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# CHARACTERIZATIONS OF WEIGHT CLASSES FOR MULTILINEAR INTEGRAL OPERATORS AND SPARSE FORMS

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Abstract. The class of Muckenhoupt weights has been extended to the multilinear setting in several contexts of real and Fourier analysis. In this work, we provide new characterizations of these multilinear weight classes through a unifying principle based on pointwise, rather than averaged, evaluation of the weights.

## 1. Introduction and main results

Since the pioneering works of R. Coifman and Y. Meyer [5, 6, 7] and the subsequent systematic studies by C. Kenig and E. Stein [15], M. Lacey and C. Thiele [16, 17], and L. Grafakos and R. Torres [12, 13], the areas of multilinear Fourier and real analysis have continued to advance in a wide range of directions. One particularly active direction concerns the study of weighted estimates for multilinear counterparts of fundamental linear operators such as the *multi(sub)linear maximal operator* $\mathcal{M}$, defined for a vector function $\vec{f} := (f_1, \dots, f_m) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)^m$ as

$$\mathcal{M}(\vec{f})(x) := \sup_{Q \ni x} \prod_{j=1}^{m} \left( ⨍_Q |f_j(y_j)|\, dy_j \right) \quad \forall x \in \mathbb{R}^n, \tag{1.1}$$

where $Q \subset \mathbb{R}^n$ is a cube (always with sides parallel to the coordinate axes), and where $⨍_Q u$ stands for the average value $\frac{1}{|Q|}\int_Q u(x)\, dx$; the *multi(sub)linear fractional maximal and integral operators* $\mathcal{M}_\alpha$ and $\mathcal{I}_\alpha$, defined for $0 < \alpha < mn$ by

$$\mathcal{M}_\alpha(\vec{f})(x) := \sup_{Q \ni x} \prod_{j=1}^{m} \left( |Q|^{\frac{\alpha}{mn}} ⨍_Q |f_j(y_j)|\, dy_j \right) \quad \forall x \in \mathbb{R}^n \tag{1.2}$$

and

$$\mathcal{I}_\alpha(\vec{f})(x) := \int_{(\mathbb{R}^n)^m} \left( \sum_{j=1}^{m} |x - y_j| \right)^{\alpha - mn} \prod_{j=1}^{m} f_j(y_j) dy_j \quad \forall x \in \mathbb{R}^n, \tag{1.3}$$

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as well as *multilinear sparse forms* of the type

$$\Lambda_{\mathcal{S},\vec{r}}(f_1,\dots,f_m,h) := \sum_{Q\in\mathcal{S}} \left(\fint_Q |h(x)|^{r_{m+1}}\,dx\right)^{\frac{1}{r_{m+1}}} \prod_{j=1}^m \left(\fint_Q |f_j(x)|^{r_j}\,dx\right)^{\frac{1}{r_j}}, \tag{1.4}$$

where $\vec{r} := (r_1,\dots,r_{m+1}) \in [1,\infty)^{m+1}$ with $\sum_{j=1}^{m+1} 1/r_j > 1$, and $\mathcal{S}$ is a sparse family of cubes.

The multilinear weight classes that govern the weighted-norm estimates for $\mathcal{M}$, $\mathcal{M}_\alpha$, $\mathcal{I}_\alpha$, and $\Lambda_{\mathcal{S},\vec{r}}$ are described in Sections 1.1, 1.2, and 1.3 below, and the purpose of this note is to provide new, simpler characterizations of such classes. In practice, our characterizations allow us to check whether an $m$-tuple $\vec{w} := (w_1,\dots,w_m)$ belongs to one or more of the above classes by inspecting a condition of the type "there exists some $p \in (1,\infty)$ such that $u \in A_p(\mathbb{R}^n)$", instead of one of the type "$u \in A_p(\mathbb{R}^n)$ for a specific value of $p$", where $u$ is a power of some $w_j$.

Recall that for $1 < p < \infty$, a weight $w$ in $\mathbb{R}^n$ (that is, $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $w \geq 0$ a.e. in $\mathbb{R}^n$) is said to belong to the *Muckenhoupt class* $A_p(\mathbb{R}^n)$ if

$$[w]_{A_p} := \sup_Q \left(\fint_Q w\right)\left(\fint_Q w^{-\frac{1}{(p-1)}}\right)^{p-1} < \infty.$$

The endpoint classes for $p = 1$ and $p = \infty$ are defined by

$$[w]_{A_1} := \sup_Q \left(\fint_Q w\right)\left(\operatorname*{ess\,inf}_Q w\right)^{-1} < \infty$$

and

$$[w]_{A_\infty} := \sup_Q \left(\fint_Q w\right)\exp\left(-\fint_Q \ln w\right) < \infty.$$

As it turns out (see, for instance, [10, Section 9.3]), $A_\infty(\mathbb{R}^n) = \bigcup_{1\leq p<\infty} A_p(\mathbb{R}^n)$.

1.1. **The multilinear weight classes $A_{p,\vec{P}}$ and $A_{\vec{P}}$.** At the core of the multilinear Fourier analysis theory developed by A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, and R. Trujillo-González in [18] lies a multilinear version of the $A_p$-condition that can be introduced as follows. Let $\nu, w_1,\dots,w_m$ be nonnegative measurable functions defined in $\mathbb{R}^n$, and set $\vec{w} := (w_1,\dots,w_m)$. Given $0 < p < \infty$ and $\vec{P} := (p_1,\dots,p_m) \in [1,\infty)^m$, we write $(\nu,\vec{w}) \in A_{p,\vec{P}}$ if

$$[(\nu,\vec{w})]_{A_{p,\vec{P}}} := \sup_Q \left(\fint_Q \nu\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\fint_Q w_j^{1-p_j'}\right)^{1/p_j'} < \infty, \tag{1.5}$$

and, if some $p_j = 1$, then $\left(\fint_Q w_j^{1-p_j'}\right)^{1/p_j'}$ stands for $1/\inf_Q w_j$; see [18, Section 3.2]. In other words, if the indices $1,\dots,m$ are split as

$$\begin{aligned} I_{=1} &:= \{1 \leq j \leq m : p_j = 1\}, \\ I_{>1} &:= \{1 \leq j \leq m : p_j > 1\}, \end{aligned} \tag{1.6}$$

then $(\nu,\vec{w}) \in A_{p,\vec{P}}$ means

$$[(\nu,\vec{w})]_{A_{p,\vec{P}}} := \sup_Q \left(\fint_Q \nu\right)^{\frac{1}{p}} \left(\prod_{j\in I_{=1}} \frac{1}{\inf_Q w_j}\right) \prod_{j\in I_{>1}} \left(\fint_Q w_j^{1-p_j'}\right)^{1/p_j'} < \infty. \quad (1.7)$$

Notice that the condition (1.7) along with Lebesgue's differentiation theorem yields

$$\nu(x) \le [(\nu,\vec{w})]_{A_{p,\vec{P}}}^p \prod_{j=1}^m w_j(x)^{p/p_j} \quad \text{a.e. } x \in \mathbb{R}^n. \quad (1.8)$$

Following [18, Section 3.2], if the exponent $0 < p < \infty$ is related to $\vec{P}$ by means of the Hölder relation

$$\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j} \quad (1.9)$$

and $\nu$ is chosen as the product weight

$$\nu_{\vec{w}} := \prod_{j=1}^m w_j^{p/p_j}, \quad (1.10)$$

then $\vec{w}$ is said to satisfy the $A_{\vec{P}}$-condition, in symbols $\vec{w} \in A_{\vec{P}}$, if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\fint_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\prod_{j\in I_{=1}} \frac{1}{\inf_Q w_j}\right) \prod_{j\in I_{>1}} \left(\fint_Q w_j^{1-p_j'}\right)^{1/p_j'} < \infty.$$

The key roles of the classes $A_{p,\vec{P}}$ and $A_{\vec{P}}$ in multilinear Fourier and real analysis stem from two fundamental results for the multi(sub)linear maximal function $\mathcal{M}$ defined in (1.1): firstly, for $1 \le p_1, \dots, p_m < \infty$ and $p$ as in (1.9), by [18, Theorem 3.3] the multilinear weighted weak-type estimate

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)} \quad (1.11)$$

holds true if and only if $(\nu,\vec{w}) \in A_{p,\vec{P}}$. Secondly, given $1 < p_1, \dots, p_m < \infty$, and with $p$ still as in (1.9), by [18, Theorem 3.7] the multilinear weighted inequality

$$\|\mathcal{M}(\vec{f})\|_{L^{p}(\nu_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)} \quad (1.12)$$

holds true if and only if $\vec{w} \in A_{\vec{P}}$. Regarding sharp versions of (1.12), K. Li, K. Moen, and W. Sun established in [20, Theorem 1.2] the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^{p}(\nu_{\vec{w}})} \le C_{m,n,\vec{P}}\, [\vec{w}]_{A_{\vec{P}}}^{\max\{p_1'/p,\dots,p_m'/p\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

and proved that $\max\{p_1'/p,\dots,p_m'/p\}$ is the best possible exponent.

The classes $A_{p,\vec{P}}$ and $A_{\vec{P}}$ also shape weighted inequalities such as (1.11) and (1.12) for a number of other operators, including multilinear Calderón–Zygmund operators [9, Theorem 1.5], [18, Corollary 3.9], [20, Theorem 1.4]; multilinear square functions [11, Theorem 3.1]; multilinear singular integrals with rough kernels [2, Theorem 3.2]; commutators with singular integrals [2, Theorem 3.4]; multilinear Fourier multipliers [22, Theorem 1.2]; multilinear pseudodifferential operators; extrapolation estimates for multilinear operators [14, Theorem 2.9]; etc.

The first characterization of the multilinear class $A_{\vec{P}}$ comes from [18, Theorem 3.6].

**Theorem A** ([18, Theorem 3.6])**.** *Fix $1 \leq p_1, \ldots, p_m < \infty$ and let $p > 0$ be related to $\vec{P}$ by means of the Hölder relation* (1.9)*. Then $\vec{w} \in A_{\vec{P}}$ if and only if*

(i) $\nu_{\vec{w}} \in A_{mp}(\mathbb{R}^n)$, *and*

(ii) $w_j^{1-p_j'} \in A_{mp_j'}(\mathbb{R}^n)$ *for $j = 1, \ldots, m$. If $p_j = 1$, then the condition $w_j^{1-p_j'} \in A_{mp_j'}(\mathbb{R}^n)$ is understood as $w_j^{1/m} \in A_1(\mathbb{R}^n)$.*

Later on, and always under the Hölder relation (1.9), D. Cruz-Uribe and K. Moen extended Theorem A, within the case $p_j > 1$ for every $j = 1, \ldots, m$, as follows.

**Theorem B** ([8, Theorem 1.8])**.** *Fix $1 < p_1, \ldots, p_m < \infty$ and let $p > 0$ be related to $\vec{P}$ by means of the Hölder relation* (1.9)*. Then $\vec{w} \in A_{\vec{P}}$ if and only if*

(i) $\nu_{\vec{w}} \in A_\infty(\mathbb{R}^n)$, *and*

(ii) $w_j^{1-p_j'} \in A_\infty(\mathbb{R}^n)$ *for $j = 1, \ldots, m$.*

Along these lines, we now extend Theorems A and B so as to accommodate any $A_\infty(\mathbb{R}^n)$-weight $\nu$ satisfying $\nu \lesssim \nu_{\vec{w}}$ (which, due to (1.8), is a necessary condition for (1.5) to hold), as well as the cases when $p_j = 1$ for some $j = 1, \ldots, m$, and when $p$ and $\vec{P}$ are not necessarily related through the Hölder relation (1.9). More precisely, we prove

**Theorem 1.1.** *Fix $1 \leq p_1, \ldots, p_m < \infty$ and $0 < p < \infty$, not necessarily satisfying the Hölder relation* (1.9)*. Let $\vec{w} = (w_1, \ldots, w_m)$ be a vector weight and let $\nu$ be a weight satisfying*

(i) $\nu \in A_\infty(\mathbb{R}^n)$ *and there exists a constant $C_0 > 0$ such that*

$$\nu(x) \leq C_0 \prod_{j=1}^m w_j(x)^{p/p_j} \quad a.e.\ x \in \mathbb{R}^n; \tag{1.13}$$

(ii) $w_j^{1-p_j'} \in A_\infty(\mathbb{R}^n)$ *for $j = 1, \ldots, m$. If $p_j = 1$, then the condition $w_j^{1-p_j'} \in A_\infty(\mathbb{R}^n)$ is understood as $w_j^{\alpha_j} \in A_1(\mathbb{R}^n)$ for some $\alpha_j > 0$.*

*Then $(\nu, \vec{w}) \in A_{p,\vec{P}}$ with the estimate*

$$\begin{aligned} [(\nu, \vec{w})]_{A_{p,\vec{P}}} \leq C_0^{\frac{1}{p}} [2(m+1)]^{\frac{q_{m+1}-1}{p}} [\nu]_{A_{q_{m+1}}}^{\frac{1}{p}} \left( \prod_{j \in I_{>1}} [2(m+1)]^{\frac{q_j - 1}{p_j'}} \left[ w_j^{1-p_j'} \right]_{A_{q_j}}^{1/p_j'} \right) \\ \times \prod_{j \in I_{=1}} [2(m+1)]^{1/\alpha_j} \left[ w_j^{\alpha_j} \right]_{A_1}^{\frac{1}{\alpha_j}}, \end{aligned} \tag{1.14}$$

*where the index sets* $I_{>1}$ *and* $I_{=1}$ *are as in* (1.6)*,* $1 < q_j < \infty$ *satisfies* $w_j^{1-p_j'} \in A_{q_j}(\mathbb{R}^n)$ *for each* $j \in I_{>1}$*, and* $1 \le q_{m+1} < \infty$ *such that* $\nu \in A_{q_{m+1}}(\mathbb{R}^n)$*.*

1.2. **The multilinear weight class** $A_{(\vec{P},q)}$**.** The following version of the condition (1.5) has been studied by K. Moen [24]; by S. Chen, H. Wu, and Q. Xue in [3], and by X. Chen and Q. Xue in [4]: Given $1 \le p_1, \dots, p_m < \infty$ and $0 < q < \infty$, we say that $\vec{w} := (w_1, \dots, w_m)$ belongs to $A_{(\vec{P},q)}$ if

$$[\vec{w}]_{A_{(\vec{P},q)}} := \sup_Q \left( \fint_Q \left( \prod_{j=1}^m w_j \right)^q \right)^{1/q} \left( \prod_{j \in I_{=1}} \frac{1}{\inf_Q w_j} \right) \prod_{j \in I_{>1}} \left( \fint_Q w_j^{-p_j'} \right)^{1/p_j'} < \infty, \tag{1.15}$$

where the sets $I_{=1}$ and $I_{>1}$ are as in (1.6). Here are some known facts for $\vec{w} \in A_{(\vec{P},q)}$. The multilinear class $A_{(\vec{P},q)}$ arises from the study of the boundedness properties in weighted Lebesgue spaces of the operators $\mathcal{M}_\alpha$ and $\mathcal{I}_\alpha$ defined in (1.2) and (1.3). More precisely, by [24, Theorems 3.5 and 3.6], if $0 < \alpha < nm$ and $1 < p_1, \dots, p_m < \infty$ with $p\alpha < n$, where $p$ is as in (1.9), and if $0 < q < \infty$ is defined by

$$\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{n}, \tag{1.16}$$

then the inequality

$$\left( \int_{\mathbb{R}^n} \left[ |\mathcal{T}_\alpha(\vec{f})| \prod_{j=1}^m w_j \right]^q \right)^{1/q} \le C \prod_{j=1}^m \left( \int_{\mathbb{R}^n} (|f_j| w_j)^{p_j} \right)^{1/p_j}, \tag{1.17}$$

where $\mathcal{T}_\alpha = \mathcal{M}_\alpha$ or $\mathcal{I}_\alpha$, holds for every $\vec{f} := (f_1, \dots, f_m) \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ if and only if $\vec{w} \in A_{(\vec{P},q)}$. Sharp versions of the inequality (1.17) in terms of powers of $[\vec{w}]_{A_{(\vec{P},q)}}$ are proved in [21, Section 1]. The class $A_{(\vec{P},q)}$ also models weighted estimates for multilinear fractional maximal and integral operators with homogeneous kernels [4, Theorems 2.5 and 2.6]; fractional Leibniz rules [1, Corollary 15], extrapolation estimates for multilinear operators [14, Theorem 2.9]; etc.

A necessary condition for membership in $A_{(\vec{P},q)}$ comes from [24, Theorem 3.4].

**Theorem C** ([24, Theorem 3.4])**.** *Let* $1 < p_1, \dots, p_m < \infty$*, let* $0 < p < \infty$ *be defined by the Hölder relation* (1.9)*, and let* $p \le q < \infty$*. Then* $\vec{w} \in A_{(\vec{P},q)}$ *implies*

(i) $\left( \prod_{j=1}^m w_j \right)^q \in A_{mq}(\mathbb{R}^n)$*, and*

(ii) $w_j^{-p_j'} \in A_{mp_j'}(\mathbb{R}^n)$ *for* $j = 1, \dots, m$*.*

Later on, in [3], S. Chen, H. Wu, and Q. Xue extended Theorem C by proving the following characterization.

**Theorem D** ([3, Theorem 3.5])**.** *Fix* $1 \le p_1, \dots, p_m < \infty$*, let* $0 < p < \infty$ *be defined by the Hölder relation* (1.9)*, and let* $p \le q < \infty$*. Then* $\vec{w} \in A_{(\vec{P},q)}$ *if and only if*

(i) $\left(\prod_{j=1}^m w_j\right)^q \in A_{mq}(\mathbb{R}^n)$, *and*

(ii) $w_j^{-p'_j} \in A_{mp'_j}(\mathbb{R}^n)$ *for* $j = 1, \dots, m$. *If* $p_j = 1$, *then the condition* $w_j^{-p'_j} \in A_{mp'_j}(\mathbb{R}^n)$ *is understood as* $w_j^{1/m} \in A_1(\mathbb{R}^n)$.

Our next result weakens the assumptions (i) and (ii) from Theorem D (considered as testing conditions for membership in $A_{(\vec{P},q)}$) by replacing the classes $A_p(\mathbb{R}^n)$ for specific values of $p$ with the class $A_\infty(\mathbb{R}^n)$ and, in the case $p_j = 1$, by replacing the exponent $1/m$ with any $0 < \alpha_j < \infty$. Moreover, the Hölder relation (1.9) is not assumed. Namely, we prove

**Theorem 1.2.** *Let* $1 \leq p_1, \dots, p_m < \infty$ *and* $0 < q < \infty$ *(notice there's no "p" involved) and suppose that*

(i) $\left(\prod_{j=1}^m w_j\right)^q \in A_\infty(\mathbb{R}^n)$, *and*

(ii) $w_j^{-p'_j} \in A_\infty(\mathbb{R}^n)$ *for* $j = 1, \dots, m$. *If* $p_j = 1$, *then the condition* $w_j^{-p'_j} \in A_\infty(\mathbb{R}^n)$ *is understood as* $w_j^{\alpha_j} \in A_1(\mathbb{R}^n)$ *for some* $\alpha_j > 0$.

*Then* $\vec{w} \in A_{(\vec{P},q)}$ *with the estimate*

$$[\vec{w}]_{A_{(\vec{P},q)}} \leq [2(m+1)]^{\frac{q_{m+1}-1}{q}} \left[\left(\prod_{j=1}^m w_j\right)^q\right]_{A_{q_{m+1}}}^{\frac{1}{q}} \left(\prod_{j \in I_{>1}} [2(m+1)]^{\frac{q_j-1}{p'_j}} \left[w_j^{-p'_j}\right]_{A_{q_j}}^{1/p'_j}\right) \times \prod_{j \in I_{=1}} [2(m+1)]^{1/\alpha_j} \left[w_j^{\alpha_j}\right]_{A_1}^{\frac{1}{\alpha_j}}, \tag{1.18}$$

*where* $1 < q_j < \infty$ *satisfies* $w_j^{-p'_j} \in A_{q_j}(\mathbb{R}^n)$ *for each* $j \in I_{>1}$, *and* $1 \leq q_{m+1} < \infty$ *is such that* $\nu \in A_{q_{m+1}}(\mathbb{R}^n)$.

By combining Theorems D and 1.2 we immediately obtain the following characterization of the multilinear class $A_{(\vec{P},q)}$.

**Corollary 1.3.** *Fix* $1 \leq p_1, \dots, p_m < \infty$, *let* $0 < p < \infty$ *be defined by the Hölder relation* (1.9), *and let* $p \leq q < \infty$. *Then* $\vec{w} \in A_{(\vec{P},q)}$ *if and only if*

(i) $\left(\prod_{j=1}^m w_j\right)^q \in A_\infty(\mathbb{R}^n)$ *and*

(ii) $w_j^{-p'_j} \in A_\infty(\mathbb{R}^n)$ *for* $j = 1, \dots, m$. *If* $p_j = 1$, *then the condition* $w_j^{-p'_j} \in A_{mp'_j}(\mathbb{R}^n)$ *is understood as* $w_j^{1/m} \in A_1(\mathbb{R}^n)$.

Notice that there is no "$\alpha$" involved in Theorems C, D, 1.2, or Corollary 1.3. Now, regarding $0 < \alpha < mn$, X. Chen and Q. Xue [4] proved the following necessary condition for $\vec{w} \in A_{(\vec{P},q)}$ with $0 < p < \infty$ as in (1.9) and $0 < q < \infty$ as in (1.16).

**Theorem E** ([4, Theorem 2.2])**.** *Fix* $0 < \alpha < mn$ *and* $1 \leq p_1, \dots, p_m < \infty$. *Let* $0 < p < \infty$ *be defined by the Hölder relation* (1.9), *and let* $0 < q < \infty$ *as in* (1.16). *Then* $\vec{w} \in A_{(\vec{P},q)}$ *implies*

(i) $\left(\prod_{j=1}^m w_j\right)^q \in A_{q(m-\alpha/n)}(\mathbb{R}^n)$, *and*

(ii) $w_j^{-p_j'} \in A_{p_j'(m-\alpha/n)}(\mathbb{R}^n)$ *for* $j = 1, \dots, m$ *whenever* $\alpha/n < (m-2)+1/p_i + 1/p_j$ *for any* $1 \le i, j \le m$. *If* $p_j = 1$, *the condition* $w_j^{-p_j'} \in A_{p_j'(m-\alpha/n)}(\mathbb{R}^n)$ *is understood as* $w_j^{n/(mn-\alpha)} \in A_1(\mathbb{R}^n)$.

Later in [3], S. Chen, H. Wu, and Q. Xue completed the characterization of $\vec{w} \in A_{(\vec{P},q)}$ with $q$ as in (1.16) by proving

**Theorem F** ([3, Theorem 3.7]). *Fix* $0 < \alpha < mn$ *and* $1 \le p_1, \dots, p_m < \infty$. *Let* $0 < p < \infty$ *be defined by the Hölder relation* (1.9), *and let* $0 < q < \infty$ *be defined by* (1.16). *Then* $\vec{w} \in A_{(\vec{P},q)}$ *if and only if*

(i) $\left(\prod_{j=1}^m w_j\right)^q \in A_{q(m-\alpha/n)}(\mathbb{R}^n)$, *and*

(ii) $w_j^{-p_j'} \in A_{p_j'(m-\alpha/n)}(\mathbb{R}^n)$ *for* $j = 1, \dots, m$. *If* $p_j = 1$, *the condition* $w_j^{-p_j'} \in A_{p_j'(m-\alpha/n)}(\mathbb{R}^n)$ *is understood as* $w_j^{n/(mn-\alpha)} \in A_1(\mathbb{R}^n)$.

As a consequence of Theorem 1.2 and Theorem F we now obtain the following characterization of $\vec{w} \in A_{(\vec{P},q)}$, with $0 < p < \infty$ as in (1.9) and $0 < q < \infty$ as in (1.16), that extends both Theorem E and Theorem F by replacing the specific $A_p(\mathbb{R}^n)$ with $A_\infty(\mathbb{R}^n)$.

**Corollary 1.4.** *Fix* $0 < \alpha < mn$ *and* $1 \le p_1, \dots, p_m < \infty$. *Let* $0 < p < \infty$ *be as in* (1.9) *and* $q > 0$ *as in* (1.16). *Then* $\vec{w} \in A_{(\vec{P},q)}$ *if and only if*

(i) $\left(\prod_{j=1}^m w_j\right)^q \in A_\infty(\mathbb{R}^n)$, *and*

(ii) $w_j^{-p_j'} \in A_\infty(\mathbb{R}^n)$ *for* $j = 1, \dots, m$. *If* $p_j = 1$, *the condition* $w_j^{-p_j'} \in A_\infty(\mathbb{R}^n)$ *is understood as* $w_j^{n/(mn-\alpha)} \in A_1(\mathbb{R}^n)$.

*Proof.* If $\vec{w} \in A_{(\vec{P},q)}$, then (i) and (ii) follow from Theorem F; in turn, the fact that (i) and (ii) imply $\vec{w} \in A_{(\vec{P},q)}$ is provided by Theorem 1.2. ☐

1.3. **The multilinear weight class $A_{\vec{P},\vec{r}}$.** In [19], K. Li, J. M. Martell, and S. Ombrosi introduced an extension of the multilinear class of weights $A_{\vec{P}}$, defined as follows. Given $\vec{P} := (p_1, \dots, p_m) \in [1,\infty)^m$ and $\vec{r} := (r_1, \dots, r_{m+1}) \in [1,\infty)^{m+1}$, let $0 < p < \infty$ be defined by the Hölder relation (1.9), and write $\vec{r} \preceq \vec{P}$ if $r_j \le p_j$ for every $j = 1, \dots, m$ and $r_{m+1}' > p$. Also, $\vec{r} \prec \vec{P}$ if $\vec{r} \preceq \vec{P}$ and $r_j < p_j$ for every $j = 1, \dots, m$. Then a vector weight $\vec{w} := (w_1, \dots, w_m)$ belongs to $A_{\vec{P},\vec{r}}$ if

$$[\vec{w}]_{A_{\vec{P},\vec{r}}} := \sup_Q \left( \fint_Q \nu_{\vec{w}}^{\frac{r_{m+1}'}{r_{m+1}'-p}} \right)^{\frac{1}{p} - \frac{1}{r_{m+1}'}} \prod_{j=1}^m \left( \fint_Q w_j^{\frac{r_j}{r_j - p_j}} \right)^{\frac{1}{r_j} - \frac{1}{p_j}} < \infty, \tag{1.19}$$

where, if $r_{m+1} = 1$, the factor corresponding to $\nu_{\vec{w}}$ is replaced by $\left(\fint_Q \nu_{\vec{w}}\right)^{\frac{1}{p}}$ (always with $\nu_{\vec{w}}$ standing for the product weight defined in (1.10)), and if $p_j = r_j$ for some $j = 1, \dots, m$, then the corresponding term involving $w_j$ is replaced with

$1/\inf_Q w_j^{1/p_j}$. Notice that $A_{\vec{P}} = A_{\vec{P},(1,\dots,1)}$. Also, as described in [19, p. 23], by introducing

$$\frac{1}{p_{m+1}} := 1 - \frac{1}{p} \quad \text{and} \quad \frac{1}{\delta_j} := \frac{1}{r_j} - \frac{1}{p_j},\ j = 1, \dots, m+1, \tag{1.20}$$

as well as the sets

$$\mathcal{I}_< := \{1 \le j \le m : r_j < p_j\} \quad \text{and} \quad \mathcal{I}_= := \{1 \le j \le m : r_j = p_j\}, \tag{1.21}$$

the relation $\vec{r} \preceq \vec{P}$ means $r_j \le p_j$ for $j = 1, \dots, m$ and $r_{m+1} < p_{m+1}$, and the condition $\vec{w} \in A_{\vec{P},\vec{r}}$ defined in (1.19) can be recast as

$$[\vec{w}]_{A_{\vec{P},\vec{r}}} := \sup_Q \left( \fint_Q \nu_{\vec{w}}^{\delta_{m+1}/p} \right)^{\frac{1}{\delta_{m+1}}} \left( \prod_{j\in\mathcal{I}_=} \inf_Q w_j^{-\frac{1}{p_j}} \right) \prod_{j\in\mathcal{I}_<} \left( \fint_Q w_j^{-\delta_j/p_j} \right)^{\frac{1}{\delta_j}} < \infty. \tag{1.22}$$

The class $A_{\vec{P},\vec{r}}$ shapes weighted norm inequalities for the multilinear sparse form $\Lambda_{\mathcal{S},\vec{r}}$ defined in (1.4). Indeed, by [19, Remark 2.14], when $\vec{r} \prec \vec{P}$ the condition $\vec{w} \in A_{\vec{P},\vec{r}}$ is necessary and sufficient for the estimate

$$\Lambda_{\mathcal{S},\vec{r}}(f_1, \dots, f_m, h) \le C[\vec{w}]_{A_{\vec{P},\vec{r}}}^{\frac{1}{r-1}} \|h\|_{L^{p'}(\nu_{\vec{w}}^{1-p'})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)} \tag{1.23}$$

to hold for every measurable $f_1, \dots, f_m$, and $h$. As a consequence of (1.23), if a multilinear operator $T$ satisfies the inequality

$$\left| \int_{\mathbb{R}^n} hT(f_1, \dots, f_m) \right| \le C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},\vec{r}}(f_1, \dots, f_m, h) \tag{1.24}$$

for every $f_1, \dots, f_m, h \in C_c^\infty(\mathbb{R}^n)$, where the supremum is taken over all sparse families with a given sparsity constant (see [19, Section 2] for details), then the following weighted estimate for $T$ holds:

$$\|T(f_1, \dots, f_m)\|_{L^p(\nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P},\vec{r}}}^{\frac{1}{r-1}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Examples of operators $T$ satisfying (1.24) under suitable choices of indices $\vec{r}$ include bilinear rough singular integral operators, the bilinear Hilbert transform, and commutators with BMO functions; see [19, Sections 2.4, 2.5, 2.6].

In [19, Lemma 3.2], K. Li, J. M. Martell, and S. Ombrosi proved the following characterization of the weights in $A_{\vec{P},\vec{r}}$.

**Theorem G** ([19, Lemma 3.2]). *Let $\vec{P} := (p_1, \dots, p_m) \in [1,\infty)^m$ and $\vec{r} := (r_1, \dots, r_{m+1}) \in [1,\infty)^{m+1}$ be such that $\vec{r} \preceq \vec{P}$. Let $0 < p < \infty$ be defined by the Hölder relation* (1.9)*. For $j = 1, \dots, m+1$, let $\delta_j$ be as in* (1.20)*, and introduce*

$$\frac{1}{r} := \sum_{j=1}^{m+1} \frac{1}{r_j}, \quad \frac{1}{\varrho} := \frac{1}{\delta_m} + \frac{1}{\delta_{m+1}}, \quad \text{and} \quad \frac{1}{\theta_j} := \frac{1-r}{r} - \frac{1}{\delta_j} \ \text{for } j = 1, \dots, m. \tag{1.25}$$

*Then the following hold:*

(i) *Given* $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P},\vec{r}}$, *set*

$$\widehat{w} := \left( \prod_{j=1}^{m-1} w_j^{\frac{1}{p_j}} \right)^{\varrho} \quad \text{and} \quad W := \nu_{\vec{w}}^{\frac{r_m}{p}} \, \widehat{w}^{-\frac{r_m}{\delta_{m+1}}}.$$

*Then*

(i.1) $w_j^{\theta_j/p_j} \in A_{\frac{1-r}{r}\theta_j}$ *with* $\left[w_j^{\theta_j/p_j}\right]_{A_{\frac{1-r}{r}\theta_j}} \le [\vec{w}]_{A_{\vec{P},\vec{r}}}^{\theta_j}$ *for* $j = 1, \dots, m$;

(i.2) $\widehat{w} \in A_{\frac{1-r}{r}\varrho}$ *with* $[\widehat{w}]_{A_{\frac{1-r}{r}\varrho}} \le [\vec{w}]_{A_{\vec{P},\vec{r}}}^{\varrho}$; *and*

(i.3) $W \in A_{\frac{p_m}{r_m}, \frac{\delta_{m+1}}{r_m}}(\widehat{w})$ *with* $[W]_{A_{\frac{p_m}{r_m}, \frac{\delta_{m+1}}{r_m}}(\widehat{w})} \le [\vec{w}]_{A_{\vec{P},\vec{r}}}^{\delta_{m+1}}$.

(ii) *Given* $w_j^{\theta_j/p_j} \in A_{\frac{1-r}{r}\theta_j}$ *for* $j = 1, \dots, m-1$, *such that*

$$\widehat{w} := \left( \prod_{j=1}^{m-1} w_j^{\frac{1}{p_j}} \right)^{\varrho} \in A_{\frac{1-r}{r}\varrho} \quad \text{and} \quad W \in A_{\frac{p_m}{r_m}, \frac{\delta_{m+1}}{r_m}}(\widehat{w}),$$

*let us set* $w_m := W^{\frac{p_m}{r_m}} \, \widehat{w}^{-\frac{p_m}{\delta_m}}$. *Then* $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P},\vec{r}}$ *with*

$$[\vec{w}]_{A_{\vec{P},\vec{r}}} \le [W]_{A_{\frac{p_m}{r_m}, \frac{\delta_{m+1}}{r_m}}(\widehat{w})}^{\frac{1}{\delta_{m+1}}} [\widehat{w}]_{A_{\frac{1-r}{r}\varrho}}^{\frac{1}{\varrho}} \prod_{j=1}^{m-1} \left[w_j^{\theta_j/p_j}\right]_{A_{\frac{1-r}{r}\theta_j}}^{\frac{1}{\theta_j}}.$$

**Remark 1.5.** The definition of $A_{\frac{p_m}{r_m}, \frac{\delta_{m+1}}{r_m}}(\widehat{w})$ can be found in [19, p. 22] and it will not be used in what follows. The actual item (i.1) in [19, Lemma 3.2, p. 24] states the condition $w_j^{\theta_j/p_j} \in A_{\frac{1-r}{r}\theta_j}$ only for $j = 1, \dots, m-1$, but from its proof on [19, p. 26] it follows that the condition holds for $j = m$ as well.

Based directly on the condition (1.22), our next result provides a characterization of the class $A_{\vec{P},\vec{r}}$ in terms of $A_\infty(\mathbb{R}^n)$. More precisely, we prove

**Theorem 1.6.** *Let* $\vec{P} := (p_1, \dots, p_m) \in [1, \infty)^m$ *and* $\vec{r} := (r_1, \dots, r_{m+1}) \in [1, \infty)^{m+1}$ *be such that* $\vec{r} \preceq \vec{P}$. *With* $0 < p < \infty$ *defined by the Hölder relation* (1.9), *along with all the preceding notation, the following are equivalent:*

(i) $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P},\vec{r}}$;

(ii) $\nu_{\vec{w}}^{\delta_{m+1}/p} \in A_\infty(\mathbb{R}^n)$ *and* $w_j^{-\delta_j/p_j} \in A_\infty(\mathbb{R}^n)$ *for every* $j = 1, \dots, m$. *In the case* $\delta_j^{-1} = 0$ *(that is, if* $p_j = r_j$*), the condition* $w_j^{-\delta_j/p_j} \in A_\infty(\mathbb{R}^n)$ *is understood as* $w_j^{\theta_j/p_j} \in A_1(\mathbb{R}^n)$.

**Remark 1.7.** Let us illustrate an application of Theorem 1.6 with $m = 2$ and the power weights $w_1(x) := |x|^{-\alpha_1}$ and $w_2(x) := |x|^{-\alpha_2}$. Given $1 \le p_1, p_2 < \infty$ and $1 \le r_1, r_2, r_3 < \infty$ with $\vec{r} \prec \vec{P}$, that is, $r_1 < p_1$, $r_2 < p_2$, and $r_3 < p_3$, where

$$\frac{1}{p_3} := 1 - \frac{1}{p} = 1 - \frac{1}{p_1} - \frac{1}{p_2},$$

set $1/\delta_j := 1/r_j - 1/p_j$ for $j = 1, 2, 3$. Then $w_j(x)^{-\frac{\delta_j}{p_j}} = |x|^{\frac{\alpha_j \delta_j}{p_j}}$ for $j = 1, 2$, and $\nu_{\vec{w}}(x)^{\frac{\delta_3}{p}} = |x|^{-\delta_3\left(\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2}\right)}$. Recalling that a weight $|x|^\beta \in A_\infty(\mathbb{R}^n)$ if and only if

$\beta > -n$ (see, for instance, [10, Example 9.1.7]), it follows from Theorem 1.6 that $\vec{w} := (|x|^{-\alpha_1}, |x|^{-\alpha_2}) \in A_{\vec{P},\vec{r}}$ if and only if

$$\frac{\alpha_1}{p_1} > -\frac{n}{\delta_1}, \quad \frac{\alpha_2}{p_2} > -\frac{n}{\delta_2}, \quad \text{and} \quad \left(\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2}\right) < \frac{n}{\delta_3}. \tag{1.26}$$

As proved in [19, pp. 13–14], when $m = 2$ the condition

$$\frac{1}{\min\{r_1, 2\}} + \frac{1}{\min\{r_2, 2\}} + \frac{1}{\min\{r_3, 2\}} < 2 \tag{1.27}$$

for $\vec{r} := (r_1, r_2, r_3) \in (1,\infty)^3$ along with $\vec{r} \prec \vec{P}$, $p > 1$, and $\vec{w} = (w_1, w_2) \in A_{\vec{P},\vec{r}}$ guarantees the $L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(w)$ boundedness, with $w := w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}}$, of the bilinear Hilbert transform

$$\mathrm{BH}(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)g(x+t)\frac{dt}{t}.$$

In particular, it follows that

$$\mathrm{BH} : L^{p_1}(|x|^{-\alpha_1}) \times L^{p_2}(|x|^{-\alpha_2}) \to L^p(|x|^{-\frac{\alpha_1 p}{p_1}}|x|^{-\frac{\alpha_2 p}{p_2}}),$$

for indices $\vec{r} := (r_1, r_2, r_3) \in (1,\infty)^3$ satisfying (1.27), $1 < p, p_1, p_2 < \infty$ with $r_1 < p_1$, $r_2 < p_2$, $1/p = 1/p_1 + 1/p_2$, and $\alpha_1, \alpha_2$ as in (1.26).

## 2. Preliminaries

Our proofs will be based on the following lemma from [23].

**Lemma 2.1.** *Fix $1 \leq p < \infty$ and $w \in A_p(\mathbb{R}^n)$. Then, for every cube $Q$ and $N > 1$, there exists a set $F_N \subset Q$ such that*

$$|Q| \leq \frac{(N+1)}{N-1}|F_N| \tag{2.1}$$

*and*

$$\operatorname*{ess\,sup}_{F_N} w \leq (N+1) \fint_Q w\, dx \leq (N+1)^p [w]_{A_p} \operatorname*{ess\,inf}_{F_N} w. \tag{2.2}$$

**Remark 2.2.** Notice that if $w \in A_\infty(\mathbb{R}^n)$ and $p^*(w) := \inf\{p \in [1,\infty) : w \in A_p(\mathbb{R}^n)\}$, then by letting $p \to p^*(w)$ the inequality (2.2) also holds with $p^*(w)$ instead of $p$.

**Remark 2.3.** In the case of multiple $A_\infty(\mathbb{R}^n)$-weights, Lemma 2.1 will be used as follows. Fix $K$ weights $w_1, \dots, w_K \in A_\infty(\mathbb{R}^n)$, a cube $Q$, and $N > 1$. For $j = 1, \dots, K$, let $1 \leq q_j < \infty$ be an index such that $w_j \in A_{q_j}(\mathbb{R}^n)$ and let $F_{N,w_j}$ be the set $F_N$ from Lemma 2.1 applied to $w_j$, $Q$, and $N$. Then (2.1) implies

$$|Q| \leq \frac{(N+1)}{N-1}|F_{N,w_j}| \tag{2.3}$$

for each $j = 1, \dots, K$. Moreover, we have

$$\left|Q \setminus \bigcap_{j=1}^K F_{N,w_j}\right| = \left|\bigcup_{j=1}^K (Q \setminus F_{N,w_j})\right| \leq \sum_{j=1}^K |Q \setminus F_{N,w_j}| \leq \frac{2K}{N+1}|Q|,$$

since $|Q \setminus F_{N,w_j}| \le \frac{2}{N+1}|Q|$ due to (2.3). Thus, by setting $F_{N,K} := \bigcap_{j=1}^{K} F_{N,w_j}$ and choosing $N > 2K-1$, we obtain

$$|F_{N,K}| \ge \left(1 - \frac{2K}{N+1}\right)|Q| > 0. \tag{2.4}$$

In particular, the inequality (2.2) yields

$$\operatorname*{ess\,sup}_{F_{N,K}} w_j \le (N+1) \fint_Q w_j\, dx \le (N+1)^{q_j}[w_j]_{A_{q_j}} \operatorname*{ess\,inf}_{F_{N,K}} w \quad \forall j = 1, \dots, K. \tag{2.5}$$

In the proof of Theorem 1.6 we will use the following well-known characterization of the class $A_\infty(\mathbb{R}^n)$ (see, for instance, [10, Section 9.3.2]).

**Theorem H.** *Let $w$ be a weight in $\mathbb{R}^n$. The following are equivalent:*

(i) $w \in A_\infty(\mathbb{R}^n)$;

(ii) *there exist constants $\gamma, \beta \in (0,1)$ such that $\left|\left\{x \in Q : w(x) \ge \gamma \fint_Q w\right\}\right| \ge \beta|Q|$ for every cube $Q \subset \mathbb{R}^n$.*

## 3. Proofs of Theorems 1.1, 1.2, and 1.6

3.1. **Proof of Theorem 1.1.** In keeping with the notation $I_{>1}$ and $I_{=1}$ from (1.6), set $\sigma_j := w_j^{1-p_j'} \in A_\infty(\mathbb{R}^n)$ for $j \in I_{>1}$, $\sigma_j := w_j^{\alpha_j} \in A_1(\mathbb{R}^n)$ for $j \in I_{=1}$, and $\sigma_{m+1} := \nu \in A_\infty(\mathbb{R}^n)$. Given $N > 2(m+1)-1$ and a cube $Q \subset \mathbb{R}^n$, let $F_{N,\sigma_j}$ be the set $F_N \subset Q$ from Lemma 2.1 corresponding to each $\sigma_j \in A_\infty(\mathbb{R}^n)$, $j = 1, \dots, m+1$, and define $\widetilde{F_N} := \bigcap_{j=1}^{m+1} F_{N,\sigma_j}$ so that (2.4) (used with $K = m+1$) implies $|\widetilde{F_N}| \ge [1 - 2(m+1)/(N+1)]|Q| > 0$. Thus, from (2.5) we get

$$\fint_Q w_j^{1-p_j'} \le (N+1)^{q_j - 1}\big[w_j^{1-p_j'}\big]_{A_{q_j}} w_j(x)^{1-p_j'} \quad \text{a.e. } x \in F_{N,\sigma_j},\ j \in I_{>1}, \tag{3.1}$$

where $1 < q_j < \infty$ satisfies $w_j^{1-p_j'} \in A_{q_j}(\mathbb{R}^n)$, and (when $p_j = 1$)

$$\operatorname*{ess\,sup}_{F_{N,\sigma_j}} w_j^{\alpha_j} \le (N+1) \fint_Q w_j^{\alpha_j} \le (N+1)[w_j^{\alpha_j}]_{A_1} \operatorname*{ess\,inf}_{Q} w_j^{\alpha_j} \quad \forall j \in I_{=1}, \tag{3.2}$$

as well as (for $j = m+1$)

$$\fint_Q \nu \le (N+1)^{q_{m+1}-1}[\nu]_{A_{q_{m+1}}} \nu(x) \quad \text{a.e. } x \in F_{N,\sigma_{m+1}}, \tag{3.3}$$

with $1 \le q_{m+1} < \infty$ such that $\nu \in A_{q_{m+1}}(\mathbb{R}^n)$. Then, by raising (3.1) to the power $1/p_j'$,

$$\left(\fint_Q w_j^{1-p_j'}\right)^{1/p_j'} \le (N+1)^{(q_j-1)/p_j'}\big[w_j^{1-p_j'}\big]_{A_{q_j}}^{1/p_j'} w_j(x)^{(1-p_j')/p_j'} \tag{3.4}$$

for a.e. $x \in F_{N,\sigma_j}$ and every $j \in I_{>1}$. By raising (3.2) to the power $1/\alpha_j$, we get

$$\operatorname*{ess\,sup}_{F_{N,\sigma_j}} w_j \le (N+1)^{1/\alpha_j}\big[w_j^{\alpha_j}\big]_{A_1}^{1/\alpha_j} \inf_Q w_j \quad \forall j \in I_{=1}. \tag{3.5}$$

Now, for a.e. $x \in \widetilde{F_N}$, by multiplying (3.4) over $j \in I_{>1}$ and using (3.3) raised to the power $1/p$ we obtain

$$\begin{aligned}
&\left(\fint_Q \nu\right)^{\frac{1}{p}} \prod_{j\in I_{>1}} \left(\fint_Q w_j^{1-p_j'}\right)^{1/p_j'} \\
&\le (N+1)^{\frac{q_{m+1}-1}{p}} [\nu]_{A_{q_{m+1}}}^{\frac{1}{p}} \nu(x)^{\frac{1}{p}} \prod_{j\in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{1-p_j'}\right]_{A_{q_j}}^{1/p_j'} w_j(x)^{\frac{1-p_j'}{p_j'}} \\
&\le C_0^{\frac{1}{p}} (N+1)^{\frac{q_{m+1}-1}{p}} [\nu]_{A_{q_{m+1}}}^{\frac{1}{p}} \left(\prod_{j=1}^m w_j(x)^{\frac{1}{p_j}}\right) \\
&\quad \times \prod_{j\in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{1-p_j'}\right]_{A_{q_j}}^{1/p_j'} w_j(x)^{\frac{1-p_j'}{p_j'}} \\
&= C_0^{\frac{1}{p}} (N+1)^{\frac{q_{m+1}-1}{p}} [\nu]_{A_{q_{m+1}}}^{\frac{1}{p}} \left(\prod_{j\in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{1-p_j'}\right]_{A_{q_j}}^{1/p_j'}\right) \prod_{j\in I_{=1}} w_j(x)^{\frac{1}{p_j}},
\end{aligned}$$

where for the second inequality we used the hypothesis (1.13). Next, since $w_j^{\frac{1}{p_j}} = w_j$ for $j \in I_{=1}$ (because then $p_j = 1$), from (3.5) we get

$$\prod_{j\in I_{=1}} w_j(x)^{\frac{1}{p_j}} = \prod_{j\in I_{=1}} w_j(x) \le \prod_{j\in I_{=1}} (N+1)^{1/\alpha_j} \left[w_j^{\alpha_j}\right]_{A_1}^{\frac{1}{\alpha_j}} \inf_Q w_j, \tag{3.6}$$

and (1.7) follows with the estimate

$$\begin{aligned}
[(\nu,\vec{w})]_{A_{p,\vec{P}}} \le C_0^{\frac{1}{p}} (N+1)^{\frac{q_{m+1}-1}{p}} [\nu]_{A_{q_{m+1}}}^{\frac{1}{p}} &\left(\prod_{j\in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{1-p_j'}\right]_{A_{q_j}}^{1/p_j'}\right) \\
&\times \prod_{j\in I_{=1}} (N+1)^{1/\alpha_j} \left[w_j^{\alpha_j}\right]_{A_1}^{\frac{1}{\alpha_j}},
\end{aligned}$$

and (1.14) follows after letting $N \to 2(m+1)-1$. □

3.2. **Proof of Theorem 1.2.** The proof is similar to that of Theorem 1.1. For $j \in I_{>1}$, set $\sigma_j := w_j^{-p_j'} \in A_\infty(\mathbb{R}^n)$; for $j \in I_{=1}$, let $\sigma_j := w_j^{\alpha_j} \in A_1(\mathbb{R}^n)$ and $\sigma_{m+1} := \left(\prod_{j=1}^m w_j\right)^q \in A_\infty(\mathbb{R}^n)$. As before, given $N > 2(m+1)-1$ and a cube $Q \subset \mathbb{R}^n$, let $F_{N,\sigma_j}$ be the set $F_N \subset Q$ from Lemma 2.1 corresponding to each $\sigma_j \in A_\infty(\mathbb{R}^n)$, $j = 1,\dots,m+1$. Set $\widetilde{F_N} := \bigcap_{j=1}^{m+1} F_{N,\sigma_j}$ so that $|\widetilde{F_N}| \ge [1-2(m+1)/(N+1)]|Q| > 0$. Then (2.5) gives

$$\fint_Q w_j^{-p_j'} \le (N+1)^{q_j-1} \left[w_j^{-p_j'}\right]_{A_{q_j}} w_j(x)^{-p_j'} \quad \text{a.e. } x \in F_{N,\sigma_j},\ j \in I_{>1}, \tag{3.7}$$

where $1 < q_j < \infty$ satisfies $w_j^{-p_j'} \in A_{q_j}(\mathbb{R}^n)$, as well as (3.5) and, for $j = m+1$,

$$\fint_Q \left(\prod_{j=1}^m w_j\right)^q \leq (N+1)^{q_{m+1}-1} \left[\left(\prod_{j=1}^m w_j\right)^q\right]_{A_{q_{m+1}}} \left(\prod_{j=1}^m w_j(x)\right)^q \tag{3.8}$$

for a.e. $x \in F_{N,\sigma_{m+1}}$, with $1 \leq q_{m+1} < \infty$ such that $\left(\prod_{j=1}^m w_j\right)^q \in A_{q_{m+1}}(\mathbb{R}^n)$. By raising (3.7) to the power $1/p_j'$ and by multiplying over $j \in I_{>1}$ for a.e. $x \in \widetilde{F_N}$, we get

$$\prod_{j \in I_{>1}} \left(\fint_Q w_j^{-p_j'}\right)^{1/p_j'} \leq \prod_{j \in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{-p_j'}\right]_{A_{q_j}}^{1/p_j'} w_j(x)^{-1},$$

which multiplied by (3.8) (raised to the power $1/q$) gives, always for a.e. $x \in \widetilde{F_N}$,

$$\begin{aligned}
&\left(\fint_Q \left(\prod_{j=1}^m w_j\right)^q\right)^{\frac{1}{q}} \prod_{j \in I_{>1}} \left(\fint_Q w_j^{-p_j'}\right)^{1/p_j'} \\
&\leq (N+1)^{\frac{q_{m+1}-1}{q}} \left[\left(\prod_{j=1}^m w_j\right)^q\right]_{A_{q_{m+1}}}^{\frac{1}{q}} \left(\prod_{j=1}^m w_j(x)\right) \\
&\quad \times \prod_{j \in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{-p_j'}\right]_{A_{q_j}}^{1/p_j'} w_j(x)^{-1} \\
&= (N+1)^{\frac{q_{m+1}-1}{q}} \left[\left(\prod_{j=1}^m w_j\right)^q\right]_{A_{q_{m+1}}}^{\frac{1}{q}} \left(\prod_{j \in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{-p_j'}\right]_{A_{q_j}}^{1/p_j'}\right) \prod_{j \in I_{=1}} w_j(x)
\end{aligned}$$

and, by recalling (3.6), the condition (1.15) holds with

$$\begin{aligned}
[\vec{w}]_{A_{(\vec{P},q)}} &\leq (N+1)^{\frac{q_{m+1}-1}{q}} \left[\left(\prod_{j=1}^m w_j\right)^q\right]_{A_{q_{m+1}}}^{\frac{1}{q}} \left(\prod_{j \in I_{>1}} (N+1)^{\frac{q_j-1}{p_j'}} \left[w_j^{-p_j'}\right]_{A_{q_j}}^{1/p_j'}\right) \\
&\quad \times \prod_{j \in I_{=1}} (N+1)^{1/\alpha_j} \left[w_j^{\alpha_j}\right]_{A_1}^{\frac{1}{\alpha_j}},
\end{aligned}$$

and (1.18) follows after letting $N \to 2(m+1)-1$. □

3.3. **Proof of Theorem 1.6.** (i) ⇒ (ii). Recall the definition of the sets $\mathcal{I}_<$ and $\mathcal{I}_=$ from (1.21). By (i.1) of Theorem G, $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P},\vec{r}}$ implies that $w_j^{\theta_j/p_j} \in A_{\frac{1-r}{r}\theta_j}$ for $j \in \mathcal{I}_<$, and the definition of $\theta_j$ from (1.25) yields $\frac{1-r}{r}\theta_j = 1 + \theta_j/\delta_j$. Hence $w_j^{\theta_j/p_j} \in A_{1+\theta_j/\delta_j}$, which means

$$\left[w_j^{\theta_j/p_j}\right]_{A_{1+\theta_j/\delta_j}} = \sup_Q \left(\frac{1}{|Q|}\int_Q w_j^{\theta_j/p_j}\right)\left(\frac{1}{|Q|}\int_Q \left(w_j^{\theta_j/p_j}\right)^{-\frac{1}{\theta_j/\delta_j}}\right)^{\theta_j/\delta_j} < \infty,$$

with

$$\begin{aligned}\big[w_j^{\theta_j/p_j}\big]_{A_{1+\theta_j/\delta_j}} &= \sup_Q \left(\fint_Q w_j^{\theta_j/p_j}\right)\left(\fint_Q w_j^{-\delta_j/p_j}\right)^{\theta_j/\delta_j} \\ &= \sup_Q \left(\fint_Q \left(w_j^{-\delta_j/p_j}\right)^{-\frac{\frac{1}{\delta_j}}{\frac{\delta_j}{\theta_j}}}\right)\left(\fint_Q w_j^{-\delta_j/p_j}\right)^{\theta_j/\delta_j} = \big[w_j^{-\delta_j/p_j}\big]^{\frac{\theta_j}{\delta_j}}_{A_{1+\frac{\delta_j}{\theta_j}}}.\end{aligned}$$

Consequently, $w_j^{-\delta_j/p_j} \in A_\infty(\mathbb{R}^n)$ for $j \in \mathcal{I}_<$. Now, if $j \in \mathcal{I}_=$, the proof of [19, Lemma 3.2 (p. 26, line 8)] yields $w_j^{\theta_j/p_j} \in A_1(\mathbb{R}^n)$ with $[w_j^{\theta_j/p_j}]_{A_1(\mathbb{R}^n)} \leq [\vec{w}]^{\theta_j}_{A_{\vec{P},\vec{r}}}$.

Next, let us see that $\nu_{\vec{w}}^{\delta_{m+1}/p} \in A_\infty(\mathbb{R}^n)$. Given a cube $Q \subset \mathbb{R}^n$, fix $N > 2|\mathcal{I}_<|-1$ (if $|\mathcal{I}_<| = 0$, take $N > 1$) and let $F_{N,\sigma_j}$ be the set $F_N \subset Q$ from Lemma 2.1 corresponding to each $w_j^{-\delta_j/p_j} \in A_\infty(\mathbb{R}^n)$ for $j \in \mathcal{I}_<$. Set $\widetilde{F_N} := \bigcap_{j\in\mathcal{I}_<} F_{N,\sigma_j}$ so that

$$|\widetilde{F_N}| \geq [1 - 2|\mathcal{I}_<|/(N+1)]|Q| > 0. \tag{3.9}$$

We will prove that Theorem H (ii) holds true for $\nu_{\vec{w}}^{\delta_{m+1}/p}$ and therefore $\nu_{\vec{w}}^{\delta_{m+1}/p} \in A_\infty(\mathbb{R}^n)$. Now, the first inequality from (2.2) implies

$$\operatorname*{ess\,sup}_{\widetilde{F_N}} w_j^{-\delta_j/p_j} \leq (N+1)\fint_Q w_j^{-\delta_j/p_j} \quad \forall j \in \mathcal{I}_<. \tag{3.10}$$

Next, for a.e. $x \in \widetilde{F_N} \subset Q$, the definition of $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P},\vec{r}}$ in (1.22) and the inequality (3.10) (raised to the power $-1/\delta_j$) yield

$$\begin{aligned}\left(\fint_Q \nu_{\vec{w}}^{\delta_{m+1}/p}\right)^{1/\delta_{m+1}} &\leq [\vec{w}]_{A_{\vec{P},\vec{r}}} \prod_{j\in\mathcal{I}_<}\left(\fint_Q w_j^{-\delta_j/p_j}\right)^{-\frac{1}{\delta_j}}\left(\prod_{j\in\mathcal{I}_=}\inf_Q w_j^{\frac{1}{p_j}}\right) \\ &\leq [\vec{w}]_{A_{\vec{P},\vec{r}}} \prod_{j\in\mathcal{I}_<}(N+1)^{1/\delta_j} w_j(x)^{\frac{1}{p_j}}\left(\prod_{j\in\mathcal{I}_=}\inf_Q w_j^{\frac{1}{p_j}}\right) \\ &\leq [\vec{w}]_{A_{\vec{P},\vec{r}}}\left(\prod_{j\in\mathcal{I}_<}(N+1)^{1/\delta_j}\right)\nu_{\vec{w}}(x)^{\frac{1}{p}}.\end{aligned}$$

Hence, Theorem H (ii) holds true for $\nu_{\vec{w}}^{\delta_{m+1}/p}$ with $\beta, \gamma \in (0,1)$ defined by

$$1/\gamma := [\vec{w}]^{\delta_{m+1}}_{A_{\vec{P},\vec{r}}}\left(\prod_{j\in\mathcal{I}_<}(N+1)^{1/\delta_j}\right)^{\delta_{m+1}}$$

and, by recalling (3.9), $\beta := 1 - 2|\mathcal{I}_<|/(N+1)$; consequently, $\nu_{\vec{w}}^{\delta_{m+1}/p} \in A_\infty(\mathbb{R}^n)$, as claimed.

(ii) $\Rightarrow$ (i). Set $\sigma_j := w_j^{-\delta_j/p_j} \in A_\infty(\mathbb{R}^n)$ for $j \in \mathcal{I}_<$, $\sigma_j := w_j^{\theta_j/p_j} \in A_1(\mathbb{R}^n)$ for $j \in \mathcal{I}_=$, and $\sigma_{m+1} := \nu_{\vec{w}}^{\delta_{m+1}/p} \in A_\infty(\mathbb{R}^n)$. Given $N > 2(m+1) - 1$ and a cube $Q \subset \mathbb{R}^n$, let $F_{N,\sigma_j}$ be the set $F_N \subset Q$ from Lemma 2.1 corresponding to each $\sigma_j \in A_\infty(\mathbb{R}^n)$, $j = 1, \dots, m+1$, and set $\widetilde{F_N} := \bigcap_{j=1}^{m+1} F_{N,\sigma_j}$ so that $|\widetilde{F_N}| \geq [1 - 2(m+1)/(N+1)]|Q| > 0$. Then, for a.e. $x \in \widetilde{F_N}$, we have

$$\fint_Q w_j^{-\delta_j/p_j} \leq (N+1)^{q_j-1} \big[w_j^{-\delta_j/p_j}\big]_{A_{q_j}} w_j(x)^{-\delta_j/p_j} \quad \forall j \in \mathcal{I}_<, \tag{3.11}$$

where $1 < q_j < \infty$ satisfies $w_j^{-\delta_j/p_j} \in A_{q_j}(\mathbb{R}^n)$,

$$\operatorname*{ess\,sup}_{\widetilde{F_N}} w_j^{\theta_j/p_j} \leq (N+1) \fint_Q w_j^{\theta_j/p_j} \leq (N+1)[w_j^{\theta_j/p_j}]_{A_1} \operatorname*{ess\,inf}_Q w_j^{\theta_j/p_j} \quad \forall j \in \mathcal{I}_=, \tag{3.12}$$

and

$$\fint_Q \nu_{\vec{w}}^{\delta_{m+1}/p} \leq (N+1)^{q_{m+1}-1} \big[\nu_{\vec{w}}^{\delta_{m+1}/p}\big]_{A_{q_{m+1}}} \nu_{\vec{w}}(x)^{\delta_{m+1}/p}, \tag{3.13}$$

with $1 \leq q_{m+1} < \infty$ such that $\nu_{\vec{w}}^{\delta_{m+1}/p} \in A_{q_{m+1}}(\mathbb{R}^n)$. Then, always for a.e. $x \in \widetilde{F_N}$, (3.11) and (3.13) give

$$\begin{aligned}
&\left(\fint_Q \nu_{\vec{w}}^{\delta_{m+1}/p}\right)^{\frac{1}{\delta_{m+1}}} \prod_{\mathcal{I}_<} \left(\fint_Q w_j^{-\delta_j/p_j}\right)^{\frac{1}{\delta_j}} \\
&\leq (N+1)^{\frac{q_{m+1}-1}{\delta_{m+1}}} \big[\nu_{\vec{w}}^{\delta_{m+1}/p}\big]_{A_{q_{m+1}}}^{\frac{1}{\delta_{m+1}}} \nu_{\vec{w}}(x)^{1/p} \prod_{\mathcal{I}_<} (N+1)^{\frac{q_j-1}{\delta_j}} \big[w_j^{-\delta_j/p_j}\big]_{A_{q_j}}^{\frac{1}{\delta_j}} w_j(x)^{-1/p_j} \\
&= (N+1)^{\frac{q_{m+1}-1}{\delta_{m+1}}} \big[\nu_{\vec{w}}^{\delta_{m+1}/p}\big]_{A_{q_{m+1}}}^{\frac{1}{\delta_{m+1}}} \left(\prod_{\mathcal{I}_<} (N+1)^{\frac{q_j-1}{\delta_j}} \big[w_j^{-\delta_j/p_j}\big]_{A_{q_j}}^{\frac{1}{\delta_j}}\right) \prod_{\mathcal{I}_=} w_j(x)^{\frac{1}{p_j}},
\end{aligned}$$

whereas (3.12) implies

$$\prod_{\mathcal{I}_=} w_j(x)^{\frac{1}{p_j}} \leq \prod_{\mathcal{I}_=} (N+1)^{1/\theta_j} \big[w_j^{\theta_j/p_j}\big]_{A_1}^{1/\theta_j} \operatorname*{ess\,inf}_Q w_j^{1/p_j},$$

and then (1.22) holds with

$$\begin{aligned}
[\vec{w}]_{A_{\vec{P},\vec{r}}} \leq (N+1)^{\frac{q_{m+1}-1}{\delta_{m+1}}} \big[\nu_{\vec{w}}^{\delta_{m+1}/p}\big]_{A_{q_{m+1}}}^{\frac{1}{\delta_{m+1}}} &\left(\prod_{\mathcal{I}_<} (N+1)^{\frac{q_j-1}{\delta_j}} \big[w_j^{-\delta_j/p_j}\big]_{A_{q_j}}^{\frac{1}{\delta_j}}\right) \\
&\times \prod_{\mathcal{I}_=} (N+1)^{1/\theta_j} \big[w_j^{\theta_j/p_j}\big]_{A_1}^{1/\theta_j},
\end{aligned}$$

where $N+1$ above can be replaced by $2(m+1)$ by letting $N \to 2(m+1) - 1$. □

**Remark 3.1.** Notice that the Hölder relation (1.9) has not been used in the proof of the implication (ii) ⇒ (i) from Theorem 1.6. Moreover, that implication still holds true if, in the case $r_j = p_j$, the assumption $w_j^{\theta_j/p_j} \in A_1(\mathbb{R}^n)$ in (ii) is replaced with $w_j^{\alpha_j} \in A_1(\mathbb{R}^n)$ for some $\alpha_j > 0$, since the latter also implies (3.12) with $w_j^{\theta_j/p_j}$ replaced by $w_j^{\alpha_j}$.

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