in V$ and $h \in H$. If $\varphi \in \operatorname{End}_k(V)$, we say that φ is *self-adjoint* when $\varphi = \varphi^*$. The existence and uniqueness of the adjoint φ^* of a bounded (or equivalently a continuous) operator on arbitrary Hilbert spaces is immediately deduced form the Riesz Representation Theorem. Moreover, the adjoint of a bounded linear map is also bounded. In [21, Section 4], the author studied the structure of the adjoint of a bounded finite potent endomorphism. Let us recall some of the results presented there. If $\varphi \in B_{\mathrm{fp}}(\mathcal{H})$, with $\mathcal{H} = W_\varphi \oplus U_\varphi$ the AST decomposition induced by φ, and $\varphi = \varphi_1 + \varphi_2$ the CN decomposition, then the adjoint operator φ^* has the following properties:

(I) $\varphi^* \in B_{\mathrm{fp}}(\mathcal{H})$;
(II) $\mathrm{i}(\varphi^*) = \mathrm{i}(\varphi)$;
(III) $\varphi^* = (\varphi_1)^* + (\varphi_2)^*$ is the CN decomposition of φ^*;

(IV) If $\mathcal{H} = W_{\varphi^*} \oplus U_{\varphi^*}$ is the AST decomposition induced by φ^* (notice that this makes sense due to (I)), then one has that $W_{\varphi^*} = [U_\varphi]^\perp$ and $U_{\varphi^*} = [W_\varphi]^\perp$.

2.3. **Generalized inverses.** If $A \in \mathrm{Mat}_{n\times m}(k)$ is a matrix with entries in an arbitrary field k, a matrix $A^- \in \mathrm{Mat}_{m\times n}(k)$ is a $\{1\}$-inverse of A when $AA^-A = A$, and it is a $\{2\}$-inverse of A when $A^-AA^- = A^-$. Moreover, we say that a matrix $A^+ \in \mathrm{Mat}_{m\times n}(k)$ is a reflexive generalized inverse of A when A^+ is a $\{1\}$-inverse of A and A is a $\{1\}$-inverse of A^+, that is, $AA^+A = A$ and $A^+AA^+ = A^+$. Similarly, given two k-vector spaces V and W and a linear map $\varphi\colon V \to W$, we will say thay a morphism $\varphi^-\colon W \to V$ is a $\{1\}$-inverse of φ when $\varphi \circ \varphi^- \circ \varphi = \varphi$, and it is a $\{2\}$-inverse of φ when $\varphi^- \circ \varphi \circ \varphi^- = \varphi^-$. Similarly, considering again a linear map $\varphi : V \to W$, a linear map $\varphi^+\colon W \to V$ is a reflexive generalized inverse of φ when φ^+ is a $\{1\}$-inverse of φ and φ is a $\{1\}$-inverse of φ^+. Given any linear operator φ, we will denote by $X_\varphi(1)$, $X_\varphi(2)$, $X_\varphi(1,2)$ the sets of $\{1\}$-inverses, $\{2\}$-inverses, and reflexive generalized inverses of φ, respectively.

2.4. **Group inverse of finite potent endomorphisms.** Let V be an arbitrary k-vector space and let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism of V. We say that a linear map $\varphi^\# \in \mathrm{End}_k(V)$ is a group inverse of φ when it satisfies the following properties:

- $\varphi \circ \varphi^\# \circ \varphi = \varphi$;
- $\varphi^\# \circ \varphi \circ \varphi^\# = \varphi^\#$;
- $\varphi^\# \circ \varphi = \varphi \circ \varphi^\#$.

According to [20, Lemma 3.4], we know that if there exists a group inverse $\varphi^\# \in \mathrm{End}_k(V)$, then $\mathrm{i}(\varphi) \leq 1$. Moreover, [20, Theorem 3.5] shows that $\varphi^D = \varphi^\#$ is the unique group inverse of φ, where φ^D is its Drazin inverse.

The group inverse $\varphi^\#$ satisfies the following properties:

- $(\varphi^\#)^\# = \varphi$;
- $\varphi = \varphi^\#$ if and only if $(\varphi_{|W_\varphi})^2 = \mathrm{Id}_{|W_\varphi}$;
- if $n \in \mathbb{Z}^+$, then $(\varphi^n)^\# = (\varphi^\#)^n$.

2.5. **Moore–Penrose inverse of a bounded linear map.** Let (V, g) and $(H, \bar{g})$ be inner product spaces over k, with $k = \mathbb{C}$ or $k = \mathbb{R}$.

Definition 2.2. Given a linear map $\varphi\colon V \to H$, we say that φ is *admissible* for the Moore–Penrose inverse when $V = \mathrm{Ker}(\varphi) \oplus [\mathrm{Ker}(\varphi)]^\perp$ and $H = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^\perp$.

According to [5, Theorem 3.12], if (V, g) and $(H, \bar{g})$ are inner product spaces over k, then $\varphi\colon V \to H$ is a linear map admissible for the Moore–Penrose inverse if and only if there exists a unique linear map $\varphi^\dagger\colon H \to V$ such that

(I) $\varphi^\dagger$ is a reflexive generalized inverse of φ;
(II) $\varphi^\dagger \circ \varphi$ and $\varphi \circ \varphi^\dagger$ are self-adjoint, that is,
  (a) $g([\varphi^\dagger \circ \varphi](v), v') = g(v, [\varphi^\dagger \circ \varphi](v'))$,
  (b) $\bar{g}([\varphi \circ \varphi^\dagger](h), h') = \bar{g}(h, [\varphi \circ \varphi^\dagger](h'))$

for all $v, v' \in V$ and $h, h' \in H$. The operator $\varphi^\dagger$ is called the *Moore–Penrose inverse* of φ and it is the unique linear map satisfying

$$\varphi^\dagger(h) = \begin{cases} (\varphi_{|[\mathrm{Ker}(\varphi)]^\perp})^{-1}(h) & \text{if } h \in \mathrm{Im}(\varphi), \\ 0 & \text{if } h \in [\mathrm{Im}(\varphi)]^\perp. \end{cases}$$

The Moore–Penrose inverse $\varphi^\dagger \colon H \to V$ also satisfies the following properties:

- $\varphi^\dagger$ is also admissible for the Moore–Penrose and $(\varphi^\dagger)^\dagger = \varphi$;
- if $\varphi \in \mathrm{End}_k(V)$ and φ is an isomorphism, then $\varphi^\dagger = \varphi^{-1}$;
- $\varphi^\dagger \circ \varphi = P_{[\mathrm{Ker}(\varphi)]^\perp}$;
- $\varphi \circ \varphi^\dagger = P_{\mathrm{Im}(\varphi)}$;

where $P_{[\mathrm{Ker}(\varphi)]^\perp}$ and $P_{\mathrm{Im}(\varphi)}$ are the projections induced by the decompositions $V = \mathrm{Ker}(\varphi) \oplus [\mathrm{Ker}(\varphi)]^\perp$ and $H = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^\perp$, respectively.

2.5.1. *Moore–Penrose inverse of a bounded linear map.* Finally, let us recall some properties of the Moore–Penrose inverse of a bounded linear map between two Hilbert spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert spaces. Given a linear map $\varphi \colon \mathcal{H}_1 \to \mathcal{H}_2$ that is admissible for the Moore–Penrose inverse, that is, $\mathcal{H}_1 = \mathrm{Ker}(\varphi) \oplus [\mathrm{Ker}(\varphi)]^\perp$ and $\mathcal{H}_2 = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^\perp$, the following is well known.

Lemma 2.3. *If $\varphi \in B(\mathcal{H}_1, \mathcal{H}_2)$, then φ is admissible for the Moore–Penrose inverse if and only if* $\mathrm{Im}(\varphi)$ *is a closed subspace of $\mathcal{H}_1$.*

Also, it is well known that

- if $\varphi \in B(\mathcal{H}_1, \mathcal{H}_2)$ is admissible for the Moore–Penrose inverse, then $\varphi^\dagger \in B(\mathcal{H}_2, \mathcal{H}_1)$;
- if $\varphi \in B(\mathcal{H}_1, \mathcal{H}_2)$ is admissible for the Moore–Penrose inverse, then φ^* is also admissible for the Moore–Penrose inverse and $(\varphi^*)^\dagger = (\varphi^\dagger)^*$.

From the properties of the Moore–Penrose inverse of a linear map, if $\varphi \in B(\mathcal{H}_1, \mathcal{H}_2)$ with $\mathrm{Im}(\varphi)$ being a closed subspace of $\mathcal{H}_2$, one has that

- $\varphi^* \circ (\varphi^*)^\dagger = P_{[\mathrm{Ker}(\varphi)]^\perp}$;
- $(\varphi^*)^\dagger \circ \varphi^* = P_{\mathrm{Im}(\varphi)}$;

where $P_{[\mathrm{Ker}(\varphi)]^\perp}$ and $P_{\mathrm{Im}(\varphi)}$ are the projections induced by the decompositions $\mathcal{H}_1 = \mathrm{Ker}(\varphi) \oplus [\mathrm{Ker}(\varphi)]^\perp$ and $\mathcal{H}_2 = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^\perp$, respectively. Moreover, the following usual relations between the adjoint and the Moore–Penrose hold.

Lemma 2.4. *If $\varphi \in B(\mathcal{H}_1, \mathcal{H}_2)$ is such that* $\mathrm{Im}(\varphi)$ *is a closed subspace of $\mathcal{H}_2$, then one has:*

(I) $\varphi^* \circ \varphi \circ \varphi^\dagger = \varphi^*$;
(II) $\varphi^\dagger \circ \varphi \circ \varphi^* = \varphi^*$;
(III) $(\varphi^*)^\dagger \circ \varphi^* \circ \varphi = \varphi$;
(IV) $\varphi \circ \varphi^* \circ (\varphi^*)^\dagger = \varphi$.

2.6. **Space pre-order.** The space pre-order was introduced in [14, Section 3.2] as a tool to study most of the matrix partial orders that include $\{1\}$-inverses on their definitions. In [23], the definition of space pre-order was extended to the class of bounded linear operators on Banach spaces.

Definition 2.5. Let $\varphi, \psi \in \operatorname{End}_k(\mathcal{B})$ be two bounded linear operators over a Banach space $\mathcal{B}$. Then φ is said to be below ψ under the space pre-order if $\operatorname{Im}(\varphi) \subseteq \operatorname{Im}(\psi)$ and $\operatorname{Ker}(\psi) \subseteq \operatorname{Ker}(\varphi)$ (or equivalently $\operatorname{Im}(\varphi^*) \subseteq \operatorname{Im}(\psi^*)$). We will denote the space pre-order by $<^s$ and we will write $\varphi <^s \psi$ whenever φ is below ψ under the space pre-order.

3. On the Moore–Penrose inverse of a bounded finite potent operator

The aim of this section is to include some new results related to the Moore–Penrose inverse of a bounded finite potent endomorphism that are not present in literature and that will be used in the rest of the paper.

3.1. **On the closedness of the image of a bounded finite potent operator.** Given a Hilbert space $\mathcal{H}$, it is well known that the closedness of the image of an operator φ is related to the solvability of the operator equation $\varphi x = y$. Moreover, the closedness of $\operatorname{Im}(\varphi)$ is equivalent to φ being relatively regular, that is, for $\varphi \in B(\mathcal{H})$ admitting a bounded $\{1\}$-inverse $\varphi^- \in B(\mathcal{H})$, i.e., $\varphi \circ \varphi^- \circ \varphi = \varphi$, so that if $\varphi x = y$ can be solved then $x = \varphi^- y$ is a solution. For details, reader is directed to [10] and [17]. There are a lot of important applications of the closedness of the image in perturbation theory and in the context of the spectral study of differential equations; see, for instance, [7]. For an interested reader let us just point out that the study of the closedness of the image is dealt within Banach spaces too, with several applications, for example, in [2]. Now notice that in [5], the authors pose the study of the Moore–Penrose inverse of a bounded finite potent operator $\varphi \in B_{\mathrm{fp}}(\mathcal{H})$ by adding the so-called admissible (for the Moore–Penrose) condition on φ (Definition 2.2), and proving Lemma 2.3 for bounded finite potent operators. However, being a bounded finite potent operator could somehow imply any condition on the closedness of the image. This question has not been studied in the framework of bounded finite potent endomorphisms, and the author considers that it is important to determine whether the admissibility condition is redundant or necessary.

We then devote this short section to solving this question. Precisely, a counterexample is given, showing that bounded finite potent operators do not have closed image in general, that is, they are not relatively regular operators and therefore, if $\varphi \in B_{\mathrm{fp}}(\mathcal{H})$, then φ is not necessarily admissible for a Moore–Penrose inverse.

Counterexample 3.1. If for every $n \in \mathbb{N}$ we denote by e_n the sequence

$$e_n = (0, \dots, \overset{(n)}{1}, 0, \dots),$$

then the family (of sequences) $\{e_n\}_{n\in\mathbb{N}}$ is a (Schauder) basis of ℓ^p, $1 \leq p < \infty$.

Now, let us consider the following operator:

$$\varphi\colon \ell^2 \to \ell^2$$
$$e_n \mapsto \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{e_{n+1}}{n+1} & \text{if } n \text{ is even,} \end{cases}$$

that is, φ is the composition of the operator consisting of multiplying by the sequence $(0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0 \dots)$ and then the operator of right traslation (or unilateral shift operator), which are both continuous. In fact, φ is continuous and compact, and it is clearly a bounded (continuous) finite potent operator (it is nilpotent).

Notice that $\varphi^2 = 0$, so $\operatorname{Im}(\varphi)^2$ is closed. However, $\operatorname{Im}(\varphi)$ is not closed: the vectors with a finite number of non-null components $(0, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \dots, \frac{1}{2n+1}, 0, 0, 0, \dots)$ for $n \geq 1$ are in the image but their limit, in ℓ^2, is not. Indeed, recall that compact operators do not have closed image unless they have finite rank.

Later in this article, we will be interested in finite potent endomorphisms with index less than or equal to one. So let us make some remarks on them.

Despite the previous counterexample, we shall point out that in the case of a finite potent endomorphism of index less than or equal to one over any inner product vector space, we shall not add any hypothesis in order to obtain the admissibility for the Moore–Penrose inverse. Notice that if (V, g) is any inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$, and $\varphi \in \operatorname{End}_k(V)$ with $\mathrm{i}(\varphi) \leq 1$, then the AST decomposition it induces is $V = W_\varphi \oplus U_\varphi = \operatorname{Im}(\varphi) \oplus \operatorname{Ker}(\varphi)$, and by definition $W_\varphi = \operatorname{Im}(\varphi)$ is a finite-dimensional k-vector subspace. Therefore, $\operatorname{Im}(\varphi)$ is a closed subspace and V also admits the decomposition $V = \operatorname{Im}(\varphi) \oplus [\operatorname{Im}(\varphi)]^\perp$. In short:

Lemma 3.2. *Let (V, g) be any inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$, and let $\varphi \in \operatorname{End}_k(V)$ be any finite potent endomorphism with $\mathrm{i}(\varphi) \leq 1$. Then $\operatorname{Im}(\varphi)$ is a closed subspace of V.*

Let $\mathcal{H}$ be a Hilbert space and let $\varphi \in \mathcal{B}(\mathcal{H})$ be a bounded finite potent endomorphism with $\mathrm{i}(\varphi) \leq 1$. It is already known that $\varphi^\dagger$ is bounded (if φ is) and that, in general, it needs not be finite potent. However, it is natural to ask whether this is changed by φ being of index less than or equal to one.

Proposition 3.3. *Given a bounded finite potent endomorphism $\varphi \in \mathcal{B}(\mathcal{H})$ with $\mathrm{i}(\varphi) \leq 1$, it follows that $\varphi^\dagger$ is also a bounded finite potent endomorphism with $\mathrm{i}(\varphi^\dagger) \leq 1$.*

Proof. This result is a consequence of the relationship between the Moore–Penrose inverse and the adjoint operator. Recall that if φ is a bounded finite potent endomorphism, so is φ^* and moreover $\mathrm{i}(\varphi) = \mathrm{i}(\varphi^*)$ (see Section 2.2.1). Therefore $\mathrm{i}(\varphi^*) \leq 1$, so $V = W_{\varphi^*} \oplus U_{\varphi^*} = \operatorname{Im}(\varphi^*) \oplus \operatorname{Ker}(\varphi^*)$. The claim is proved as $\operatorname{Ker}(\varphi^\dagger) = \operatorname{Ker}(\varphi^*)$ and $\operatorname{Im}(\varphi^\dagger) = \operatorname{Im}(\varphi^*)$, both being closed $\varphi^\dagger$-invariant subspaces of $\mathcal{H}$ with $V = \operatorname{Im}(\varphi^\dagger) \oplus \operatorname{Ker}(\varphi^\dagger)$. In fact, $(\varphi^\dagger)_{|_{\operatorname{Im}(\varphi^\dagger)}}$ is a homeomorphism with $\operatorname{Im}(\varphi^\dagger)$ being finite-dimensional and $(\varphi^\dagger)_{|_{\operatorname{Ker}(\varphi^\dagger)}}$ a bounded nilpotent operator. □

4. Core inverse of finite potent endomorphisms

The study of the core inverse can be approached in two different ways, either as a restriction of the left-Drazin–Moore–Penrose inverse ("lDMP") to index 1 endomorphisms, or using Baksalary and Trenkler's original definition (see [3]). Notice that F. Pablos [22] studied the lDMP inverse of finite potent endomorphisms, and therefore we shall use the other approach to provide a new view of the theory with new results. That is, we will start by proving the equivalence of Baksalary and Trenkler's definition with the restriction of lDMP inverses to index 1 endomorphisms, and later we shall study the properties that are not shared between core inverses and lDMP inverses in general, and thus are not already present in [22]. We must emphasize that these new properties are a consequence of the well-known equality $\varphi^D = \varphi^\#$ when $\mathrm{i}(\varphi) = 1$ (recall the obvious fact that $\varphi^D \in X_\varphi(2)$ in general and $\varphi^D = \varphi^\# \in X_\varphi(1,2)$ when $\mathrm{i}(\varphi) = 1$). All this shall be done within the framework of bounded finite potent endomorphisms (over arbitrary inner product spaces over a field) that are admissible for the Moore–Penrose inverse.

Let (V, g) be an inner product vector space over k, with $k = \mathbb{C}$ or $k = \mathbb{R}$. In particular, V can be an infinite-dimensional vector space.

Definition 4.1. Given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ admissible for the Moore–Penrose inverse, we say that a linear map $\varphi^{\text{#}} \in \mathrm{End}_k(V)$ is a *core inverse* of φ when

- $\varphi \circ \varphi^{\text{#}} = P_{\mathrm{Im}(\varphi)}$;
- $\mathrm{Im}\, \varphi^{\text{#}} \subseteq \mathrm{Im}\, \varphi$;

where $P_{\mathrm{Im}(\varphi)}$ is the orthogonal projection induced by the decomposition

$$V = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^\perp.$$

Using the above-mentioned properties of the Moore–Penrose inverse, the two conditions referred to in Definition 4.1 can be replaced by the following:

- $\varphi \circ \varphi^{\text{#}} = \varphi \circ \varphi^\dagger$;
- $P_{\mathrm{Im}\, \varphi} \circ \varphi^{\text{#}} = \varphi^{\text{#}}$, or equivalently $\varphi \circ \varphi^\dagger \circ \varphi^{\text{#}} = \varphi^{\text{#}}$.

Moreover, by substituting the first condition of the definition into the second, one gets

$$(\varphi \circ \varphi^{\text{#}}) \circ \varphi^{\text{#}} = \varphi \circ (\varphi^{\text{#}})^2 = \varphi^{\text{#}}, \tag{4.1}$$

and, by iteration, we obtain

$$\varphi^{\text{#}} = \varphi^{n-1} \circ (\varphi^{\text{#}})^n \quad \text{for every } n > 1. \tag{4.2}$$

Proposition 4.2. *Let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism admissible for the Moore–Penrose inverse. If $\varphi^{\text{#}}$ exists, then $\mathrm{i}(\varphi) \leq 1$.*

Proof. Firstly notice that from $\varphi \circ \varphi^{\text{#}} = \varphi \circ \varphi^\dagger$ and $\varphi \circ \varphi^\dagger \circ \varphi^{\text{#}} = \varphi^{\text{#}}$ one deduces that

$$\varphi \circ \varphi^{\text{#}} \circ \varphi = \varphi,$$

that is, $\varphi^{\text{#}} \in X_\varphi(1)$. Moreover, using (4.2) and substituting it on $\varphi \circ \varphi^{\text{#}} \circ \varphi = \varphi$, one gets

$$\varphi = \varphi^n \circ (\varphi^{\text{#}})^n \circ \varphi \quad \text{for every } n > 1.$$

Therefore, $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\varphi^n)$ for every $n > 1$, and as the other inclusion is true for every linear operator we conclude that

$$\mathrm{Im}(\varphi)^{\mathrm{i}(\varphi)} = W_\varphi = \mathrm{Im}(\varphi),$$

and therefore $\mathrm{i}(\varphi) \leq 1$ (recall (2.1)), as we wanted to prove. □

Corollary 4.3. *Let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism. If $\varphi^{\text{#}}$ exists, then $\varphi^{\text{#}}$ is a finite potent endomorphism with $\mathrm{i}(\varphi^{\text{#}}) \leq 1$. Moreover, $\varphi^{\text{#}}$ is unique.*

Proof. The first statement is a direct consequence of Proposition 4.2 and Definition 4.1, as $W_\varphi = \mathrm{Im}(\varphi)$ is finite-dimensional by definition of the AST decomposition and $\mathrm{Im}(\varphi^{\text{#}}) \subseteq \mathrm{Im}(\varphi)$. Now, the second statement is a consequence of the first statement and the equation shown in (4.2). To wit, as $\varphi^{\text{#}}$ is finite potent, let us consider $u \in U_{\varphi^{\text{#}}}$. Then, if $\mathrm{i}(\varphi^{\text{#}}) = m$, we have

$$\varphi^{\text{#}}(u) = \varphi^{m-1}(\varphi^{\text{#}})^m(u) = 0$$

(recall that, by definition of the AST decomposition, $(\varphi^{\text{#}})^m_{|U_{\varphi^{\text{#}}}} = 0$). Hence,

$$\mathrm{Ker}(\varphi^{\text{#}}) = U_{\varphi^{\text{#}}}$$

(as the other inclusion is always true for any finite potent endomorphism), and by (2.1) we conclude that $\mathrm{i}(\varphi^{\text{#}}) \leq 1$.

In order to see the uniqueness, let us suppose that there exist some $\varphi_1^{\text{#}}, \varphi_2^{\text{#}}$ satisfying Definition 4.1. Then

$$\varphi \circ \varphi_1^{\text{#}} = \varphi \circ \varphi^\dagger = \varphi \circ \varphi_2^{\text{#}},$$

with

$$\mathrm{Im}(\varphi_1^{\text{#}}) \subseteq \mathrm{Im}(\varphi) \quad \text{and} \quad \mathrm{Im}(\varphi_2^{\text{#}}) \subseteq \mathrm{Im}(\varphi).$$

Hence,

$$\mathrm{Im}(\varphi_1^{\text{#}} - \varphi_2^{\text{#}}) \subseteq \mathrm{Ker}(\varphi);$$
$$\mathrm{Im}(\varphi_1^{\text{#}} - \varphi_2^{\text{#}}) \subseteq \mathrm{Im}(\varphi);$$

so we have

$$\mathrm{Im}(\varphi_1^{\text{#}} - \varphi_2^{\text{#}}) \subseteq \mathrm{Ker}(\varphi) \cap \mathrm{Im}(\varphi) = \{0\};$$

as $\mathrm{i}(\varphi) \leq 1$, we conclude that $\varphi_1^{\text{#}} = \varphi_2^{\text{#}}$. □

Let us now check the equivalence between Definition 4.1 and the restriction to index 1 of the definition of the left-Drazin–Moore–Penrose inverse ([22, Theorem 3.2]). In fact, this result is the generalization to arbitrary vector spaces of [3, Theorem 1 (i)]. This will give a purely algebraic definition of the core inverse of a finite potent endomorphism.

Theorem 4.4 (Algebraic characterization of the core inverse)**.** *If $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism with $\mathrm{i}(\varphi) \leq 1$, then*

$$\varphi^{\text{#}} = \varphi^\# \circ \varphi \circ \varphi^\dagger.$$

Proof. Firstly, let us prove that Definition 4.1 implies the above expression of $\varphi^{\circledast}$. Considering the decomposition induced in $V = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^{\perp}$, one has that

$$(\varphi^{\#} \circ \varphi \circ \varphi^{\dagger})_{|_{\mathrm{Im}(\varphi)}} = (\varphi^{\#})_{|_{\mathrm{Im}(\varphi)}} = (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1}.$$

From the equality $\varphi \circ \varphi^{\circledast} = \varphi \circ \varphi^{\dagger}$, one deduces that

$$(\varphi \circ \varphi^{\circledast})_{|_{\mathrm{Im}(\varphi)}} = (\varphi \circ \varphi^{\dagger})_{|_{\mathrm{Im}(\varphi)}} = \mathrm{Id}_{|_{\mathrm{Im}(\varphi)}},$$

and therefore, as $\mathrm{i}(\varphi) \leq 1$,

$$(\varphi^{\circledast})_{|_{\mathrm{Im}(\varphi)}} = (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1}.$$

Hence,

$$(\varphi^{\circledast})_{|_{\mathrm{Im}(\varphi)}} = (\varphi^{\#} \circ \varphi \circ \varphi^{\dagger})_{|_{\mathrm{Im}(\varphi)}}.$$

On the other hand, by the expression of the Moore–Penrose inverse,

$$(\varphi \circ \varphi^{\circledast})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}} = (\varphi \circ \varphi^{\dagger})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}} = 0,$$

so $(\varphi^{\circledast})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}} \in \mathrm{Ker}(\varphi)$. Therefore,

$$(\varphi^{\circledast})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}} = 0$$

because $\mathrm{Im}(\varphi^{\circledast}) \subseteq \mathrm{Im}(\varphi)$ and $\mathrm{Im}(\varphi) \cap \mathrm{Ker}(\varphi) = \{0\}$ as $\mathrm{i}(\varphi) \leq 1$. Directly,

$$(\varphi^{\#} \circ \varphi \circ \varphi^{\dagger})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}} = 0.$$

Conversely, if $\varphi^{\circledast} = \varphi^{\#} \circ \varphi \circ \varphi^{\dagger}$, it is straightforward that

$$\varphi \circ \varphi^{\circledast} = \varphi \circ (\varphi^{\#} \circ \varphi \circ \varphi^{\dagger}) = \varphi \circ \varphi^{\dagger},$$

and it is clear that $\mathrm{Im}(\varphi^{\circledast}) = \mathrm{Im}(\varphi^{\#} \circ \varphi \circ \varphi^{\dagger}) \subseteq \mathrm{Im}(\varphi^{\#}) = W_{\varphi} = \mathrm{Im}(\varphi)$ as $\mathrm{i}(\varphi) \leq 1$, so the statement is proved. □

Theorem 4.5 (Geometric characterization of the core inverse)**.** *Let $\varphi \in \mathrm{End}_k(V)$ be a finite potent endomorphism with $\mathrm{i}(\varphi) \leq 1$. Then $\varphi^{\circledast}$ is characterized by*

$$\varphi^{\circledast}(v) = \begin{cases} (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1}(v) & \text{if } v \in \mathrm{Im}(\varphi), \\ 0 & \text{if } v \in [\mathrm{Im}(\varphi)]^{\perp}. \end{cases}$$

Proof. As in this hypothesis $\mathrm{Im}(\varphi)$ is closed (Lemma 3.2), $V = \mathrm{Im}(\varphi) \oplus [\mathrm{Im}(\varphi)]^{\perp}$. If $v \in \mathrm{Im}(\varphi)$, then

$$\varphi^{\circledast}(v) = \varphi^{\#}\varphi\varphi^{\dagger}(v) = \varphi^{\#}(v) = (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1}(v),$$

which makes sense because $\varphi_{|_{\mathrm{Im}(\varphi)}} = \varphi_{|_{W_{\varphi}}}$ is an automorphism as $\mathrm{i}(\varphi) \leq 1$. On the other hand, if $\bar{v} \in [\mathrm{Im}(\varphi)]^{\perp}$ then

$$\varphi^{\circledast}(\bar{v}) = \varphi^{\#}\varphi\varphi^{\dagger}(\bar{v}) = 0.$$

Now, let us suppose that there exists $\tilde{\varphi} \in \mathrm{End}_k(V)$ such that

$$\tilde{\varphi}(v) = \begin{cases} (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1}(v) & \text{if } v \in \mathrm{Im}(\varphi), \\ 0 & \text{if } v \in [\mathrm{Im}(\varphi)]^{\perp}. \end{cases}$$

Notice that if $v \in \operatorname{Im}(\varphi)$ then

$$\varphi\tilde{\varphi}(v) = v = \varphi\varphi^\dagger(v).$$

If $\bar{v} \in [\operatorname{Im}(\varphi)]^\perp$, we have

$$\varphi\tilde{\varphi}(\bar{v}) = 0 = \varphi\varphi^\dagger(\bar{v}).$$

Therefore,

$$\varphi \circ \tilde{\varphi} = \varphi \circ \varphi^\dagger.$$

Directly from the expression of $\tilde{\varphi}$, one deduces that $\operatorname{Im}(\tilde{\varphi}) = \operatorname{Im}(\varphi)$, and therefore Definition 4.1 is satisfied. □

Theorem 4.6. *Let us consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$. The core inverse of φ exists if and only if* $\mathrm{i}(\varphi) \leq 1$.

Proof. Let us suppose that the core inverse of φ exists. Then it was already proved in Proposition 4.2 that $\mathrm{i}(\varphi) \leq 1$. Conversely, if $\mathrm{i}(\varphi) \leq 1$, then $V = W_\varphi \oplus U_\varphi = \operatorname{Im}(\varphi) \oplus \operatorname{Ker}(\varphi)$ and $\operatorname{Im}(\varphi)$ is a finite-dimensional vector subspace and hence it is a closed subspace of V. In these conditions, $V = \operatorname{Im}(\varphi) \oplus [\operatorname{Im}(\varphi)]^\perp$. Therefore, we can now copy the reasoning presented when proving the converse of Theorem 4.5 to show that $\varphi^{\textcircled{\#}}$ exists. □

Corollary 4.7. *Let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$. *Then $\varphi^{\textcircled{\#}}$ is also admissible for the Moore–Penrose inverse. Moreover,*

$$(\varphi^{\textcircled{\#}})^\dagger(v) = \begin{cases} \varphi(v) & \text{if } v \in \operatorname{Im}(\varphi), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi)]^\perp. \end{cases}$$

Proof. As φ is admissible for the Moore–Penrose inverse, we have $V = \operatorname{Im}(\varphi) \oplus [\operatorname{Im}(\varphi)]^\perp$. Clearly, since $\operatorname{Im}(\varphi^{\textcircled{\#}}) = \operatorname{Im}(\varphi^\# \circ \varphi \circ \varphi^\dagger) = \operatorname{Im}(\varphi)$, one deduces that

$$V = \operatorname{Im}(\varphi^{\textcircled{\#}}) \oplus [\operatorname{Im}(\varphi)^{\textcircled{\#}}]^\perp = \operatorname{Im}(\varphi) \oplus [\operatorname{Im}(\varphi)]^\perp.$$

For the second statement, bearing in mind that $\operatorname{Ker}(\varphi^{\textcircled{\#}}) = \operatorname{Ker}(\varphi^\# \circ \varphi \circ \varphi^\dagger) = [\operatorname{Im}(\varphi)]^\perp$, by the characterization of the Moore–Penrose inverse one has

$$(\varphi^{\textcircled{\#}})^\dagger(v) = \begin{cases} ((\varphi^{\textcircled{\#}})_{|_{[\operatorname{Ker}(\varphi^{\textcircled{\#}})]^\perp}})^{-1}(v) & \text{if } v \in \operatorname{Im}(\varphi^{\textcircled{\#}}), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi^{\textcircled{\#}})]^\perp, \end{cases}$$

which, using the above relations, can be rewritten as

$$(\varphi^{\textcircled{\#}})^\dagger(v) = \begin{cases} ((\varphi^{\textcircled{\#}})_{|_{[\operatorname{Im}(\varphi^{\textcircled{\#}})]}})^{-1}(v) & \text{if } v \in \operatorname{Im}(\varphi), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi)]^\perp. \end{cases}$$

Finally, since $(\varphi^{\textcircled{\#}})_{|\operatorname{Im}(\varphi)} = (\varphi_{|\operatorname{Im}(\varphi)})^{-1}$, as stated in Theorem 4.5, we obtain

$$(\varphi^{\textcircled{\#}})^\dagger(v) = \begin{cases} \varphi(v) & \text{if } v \in \operatorname{Im}(\varphi), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi)]^\perp. \end{cases}$$

□

Corollary 4.8. *If* $\varphi \in \operatorname{End}_k(V)$ *is a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$, *then*

$$(\varphi^{\#})^{\#} = (\varphi^{\#})^{\dagger}.$$

Proof. Firstly we shall point out that this statement makes sense in virtue of Corollary 4.3, that is, $\mathrm{i}(\varphi^{\#}) \leq 1$.

For any $v \in V$, by Theorem 4.5,

$$(\varphi^{\#})^{\#}(v) = \begin{cases} ((\varphi^{\#})_{|_{\operatorname{Im}(\varphi^{\#})}})^{-1}(v) & \text{if } v \in \operatorname{Im}(\varphi^{\#}), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi^{\#})]^{\perp}, \end{cases}$$

as $\operatorname{Im}(\varphi^{\#}) = \operatorname{Im}(\varphi)$, and bearing in mind that

$$(\varphi^{\#})_{|_{\operatorname{Im}(\varphi)}} = (\varphi_{|_{\operatorname{Im}(\varphi)}})^{-1},$$

we can rewrite the above expression as

$$(\varphi^{\#})^{\#}(v) = \begin{cases} ((\varphi_{|_{\operatorname{Im}(\varphi)}})^{-1})^{-1}(v) & \text{if } v \in \operatorname{Im}(\varphi), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi)]^{\perp}. \end{cases}$$

Hence, we conclude by Corollary 4.7. □

Remark 4.9. Notice that the previous propositions can be abbreviated as

$$(\varphi^{\#})^{\#} = (\varphi^{\#})^{\dagger} = \varphi \circ P_{\operatorname{Im}(\varphi)},$$

which is coherent with [3, Theorem 1 (iii)]. Moreover, we highlight that $(\varphi^{\#})^{\dagger}$ is a finite potent endomorphism; in general, the Moore–Penrose inverse of a finite potent endomorphism is not finite potent. This is deduced directly from the fact that

$$((\varphi^{\#})^{\dagger})^n = (\varphi \circ P_{\operatorname{Im}(\varphi)})^n = \varphi^n \circ P_{\operatorname{Im}(\varphi)}.$$

Corollary 4.10. *Let* $\varphi \in \operatorname{End}_k(V)$ *be a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$. *Then* $\varphi^{\#}$ *is EP. Moreover,* $(\varphi^{\#})^{\dagger}$ *is also EP.*

Proof. By the expression obtained in Corollary 4.7, it is straightforward to check that

$$\varphi^{\#} \circ (\varphi^{\#})^{\dagger} = (\varphi^{\#})^{\dagger} \circ \varphi^{\#}.$$

The last statement is deduced from the fact that for any finite potent endomorphism admissible for the Moore–Penrose inverse, $((\varphi)^{\dagger})^{\dagger} = (\varphi)$. □

Corollary 4.11. *Let* $\varphi \in \operatorname{End}_k(V)$ *be a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$. *Then*

$$(\varphi^{\#})^{\dagger} = (\varphi^{\#})^{\#} = (\varphi^{\#})^{D}.$$

Proof. This is a direct consequence of [20, Proposition 3.13], which states that a finite potent endomorphism admissible for the Moore–Penrose inverse is EP if and only if $\varphi^{\#} = \varphi^{\dagger}$. The other equality holds due to the well-known fact that $\varphi^D = \varphi^{\#}$ when $\mathrm{i}(\varphi) \leq 1$. □

Corollary 4.12. *Let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$. *Then*

- $\varphi^{\textcircled{\#}} \in X_\varphi(1,2)$;
- $(\varphi^{\textcircled{\#}})^2 \circ \varphi = \varphi^{\#}$;
- $(\varphi^{\textcircled{\#}})^m = (\varphi^m)^{\textcircled{\#}}$;
- $\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\#} \circ \varphi$.

Proof. The first statement is straightforward from the algebraic expression obtained for the core inverse in Theorem 4.4. The second one can be proved from the definitions, using the commutativity of the group inverse:

$$\begin{aligned}(\varphi^{\textcircled{\#}})^2 \circ \varphi &= \varphi^{\textcircled{\#}} \circ \varphi^{\#} \circ \varphi \circ \varphi^{\dagger} \circ \varphi = \varphi^{\textcircled{\#}} \circ \varphi^{\#} \circ \varphi = \varphi^{\textcircled{\#}} \circ \varphi \circ \varphi^{\#} \\ &= \varphi^{\#} \circ \varphi \circ \varphi^{\dagger} \circ \varphi \circ \varphi^{\#} = \varphi^{\#} \circ \varphi \circ \varphi^{\#} \\ &= \varphi^{\#}.\end{aligned}$$

For the third claim, let $V = W_\varphi \oplus U_\varphi = \operatorname{Im}(\varphi) \oplus \operatorname{Ker}(\varphi)$ be the AST decomposition of V induced by φ in our conditions. For all $m \in \mathbb{Z}^+$, one has that $W_{\varphi^m} = W_\varphi$ and $U_{\varphi^m} = U_\varphi$. Therefore, the claim is deduced from the fact that

$$([\varphi^m]_{|_{\operatorname{Im}(\varphi)}})^{-1} = ([(\varphi_{|_{\operatorname{Im}(\varphi)}})]^{-1})^m.$$

The last equality is straightforward: $\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\#} \circ \varphi \circ \varphi^{\dagger} \circ \varphi = \varphi^{\#} \circ \varphi$. □

Lemma 4.13. *Let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$. *Then*

$$\varphi^{\textcircled{\#}} = \varphi^{\#} \quad \textit{if and only if } \varphi \textit{ is EP.}$$

Proof. If $\varphi^{\textcircled{\#}} = \varphi^{\#}$, then in particular $\operatorname{Ker}(\varphi^{\textcircled{\#}}) = \operatorname{Ker}(\varphi^{\#})$, so $[\operatorname{Im}(\varphi)]^{\perp} = \operatorname{Ker}(\varphi)$. As $\operatorname{Im}(\varphi)$ is closed in our hypothesis, taking orthogonals yields, as desired, $\operatorname{Im}(\varphi) = [\operatorname{Ker}(\varphi)]^{\perp} = \operatorname{Im}(\varphi^*)$.

Conversely, if $\operatorname{Im}(\varphi) = \operatorname{Im}(\varphi^*)$, then $\operatorname{Im}(\varphi) = [\operatorname{Ker}(\varphi)]^{\perp}$ and therefore $[\operatorname{Im}(\varphi)]^{\perp} = \operatorname{Ker}(\varphi)$, so the claim is deduced by the expressions of both the core inverse and the group inverse; recall that

$$\varphi^{\textcircled{\#}}(v) = \begin{cases} (\varphi_{|_{\operatorname{Im}(\varphi)}})^{-1} & \text{if } v \in \operatorname{Im}(\varphi), \\ 0 & \text{if } v \in [\operatorname{Im}(\varphi)]^{\perp}, \end{cases}$$

and

$$\varphi^{\#}(v) = \begin{cases} (\varphi_{|_{\operatorname{Im}(\varphi)}})^{-1} & \text{if } v \in \operatorname{Im}(\varphi), \\ 0 & \text{if } v \in \operatorname{Ker}(\varphi), \end{cases}$$

as $\mathrm{i}(\varphi) \leq 1$. □

In a similar way to [3, Theorem 2], let us highlight some properties of the core inverse.

Proposition 4.14. *Let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism with* $\mathrm{i}(\varphi) \leq 1$. *Then:*

(I) $\varphi^{\textcircled{\#}} = 0$ *if and only if* $\varphi = 0$;
(II) $\varphi^{\textcircled{\#}} = P_{\operatorname{Im}(\varphi)}$ *if and only if* $\varphi^2 = \varphi$;

(III) $\varphi^{\textcircled{\#}} = \varphi^{\dagger}$ *if and only if* φ *is EP;*
(IV) $\varphi^{\textcircled{\#}} = \varphi$ *if and only if* $\varphi^3 = \varphi$ *and* φ *is EP.*

Proof. For (I), notice that $\varphi^{\textcircled{\#}} = 0$ if and only if $\varphi^{\#} \circ P_{\mathrm{Im}(\varphi)} = 0$. Pre-composing and post-composing with φ one deduces that $0 = \varphi \circ \varphi^{\#} \circ \varphi = \varphi$. Conversely, if $\varphi = 0$, then it follows directly from Definition 4.1 that $\varphi^{\textcircled{\#}} = 0$.

Statement (II) follows from the fact that $\varphi^2 = \varphi$ if and only if $\varphi_{|_{\mathrm{Im}(\varphi)}} = \mathrm{Id}_{|_{\mathrm{Im}(\varphi)}}$.

The proof of (III) is analogous to the one shown in Lemma 4.13.

Claim (IV) is the restriction to index 1 of [22, Proposition 3.14]. □

4.1. **Some remarks on the core inverse for bounded finite potent endomorphisms over Hilbert spaces.** Let us now include the theorem corresponding to [26, Theorem 2.1] in the framework of our theory. Namely, we will consider $\mathcal{H}$ as a Hilbert space so that the adjoint operator of a bounded finite potent endomorphism is well defined.

Firstly, let us point out that, from the algebraic characterization of the core inverse deduced in Theorem 4.4, one immediately obtains that $\varphi^{\textcircled{\#}}$ is a continuous (bounded) finite potent operator, as it is the composition of three continuous operators.

Theorem 4.15. *If* $\varphi \in \mathcal{B}_{\mathrm{fp}}(\mathcal{H})$ *with* $\mathrm{i}(\varphi) \leq 1$*, then the core inverse* $\varphi^{\textcircled{\#}}$ *of* φ *is the unique linear operator satisfying*

(I) $\varphi \circ \varphi^{\textcircled{\#}} \circ \varphi = \varphi$;
(II) $\varphi \circ (\varphi^{\textcircled{\#}})^2 = \varphi^{\textcircled{\#}}$;
(III) $(\varphi \circ \varphi^{\textcircled{\#}})^* = \varphi \circ \varphi^{\textcircled{\#}}$.

Proof. Let us check that the operator from Definition 4.1 satisfies these three conditions. Clearly:

$$\varphi \circ \varphi^{\textcircled{\#}} \circ \varphi = \varphi \circ \varphi^{\dagger} \circ \varphi = \varphi.$$

The second condition is proved in the same way that the reasoning above (4.1), substituting into $\varphi^{\textcircled{\#}} = \varphi \circ \varphi^{\dagger} \circ \varphi^{\textcircled{\#}}$ the equality $\varphi \circ \varphi^{\textcircled{\#}} = \varphi \circ \varphi^{\dagger}$, one gets

$$\varphi \circ (\varphi^{\textcircled{\#}})^2 = \varphi^{\textcircled{\#}}.$$

Finally, the third condition is a direct consequence of the definition of the Moore–Penrose inverse, to wit:

$$(\varphi \circ \varphi^{\textcircled{\#}})^* = (\varphi \circ \varphi^{\dagger})^* = \varphi \circ \varphi^{\dagger} = \varphi \circ \varphi^{\textcircled{\#}}.$$

Finally, if there is any endomorphism $\widehat{\varphi}$ satisfying the three conditions in the statement, let us check that Definition 4.1 holds for it. From condition (II) we directly deduce that

$$\mathrm{Im}(\widehat{\varphi}) = \mathrm{Im}(\varphi \circ (\widehat{\varphi})^2) \subseteq \mathrm{Im}(\varphi).$$

From (I) we deduce that

$$(\varphi \circ \widehat{\varphi})_{|_{\mathrm{Im}(\varphi)}} = \mathrm{Id}_{|_{\mathrm{Im}(\varphi)}} = (\varphi \circ \varphi^{\dagger})_{|_{\mathrm{Im}(\varphi)}}.$$

From (III), it is clear that

$$(\varphi \circ \widehat{\varphi})^* = (\widehat{\varphi})^* \circ \varphi^* = \varphi \circ \widehat{\varphi}.$$

Therefore,

$$(\varphi \circ \widehat{\varphi})_{|_{[\mathrm{Im}(\varphi)]^\perp}} = (\varphi \circ \widehat{\varphi})_{|_{\mathrm{Ker}(\varphi^*)}} = ((\widehat{\varphi})^* \circ \varphi^*)_{|_{\mathrm{Ker}(\varphi^*)}} = 0 = (\varphi \circ \varphi^\dagger)_{|_{[\mathrm{Im}(\varphi)]^\perp}}.$$

Adding all up,

$$\varphi \circ \widehat{\varphi} = \varphi \circ \varphi^\dagger,$$

and both conditions of Definition 4.1 hold. Once the equivalence with Definition 4.1 has been proved, uniqueness is proved in the same way as in Corollary 4.3. □

Let us calculate the core inverse of the Moore–Penrose inverse of a bounded finite potent endomorphism of index less than or equal to one.

Proposition 4.16. *If* $\varphi \in \mathcal{B}_{\mathrm{fp}}(\mathcal{H})$ *with* $\mathrm{i}(\varphi) \leq 1$*, then*

$$(\varphi^\dagger)^{\text{\textcircled{\#}}} = (\varphi^\dagger)^\# \circ P_{\mathrm{Im}(\varphi^\dagger)}.$$

Proof. Firstly notice that this statement makes sense due to Proposition 3.3. By direct computation,

$$(\varphi^\dagger)^{\text{\textcircled{\#}}}(v) = \begin{cases} ((\varphi^\dagger)_{|_{\mathrm{Im}(\varphi^\dagger)}})^{-1}(v) & \text{if } v \in \mathrm{Im}(\varphi^\dagger), \\ 0 & \text{if } v \in [\mathrm{Im}(\varphi^\dagger)]^\perp, \end{cases}$$

and bearing in mind the characterization of the Moore–Penrose inverse and that $[\mathrm{Im}(\varphi^\dagger)]^\perp = [\mathrm{Im}(\varphi^*)]^\perp = \mathrm{Ker}(\varphi)$, this can be expressed as

$$(\varphi^\dagger)^{\text{\textcircled{\#}}}(v) = \begin{cases} ((\varphi)_{|_{[\mathrm{Ker}(\varphi)]^\perp}})(v) & \text{if } v \in \mathrm{Im}(\varphi^\dagger), \\ 0 & \text{if } v \in \mathrm{Ker}(\varphi), \end{cases}$$

from where the claim is deduced. □

Proposition 4.17. *Given* $\varphi \in \mathcal{B}_{\mathrm{fp}}(\mathcal{H})$ *with* $\mathrm{i}(\varphi) \leq 1$*, we have that*

$$\varphi^{\text{\textcircled{\#}}} = \varphi^* \quad \textit{if and only if } \varphi \circ \varphi^* \circ \varphi = \varphi \textit{ and } \varphi \textit{ is EP.}$$

Proof. If $\varphi^{\text{\textcircled{\#}}} = \varphi^*$, then since $\varphi \circ \varphi^{\text{\textcircled{\#}}} \circ \varphi = \varphi$ the first claim is clear. Moreover, $\mathrm{Im}(\varphi^{\text{\textcircled{\#}}}) = \mathrm{Im}(\varphi) = \mathrm{Im}(\varphi^*)$. Conversely, as $\varphi \circ \varphi^* \circ \varphi = \varphi$ and $\mathrm{i}(\varphi) \leq 1$, one has that

$$(\varphi^*)_{|_{\mathrm{Im}(\varphi)}} = (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1} = (\varphi^{\text{\textcircled{\#}}})_{|_{\mathrm{Im}(\varphi)}},$$

by Theorem 4.5. As $\mathrm{Ker}(\varphi^*) = [\mathrm{Im}(\varphi)]^\perp$, we conclude that

$$(\varphi^*)_{|_{[\mathrm{Im}(\varphi)]^\perp}} = 0 = (\varphi^{\text{\textcircled{\#}}})_{|_{[\mathrm{Im}(\varphi)]^\perp}}.$$

Therefore, we obtain $\varphi^* = \varphi^{\text{\textcircled{\#}}}$, as desired. □

To conclude this section, let us generalize [3, Theorem 3] to our case.

Theorem 4.18. *If* $\varphi \in \mathcal{B}_p(\mathcal{H})$ *with* $\mathrm{i}(\varphi) \leq 1$*, then the following are equivalent:*

(I) φ *is EP;*
(II) $(\varphi^{\text{\textcircled{\#}}})^{\text{\textcircled{\#}}} = \varphi$*;*
(III) $\varphi^{\text{\textcircled{\#}}} \circ \varphi = \varphi \circ \varphi^{\text{\textcircled{\#}}}$*;*
(IV) $(\varphi^\dagger)^{\text{\textcircled{\#}}} = \varphi$*;*
(V) $(\varphi^{\text{\textcircled{\#}}})^\dagger = (\varphi^\dagger)^\dagger$.

Proof. Firstly, let us check that (I) implies (II). Since φ is EP, we have $\varphi \circ \varphi^\dagger = \varphi^\dagger \circ \varphi$. Therefore,

$$\varphi \circ P_{\operatorname{Im}(\varphi)} = \varphi \circ (\varphi \circ \varphi^\dagger) = \varphi \circ (\varphi^\dagger \circ \varphi) = \varphi,$$

and we conclude by Corollary 4.8.

Conversely, let us see that (II) implies (I). By the expression obtained in Corollary 4.8, it is clear that $(\varphi^{\textcircled{\#}})^{\textcircled{\#}} = \varphi$ if and only if $[\operatorname{Im}(\varphi)]^\perp = \operatorname{Ker}(\varphi)$, that is, if $\operatorname{Ker}(\varphi^*) = \operatorname{Ker}(\varphi)$, which is equivalent to φ being EP.

That (I) occurs if and only if (III) occurs was proved in Lemma 4.13.

To see that (I) happens if and only if (IV), recall that $\operatorname{Im}(\varphi^\dagger) = \operatorname{Im}(\varphi^*)$. By the expression obtained in Proposition 4.16, one deduces that the equivalence occurs if and only if $\operatorname{Im}(\varphi^*) = \operatorname{Im}(\varphi)$.

Finally, in a manner similar to the previous equivalences, the equivalence between (IV) and (V) is deduced by bearing in mind the expressions obtained both in Corollary 4.7 and Proposition 4.16. The equivalence holds if and only if $[\operatorname{Im}(\varphi)]^\perp = \operatorname{Ker}(\varphi^*)$, that is, $\operatorname{Ker}(\varphi) = \operatorname{Ker}(\varphi^*)$. □

5. Core-partial order for finite potent endomorphisms

Let (V, g) be an arbitrary inner product vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$. Henceforth, we will denote by $\operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}$ the set of finite potent endomorphisms on V of index less than or equal to 1.

Definition 5.1. Let $\varphi, \psi \in \operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}$. We will say that φ *is under* ψ *for the core order*, denoted by $\varphi \leq^{\textcircled{\#}} \psi$, when

$$\varphi \circ \varphi^{\textcircled{\#}} = \psi \circ \varphi^{\textcircled{\#}}$$
$$\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\textcircled{\#}} \circ \psi.$$

Lemma 5.2. *Let* $\varphi, \psi \in \operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}$*. If* $\varphi \leq^{\textcircled{\#}} \psi$*, then*

- $\operatorname{Im}(\varphi) \subseteq \operatorname{Im}(\psi)$*;*
- $\operatorname{Ker}(\psi) \subseteq \operatorname{Ker}(\varphi)$*.*

Proof. Recall that for any $\varphi^- \in X_\varphi(1)$, one has that $\operatorname{Im}(\varphi) = \operatorname{Im}(\varphi \circ \varphi^-)$ and $\operatorname{Ker}(\varphi) = \operatorname{Ker}(\varphi^- \circ \varphi)$. Then, the claims are deduced from Corollary 4.12 and Definition 5.1:

$$\operatorname{Im}(\varphi) = \operatorname{Im}(\varphi \circ \varphi^{\textcircled{\#}}) = \operatorname{Im}(\psi \circ \varphi^{\textcircled{\#}}) \subseteq \operatorname{Im}(\psi);$$
$$\operatorname{Ker}(\psi) \subseteq \operatorname{Ker}(\varphi^{\textcircled{\#}} \circ \psi) = \operatorname{Ker}(\varphi^{\textcircled{\#}} \circ \varphi) = \operatorname{Ker}(\varphi).$$ □

Corollary 5.3. *Let* $\varphi, \psi \in \operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}$*. If* $\varphi \leq^{\textcircled{\#}} \psi$*, then* $\varphi <^s \psi$*, where* $<^s$ *denotes the space pre-order (Section* 2.6*).*

Theorem 5.4 (Characterization of the core order)**.** *Let* $\varphi, \psi \in \operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}$*. If* $V = W_\varphi \oplus U_\varphi = \operatorname{Im}(\varphi) \oplus \operatorname{Ker}(\varphi)$ *is the AST decomposition of* V*, then*

$$\varphi \leq^{\textcircled{\#}} \psi \quad \textit{if and only if } \varphi_{|_{\operatorname{Im}(\varphi)}} = \psi_{|_{\operatorname{Im}(\varphi)}} \textit{ and } \psi_{|_{\operatorname{Ker}(\varphi)}} \subseteq [\operatorname{Im}(\varphi)]^\perp.$$

Proof. Notice that if $\varphi \circ \varphi^{\#} = \psi \circ \varphi^{\#}$, then

$$\varphi_{|_{\mathrm{Im}(\varphi)}} = \psi_{|_{\mathrm{Im}(\varphi)}}.$$

On the other hand, as $(\varphi^{\#} \circ \varphi)_{|_{\mathrm{Ker}(\varphi)}} = 0 = (\varphi^{\#} \circ \psi)_{|_{\mathrm{Ker}(\varphi)}}$, we have

$$\psi_{|_{\mathrm{Ker}(\varphi)}} \subseteq \mathrm{Ker}(\varphi^{\#}) = [\mathrm{Im}(\varphi)]^{\perp}.$$

Conversely, let us suppose that both conditions in the statement hold and let us prove that the definition of the core order is satisfied. On one hand,

$$(\varphi \circ \varphi^{\#})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}} = 0 = (\psi \circ \varphi^{\#})_{|_{[\mathrm{Im}(\varphi)]^{\perp}}}$$

since $\mathrm{Ker}(\varphi^{\#}) = [\mathrm{Im}(\varphi)]^{\perp}$. Now, as $\varphi_{|_{\mathrm{Im}(\varphi)}} \in \mathrm{Aut}_k(\mathrm{Im}(\varphi))$, we have

$$\psi \circ (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1} = \varphi \circ (\varphi_{|_{\mathrm{Im}(\varphi)}})^{-1}.$$

Therefore,

$$(\varphi \circ \varphi^{\#})_{|_{\mathrm{Im}(\varphi)}} = 0 = (\psi \circ \varphi^{\#})_{|_{\mathrm{Im}(\varphi)}}$$

and we obtain

$$\varphi \circ \varphi^{\#} = \psi \circ \varphi^{\#}.$$

Moreover, $(\varphi^{\#} \circ \varphi)_{|_{\mathrm{Im}(\varphi)}} = (\varphi^{\#} \circ \psi)_{|_{\mathrm{Im}(\varphi)}}$. Since $\psi_{|_{\mathrm{Ker}(\varphi)}} \subseteq [\mathrm{Im}(\varphi)]^{\perp} = \mathrm{Ker}(\varphi^{\#})$, we have

$$(\varphi^{\#} \circ \psi)_{|_{\mathrm{Ker}(\varphi)}} = 0 = (\varphi^{\#} \circ \varphi)_{|_{\mathrm{Ker}(\varphi)}}$$

and therefore $\varphi^{\#} \circ \varphi = \varphi^{\#} \circ \psi$, as we wanted to prove. □

Corollary 5.5. *Let $\varphi, \psi \in \mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$ such that $\varphi \leq^{\#} \psi$. If φ is EP, then ψ leaves the AST decomposition of φ invariant.*

Proof. It follows from Theorem 5.4 and the fact that $[\mathrm{Im}(\varphi)]^{\perp} = \mathrm{Ker}(\varphi)$ if φ is EP. □

Lemma 5.6. *Let $\varphi, \psi \in \mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$. If $\varphi \leq^{\#} \psi$, then*

$$\mathrm{Ker}(\varphi) = \mathrm{Ker}(\psi) \oplus (\mathrm{Im}(\psi) \cap \mathrm{Ker}(\varphi)).$$

Proof. This follows immediately from φ and ψ being of index less than or equal to 1, together with the inclusion $\mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\varphi)$ (recall Lemma 5.2). □

Lemma 5.7. *Let $\varphi, \psi, \phi \in \mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$. If $\varphi \leq^{\#} \psi$ and $\psi \leq^{\#} \phi$, then*

$$\varphi_{|_{\mathrm{Im}(\varphi)}} = \phi_{|_{\mathrm{Im}(\varphi)}}.$$

Proof. By Lemma 5.2, we know that $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\psi)$. By Theorem 5.4, since $\psi \leq^{\#} \psi$, we have $\phi_{|_{\mathrm{Im}(\psi)}} = \psi_{|_{\mathrm{Im}(\psi)}}$. Therefore, since $\varphi \leq^{\#} \psi$,

$$\phi_{|_{\mathrm{Im}(\varphi)}} = \psi_{|_{\mathrm{Im}(\varphi)}} = \varphi_{|_{\mathrm{Im}(\varphi)}}$$

which concludes the proof. □

Theorem 5.8. *The core order is a partial order on the set* $\mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$.

Proof. Reflexivity holds directly. In order to prove anti-symmetry, let us consider $\varphi, \psi \in \mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$ such that $\varphi \leq^{\textcircled{\#}} \psi$ and $\psi \leq^{\textcircled{\#}} \varphi$. By Lemma 5.2, we know that $\mathrm{Im}(\varphi) = \mathrm{Im}(\psi)$ and $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\psi)$. Since both are finite potent endomorphisms of index less than or equal to 1, we conclude that $\varphi = \psi$.

For transitivity, let us consider $\varphi, \psi, \phi \in \mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$ such that $\varphi \leq^{\textcircled{\#}} \psi$ and $\psi \leq^{\textcircled{\#}} \phi$. Then, the same reasoning used in the proof of the converse of Theorem 5.4 enables us to extend the equality of Lemma 5.7 to the equality

$$\varphi \circ \varphi^{\textcircled{\#}} = \phi \circ \varphi^{\textcircled{\#}}.$$

Moreover, again by Lemma 5.7,

$$(\varphi^{\textcircled{\#}} \circ \varphi)_{|_{\mathrm{Im}(\varphi)}} = (\varphi^{\textcircled{\#}} \circ \phi)_{|_{\mathrm{Im}(\varphi)}}.$$

Further, by Lemma 5.6,

$$(\varphi^{\textcircled{\#}} \circ \phi)_{|_{\mathrm{Ker}(\varphi)}} = (\varphi^{\textcircled{\#}} \circ \phi)_{|_{\mathrm{Ker}(\psi) \oplus (\mathrm{Im}(\psi) \cap \mathrm{Ker}(\varphi))}}.$$

If $v \in (\mathrm{Im}(\psi) \cap \mathrm{Ker}(\varphi))$ with $v = \psi(v')$, then

$$\phi(v) = \phi(\psi(v')) = \psi(\psi(v')) = \psi(v),$$

and hence $\varphi^{\textcircled{\#}}\phi(v) = \varphi^{\textcircled{\#}}\psi(v)$ with $v \in \mathrm{Ker}(\varphi)$. Therefore, by Theorem 5.4,

$$\varphi^{\textcircled{\#}}\psi(v) = 0 = \varphi^{\textcircled{\#}}\phi(v),$$

and since $(\varphi^{\textcircled{\#}}\phi)_{|_{\mathrm{Ker}(\psi)}} = 0$ we deduce that $(\varphi^{\textcircled{\#}} \circ \phi)_{|_{\mathrm{Ker}(\varphi)}} = 0 = (\varphi^{\textcircled{\#}} \circ \varphi)_{|_{\mathrm{Ker}(\varphi)}}$. Finally, we have

$$\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\textcircled{\#}} \circ \phi,$$

and we conclude the proof. □

Following the discussion presented in [3, Section 3], we can prove another characterization of the core partial order.

Theorem 5.9. *Let* $\varphi, \psi \in \mathrm{End}_k^{\mathrm{fp}}(V)^{\leq 1}$. *Then:*

- $\varphi \circ \varphi^{\textcircled{\#}} = \psi \circ \varphi^{\textcircled{\#}}$ *if and only if* $\varphi^2 = \psi \circ \varphi$;
- $\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\textcircled{\#}} \circ \psi$ *if and only if* $\varphi^\dagger \circ \varphi = \varphi^\dagger \circ \psi$.

Proof. Let us begin with the first equivalence. Let us suppose that $\varphi \circ \varphi^{\textcircled{\#}} = \psi \circ \varphi^{\textcircled{\#}}$. Then, $\varphi \circ \varphi^\dagger = \psi \circ \varphi^\# \circ \varphi \circ \varphi^\dagger$ by Theorem 4.4. Post-composing with φ^2 yields $\varphi^2 = \psi \circ \varphi^\# \circ \varphi^2$, and by the commuting property of the group inverse we have

$$\varphi^2 = \psi \circ \varphi \circ \varphi^\# \circ \varphi = \psi \circ \varphi.$$

Conversely, let us suppose that $\varphi^2 = \psi \circ \varphi$. Post-composing with $\varphi^\# \circ \varphi^\dagger$ we get $\varphi^2 \circ \varphi^\# \circ \varphi^\dagger = \psi \circ \varphi \circ \varphi^\# \circ \varphi^\dagger$. Again, by the commuting property of the group inverse,

$$\varphi \circ \varphi^\# \circ \varphi \circ \varphi^\dagger = \psi \circ \varphi^\# \circ \varphi \circ \varphi^\dagger,$$

and we conclude by Theorem 4.4.

Let us prove the second statement. Let us suppose that $\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\textcircled{\#}} \circ \psi$, which can be written as $\varphi^\# \circ \varphi = \varphi^\# \circ \varphi \circ \varphi^\dagger \circ \psi$ in virtue of Theorem 4.4. Pre-composing with $\varphi^\dagger \circ \varphi$ yields $\varphi^\dagger \circ \varphi = \varphi^\dagger \circ \varphi \circ \varphi^{\#} \circ \varphi \circ \varphi^\dagger \circ \psi$. Using the definition of the group inverse, one gets

$$\varphi^\dagger \circ \varphi = \varphi^\dagger \circ \varphi \circ \varphi^\dagger \circ \psi = \varphi^\dagger \circ \psi.$$

Conversely, if $\varphi^\dagger \circ \varphi = \varphi^\dagger \circ \psi$, pre-composing with $\varphi^{\#} \circ \varphi$ we arrive at $\varphi^{\textcircled{\#}} \circ \varphi = \varphi^{\textcircled{\#}} \circ \psi$ by Theorem 4.4. □

6. A pre-order induced by the core inverse for finite potent endomorphisms

Finally, in this short section, let us include an order that makes sense in the case of finite potent endomorphisms, as it relies heavily on the notion of index and on the CN decomposition of a finite potent endomorphism (Section 2.1.1). Moreover, we state some problems for future research.

Definition 6.1. Let $\varphi, \psi \in \operatorname{End}_k^{\mathrm{fp}}(V)$, with $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$ their respective CN decompositions. We will say that φ *is below* ψ *for the general core order*, and we denote it by $\varphi <^{\textcircled{\#}} \psi$, when

$$\varphi_1 \leq^{\textcircled{\#}} \psi_1$$

for the core partial order (Definition 5.1).

Let us point out that this definition makes sense as by definition of the CN decomposition the core part of any endomorphism is of index less than or equal to one.

Remark 6.2. Let us consider any finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, with $\varphi = \varphi_1 + \varphi_2$ being its CN decomposition. Let us denote, again, by $\operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}$ the set of finite potent endomorphisms of index less than or equal to one. There always exists a surjective morphism

$$\begin{aligned} \Gamma\colon \operatorname{End}_k^{\mathrm{fp}}(V) &\to \operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1} \\ \varphi &\mapsto \varphi_1. \end{aligned}$$

It is clear that this morphism is an isomorphism when restricting to the set of finite potent endomorphisms of index less than or equal to one, that is, $\Gamma_{|\operatorname{End}_k^{\mathrm{fp}}(V)^{\leq 1}}$.

Notice that we can formulate and answer questions in the theory of matrix partial orders using this morphism. This morphism being an isomorphism when $\mathrm{i}(\varphi) \leq 1$ means that the general core order and the core partial order coincide on the set of finite potent endomorphisms of index less than or equal to one.

On the other hand, the lack of injectivity of this morphism, when applied to the general core order (Definition 6.1), is the same as saying that this relation is not anti-symmetric. Thus, bearing in mind the well-known relationship between finite matrices and endomorphisms over finite-dimensional vector spaces, any counterexample for the injectivity of this morphism is a counterexample to the anti-symmetric property of the general core order. For instance, let us consider the following matrices in $\mathrm{Mat}_{5\times 5}(\mathbb{R})$:

$$A = \begin{pmatrix} 29 & 0 & 0 & 0 & 0 \\ 0 & 33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 29 & 0 & 0 & 0 & 0 \\ 0 & 33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Notice that $\mathrm{i}(A) = 2$ and $\mathrm{i}(B) = 3$. It is evident that

$$A_1 = B_1 = \begin{pmatrix} 29 & 0 \\ 0 & 33 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

hence $A <^{\#\!\!\!\bigcirc} B$ and $B <^{\#\!\!\!\bigcirc} A$, but nevertheless $A \neq B$.

Theorem 6.3. *The general core order is a pre-order on the set* $\mathrm{End}_k^{\mathrm{fp}}(V)$*.*

Proof. Reflexivity holds directly and transitivity is guaranteed by Theorem 5.8. □

Remark 6.4. As shown in Section 4, the core inverse makes sense in the framework of finite potent endomorphisms of index less than or equal to one. This limitation, when working in the framework of finite matrices, led Manjunatha Prasad and Mohana [13] to present the "core-EP" inverse in 2014 as an extension of the core inverse for the case of matrices of arbitrary index. Moreover, in 2016, H. Wang [25] introduced a new decomposition for square matrices called the core-EP decomposition and showed some of its applications. Among them, he defined some new orders such as the core-EP order and the core-minus order. The author of the present work thinks that it shall be of mathematical interest to generalize the theory of the core-EP inverse (with the core-EP decomposition) to finite potent endomorphisms. Once this is done, it shall be applied to studying the core-EP order on the set of finite potent endomorphisms of arbitrary index. Finally, the author thinks that the relation between the "general core order" and the core-EP order shall be clarified in the framework of finite potent endomorphisms and thus, by specialization, in the case of finite square matrices.

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