

SMOOTH GEOMETRY OF DIFFUSION ALGEBRAS

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Dedicated to Professor Oswaldo Lezama on the occasion of his 70th birthday

ABSTRACT. We establish sufficient conditions to assert the differential smoothness of diffusion algebras on n generators introduced by Isaev et al. [J. Phys. A 34 (2001), pp. 5815–5834]. We present a detailed list of these algebras on four and five generators to illustrate the obtained results.

1. INTRODUCTION

Isaev et al. [20] introduced diffusion algebras in 2001 in the context of one-dimensional stochastic processes with exclusion in statistical mechanics. A year later, Pyatov and Twarock [28] presented a construction for these algebras from a mathematical point of view and proved the results formulated in [20]. Hinchcliffe, in his Ph.D. thesis [19], and other researchers have investigated several ring-theoretical and homological properties of diffusion algebras [16, 18, 42, 26, 30, 33, 32].

Our purpose in this paper is to investigate the *differential smoothness*, as defined by Brzeziński and Sitarz [12], of diffusion algebras. We begin with some preliminary remarks.

Following Brzeziński and Lomp [11, Section 1], the study of smoothness of algebras can be traced back at least to Grothendieck's EGA [17]. Grothendieck's notion of a *formally smooth commutative (topological) algebra* was later extended to the noncommutative setting by Schelter [40]. In this framework, an algebra is formally smooth precisely when the kernel of the multiplication map is projective as a bimodule; as observed by Schelter, this notion replaces a much broader smoothness condition based only on finite global dimension. Cuntz and Quillen [13] refer to such algebras as quasi-free. In the homological setting, Van den Bergh [3] called an algebra *homologically smooth* if it admits a finite resolution by finitely generated projective bimodules. The characterization of this kind of smoothness for the noncommutative pillow, the quantum teardrops, and quantum homogeneous spaces was made by Brzeziński [5, 7] and Krähmer [23].

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Brzeziński and Sitarz [12] defined another notion of smoothness of algebras, termed *differential smoothness*, due to the use of differential graded algebras of a specified dimension that admit a noncommutative version of the Hodge star isomorphism. This notion considers the existence of a top form in a differential calculus over an algebra, and a strong version of Poincaré duality realized as an isomorphism between complexes of differential and integral forms. This new notion of smoothness is different and more constructive than the homological smoothness mentioned above. We quote from [12, pp. 413–414]: “The idea behind the *differential smoothness* of algebras is rooted in the observation that a classical smooth orientable manifold, in addition to de Rham complex of differential forms, admits also the complex of *integral forms* isomorphic to the de Rham complex [27, Section 4.5]. The de Rham differential can be understood as a special left connection, while the boundary operator in the complex of integral forms is an example of a *right connection*”.

Brzeziński and several other authors [8, 9, 10, 11, 12, 15, 21, 22, 31] have characterized the differential smoothness of algebras such as the quantum two- and three-spheres, the disc, the plane, the noncommutative torus, the coordinate algebras of the quantum group $SU_q(2)$, the noncommutative pillow algebra, the quantum cone algebras, the quantum polynomial algebras, Hopf algebra domains of Gelfand–Kirillov dimension two that are not PI, families of Ore extensions, some three-dimensional skew polynomial algebras, diffusion algebras in three generators, and noncommutative coordinate algebras of deformations of several classical orbifolds, such as the pillow orbifold, singular cones, and lens spaces. Precisely, in [31] the second author presented a first approach to the differential smoothness of diffusion algebras on three generators. Since diffusion algebras on three generators are related to three-dimensional skew polynomial algebras [2, 34], skew bi-quadratic algebras [1] (see also double Ore extensions [1, p. 699] and [44]), and skew PBW extensions [16], and since the smoothness of all these families of algebras has recently been investigated in [33, 39, 35, 38, 36, 37], the purpose of this paper is to continue this line of research for diffusion algebras on four and more generators.

The article is organized as follows. Section 2 contains the key facts on diffusion algebras in order to set up notation and render this paper self-contained. We adopt the terminology presented by Pyatov and Twarock [28]. Sections 2.1 and 2.2 recall the classification of diffusion algebras on three and n generators, respectively. Next, Section 3 contains definitions and preliminaries on differential smoothness of algebras. In Section 4 we present the original results of the paper. Table 1 presents the diffusion algebras on four generators, while Table 2 contains diffusion algebras on five generators. Our key results are Theorems 4.1 and 4.2: we describe explicitly those diffusion algebras that are differentially smooth, and those for which there are no one-dimensional connected integrable calculi, respectively.

Throughout the paper, \mathbb{N} denotes the set of natural numbers, including zero. The word *ring* means an associative ring with identity, not necessarily commutative. All vector spaces and algebras (always associative and with unit) are over a fixed

field \mathbb{k} . As usual, the symbols \mathbb{R} and \mathbb{C} denote the fields of real and complex numbers, respectively.

2. DIFFUSION ALGEBRAS

Pyatov and Twarock [28] studied diffusion algebras from a mathematical point of view and proved a construction theorem for diffusion algebras. Let us recall the details of their treatment.

Let α, β be two elements belonging to the set $I_N := \{1, \dots, n\}$ with $\alpha < \beta$. Consider quadratic relations of the form

$$g_{\alpha\beta}D_\alpha D_\beta - g_{\beta\alpha}D_\beta D_\alpha = x_\beta D_\alpha - x_\alpha D_\beta, \quad (2.1)$$

with $g_{\alpha\beta} \in \mathbb{R} \setminus \{0\}$, $g_{\beta\alpha} \in \mathbb{R}$, and $x_\alpha, x_\beta \in \mathbb{C}$.

Definition 2.1 ([28, Definition 1.1]). An algebra with set of generators given by $\{D_\alpha \mid \alpha \in I_N\}$ and relations of type (2.1) is called a *diffusion algebra* if it admits a linear PBW-basis of ordered monomials of the form

$$D_{\alpha_1}^{k_1} D_{\alpha_2}^{k_2} \cdots D_{\alpha_N}^{k_N}, \quad \text{with } k_j \in \mathbb{N} \text{ and } \alpha_1 > \alpha_2 > \cdots > \alpha_N. \quad (2.2)$$

Due to physical reasons, only relations with positive coefficients $g_{\alpha\beta} \in \mathbb{R}_{>0}$ and $g_{\beta\alpha} \in \mathbb{R}_{\geq 0}$ ($\alpha < \beta$) are relevant, because they are interpreted as hopping rates in stochastic models [28, p. 3268].

Note that the requirement of having a PBW basis (2.2) implies conditions on the coefficients $g_{\alpha\beta}$ and x_α in (2.1) according to the Diamond Lemma formulated by Bergman [4]. This means that we have a criterion to verify under which conditions the relations in (2.1) are of PBW type: this is the case precisely if each subset of three generators $\{D_\alpha, D_\beta, D_\gamma\}$ with ordering $\alpha < \beta < \gamma$ is reduction unique with respect to the ordering, that is, if the two ways of reducing the monomial $D_\alpha D_\beta D_\gamma$ to the monomial $D_\gamma D_\beta D_\alpha$ lead to the same result when expressed in the PBW basis (2.2).

Just as Pyatov and Twarock [28, p. 3269] asserted, the task of deriving all diffusion algebras with N generators reduces to the following two steps:

- (1) Find all diffusion algebras with three generators.
- (2) Find all algebras with N generators such that each subset of three generators coincides with one of the cases listed before.

As can be seen, step (1) is equivalent to finding those coefficients $g_{\alpha\beta}$ and x_α in (2.1) for which a set $\{D_\gamma, D_\beta, D_\alpha\}$ of three generators is reduction unique in the above sense. The list of diffusion algebras of three generators is given in Section 2.1. The second step is a combinatorial problem: it requires one to combine, in a consistent way, the three-generator algebras listed before into algebras with N generators for general $N > 3$.

Pyatov and Twarock considered a constructive method for approaching the second step, the so-called *blending procedure* (Section 2.2), which is an inductive procedure for constructing diffusion algebras: "It uses the three generator cases and augments them to larger units by attaching further generators in accordance with the requirements of the diamond lemma, then giving a prescription of how these

larger building blocks may be glued [...] together in order to obtain a general diffusion algebra of N generators” [28, p. 3269].

2.1. Diffusion algebras on three generators. Consider a set $\{D_\alpha, D_\beta, D_\gamma\}$ of three generators with an ordering induced by the ordering of the index set $\alpha < \beta < \gamma$ and relations as in (2.1). Since $g_{\alpha\beta} \neq 0$ for all $\alpha, \beta \in I_3 = \{\alpha, \beta, \gamma\}$ with $\alpha < \beta$ by assumption, we get the relations of the algebra given by

$$\begin{aligned} D_\alpha D_\beta &= q_{\beta\alpha} D_\beta D_\alpha + x_\beta^{\alpha\beta} D_\alpha - x_\alpha^{\alpha\beta} D_\beta, \\ D_\alpha D_\gamma &= q_{\gamma\alpha} D_\gamma D_\alpha + x_\gamma^{\alpha\gamma} D_\alpha - x_\alpha^{\alpha\gamma} D_\gamma, \quad \text{and} \\ D_\beta D_\gamma &= q_{\gamma\beta} D_\gamma D_\beta + x_\gamma^{\beta\gamma} D_\beta - x_\beta^{\beta\gamma} D_\gamma, \end{aligned} \tag{2.3}$$

where $q_{ij} := \frac{g_{ji}}{g_{ij}}$, $x_\gamma^{ij} := \frac{x_\gamma}{g_{ij}}$ for $i, j, k \in \{\alpha, \beta, \gamma\}$ with $i < j$. Using (2.3), any monomial can be expressed in terms of the PBW basis (2.2), and this is well defined if we apply (2.3) in different orders and obtain the same result, that is, if the reductions

$$D_\alpha D_\beta D_\gamma \rightarrow D_\beta D_\alpha D_\gamma \rightarrow D_\beta D_\gamma D_\alpha \rightarrow D_\gamma D_\beta D_\alpha$$

and

$$D_\alpha D_\beta D_\gamma \rightarrow D_\alpha D_\gamma D_\beta \rightarrow D_\gamma D_\alpha D_\beta \rightarrow D_\gamma D_\beta D_\alpha$$

using (2.3) coincide when expressed in the PBW basis. These equalities lead to restrictions on the coefficients $g_{\alpha\beta}$ and x_α in (2.1), given by a set of six equations, and their solutions determine all diffusion algebras of three generators.

Next, we recall the list of six equations. We assume that $\alpha < \beta < \gamma$ and $x_j \neq 0$ for $j \in \{\alpha, \beta, \gamma\}$.

(1) *The case of A_I :*

$$\begin{aligned} gD_\alpha D_\beta - gD_\beta D_\alpha &= x_\beta D_\alpha - x_\alpha D_\beta, \\ gD_\alpha D_\gamma - gD_\gamma D_\alpha &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ gD_\beta D_\gamma - gD_\gamma D_\beta &= x_\gamma D_\beta - x_\beta D_\gamma, \end{aligned}$$

where $g \neq 0$.

(2) *The case of A_{II} :*

$$\begin{aligned} g_{\alpha\beta} D_\alpha D_\beta &= x_\beta D_\alpha - x_\alpha D_\beta, \\ g_{\alpha\gamma} D_\alpha D_\gamma &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ g_{\beta\gamma} D_\beta D_\gamma &= x_\gamma D_\beta - x_\beta D_\gamma, \end{aligned}$$

where $g_{ij} := g_i - g_j$ with $g_i \neq g_j$ for all $i, j \in \{\alpha, \beta, \gamma\}$ with $i < j$.

(3) *The case of $B^{(1)}$:*

$$\begin{aligned} g_\beta D_\alpha D_\beta - (g_\beta - \Lambda) D_\beta D_\alpha &= -x_\alpha D_\beta, \\ g D_\alpha D_\gamma - (g - \Lambda) D_\gamma D_\alpha &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ g_\beta D_\beta D_\gamma - (g_\beta - \Lambda) D_\gamma D_\beta &= x_\gamma D_\beta, \end{aligned}$$

where $g, g_\beta \neq 0$.

(4) The case of $B^{(2)}$:

$$\begin{aligned} g_{\alpha\beta}D_\alpha D_\beta &= -x_\alpha D_\beta, \\ g_{\alpha\gamma}D_\alpha D_\gamma - g_{\gamma\alpha}D_\gamma D_\alpha &= x_\gamma D_\alpha - x_\alpha D_\gamma, \quad \text{and} \\ g_{\beta\gamma}D_\beta D_\gamma &= x_\gamma D_\beta, \end{aligned}$$

where $g_{\alpha\beta}, g_{\alpha\gamma}, g_{\beta\gamma} \neq 0$.

(5) The case of $B^{(3)}$:

$$\begin{aligned} gD_\alpha D_\beta - (g - \Lambda)D_\beta D_\alpha &= x_\beta D_\alpha - x_\alpha D_\beta, \\ g_\gamma D_\alpha D_\gamma &= -x_\alpha D_\gamma, \quad \text{and} \\ (g_\gamma - \Lambda)D_\beta D_\gamma &= -x_\beta D_\gamma, \end{aligned}$$

where $g \neq 0$ and $g_\gamma \neq 0, \Lambda$.

(6) The case of $B^{(4)}$:

$$\begin{aligned} (g_\alpha - \Lambda)D_\alpha D_\beta &= x_\beta D_\alpha, \\ g_\alpha D_\alpha D_\gamma &= x_\gamma D_\alpha, \quad \text{and} \\ gD_\beta D_\gamma - (g - \Lambda)D_\gamma D_\beta &= x_\gamma D_\beta - x_\beta D_\gamma, \end{aligned}$$

where $g \neq 0$ and $g_\alpha \neq 0, \Lambda$.

(7) The case of $C^{(1)}$:

$$\begin{aligned} g_\beta D_\alpha D_\beta - (g_\beta - \Lambda)D_\beta D_\alpha &= -x_\alpha D_\beta, \\ g_\gamma D_\alpha D_\gamma - (g_\gamma - \Lambda)D_\gamma D_\alpha &= -x_\alpha D_\gamma, \quad \text{and} \\ g_{\beta\gamma}D_\beta D_\gamma - g_{\gamma\beta}D_\gamma D_\beta &= 0, \end{aligned}$$

where $g_\beta, g_\gamma, g_{\beta,\gamma} \neq 0$.

(8) The case of $C^{(2)}$:

$$\begin{aligned} g_{\alpha\beta}D_\alpha D_\beta - g_{\beta\alpha}D_\beta D_\alpha &= -x_\alpha D_\beta, \\ g_{\alpha\gamma}D_\alpha D_\gamma - g_{\gamma\alpha}D_\gamma D_\alpha &= -x_\alpha D_\gamma, \quad \text{and} \\ D_\beta D_\gamma &= 0, \end{aligned}$$

where $g_{\alpha\beta}, g_{\alpha\gamma} \neq 0$.

(9) The case of D : With $q_{ji} := \frac{g_{ji}}{g_{ij}}$, where $i, j \in \{\alpha, \beta, \gamma\}$ (recall that $g_{ij} \neq 0$ for $i < j$), we have that

$$\begin{aligned} D_\alpha D_\beta - q_{\beta\alpha}D_\beta D_\alpha &= 0, \\ D_\alpha D_\gamma - q_{\gamma\alpha}D_\gamma D_\alpha &= 0, \quad \text{and} \\ D_\beta D_\gamma - q_{\gamma\beta}D_\gamma D_\beta &= 0. \end{aligned}$$

The division into algebras A , B , C , and D reflects the number of coefficients x_j , $j \in \{\alpha, \beta, \gamma\}$, that are zero in the expression (2.1). As shown above, for these algebras, none, one, two, or all three of the coefficients x_i vanish, respectively.

Remark 2.2. The previous list is given, following Pyatov–Twarock, for an ordered triple $D_\alpha, D_\beta, D_\gamma$ with $\alpha < \beta < \gamma$ and with relations written in the corresponding

PBW order. Thus, different items in the list should be regarded as presentations rather than canonical representatives up to isomorphism.

More precisely, consider the transformations on generators generated by

- (i) $D_i \mapsto \kappa_i D_i$ with $\kappa_i \in \mathbb{R} \setminus \{0\}$, and
- (ii) $D_i \mapsto D_{\sigma(i)}$ with $\sigma \in S_3$.

These operations preserve the relations and yield isomorphic algebras; in particular, they can be used to normalize the nonzero parameters x_i . Accordingly, within each family A, B, C, D , the displayed relations provide convenient orbit representatives for our later constructions, rather than an isomorphism-reduced list.

2.2. Diffusion algebras on N generators. Consider the following decomposition of the index set $I_N = \{1, \dots, N\}$:

$$I_N = I \cup R, \quad \text{with } I := \{\alpha \in I_N \mid x_\alpha \neq 0\} \text{ and } R := \{\alpha \in I_N \mid x_\alpha = 0\}.$$

Definition 2.3. (1) ([28, Definition 3.1]) The normal ordering of two generators D_α and D_β is defined as

$$(D_\alpha D_\beta) := \begin{cases} D_\alpha D_\beta & \text{if } \alpha < \beta, \\ D_\beta D_\alpha & \text{if } \beta < \alpha. \end{cases}$$

- (2) ([28, Definition 3.2]) For $\alpha < \beta$, consider the notation

$$[D_\alpha, D_\beta]_{q_{\beta\alpha}} := D_\alpha D_\beta - q_{\beta\alpha} D_\beta D_\alpha,$$

where the index at the commutator is referring to the coefficients $q_{\beta\alpha}$ in terms of which the commutator is defined.

Considering the notation in Definition 2.3, the set R is subdivided into nonintersecting and nonempty subsets

$$R := R_1 \cup R_2 \cup \dots \cup R_{M_R} \tag{2.4}$$

according to the following requirements:

- Relations between generators from the sets R_a and R_b for $a \neq b$ are given by

$$(D_{r_1} D_{r_2}) = 0 \quad \text{for all } r_1 \in R_a \text{ and } r_2 \in R_b. \tag{2.5}$$

- Relations within a set R_a such that $|R_a| \geq 2$ are given by

$$[D_{r_1}, D_{r_2}]_{q_{r_2 r_1}} = 0 \quad \text{for all } r_1, r_2 \in R_a \text{ with } r_1 < r_2, \tag{2.6}$$

where the coefficients in (2.6) are subject to the condition opposite to (2.5), that is: for any subdivision $R_a = R' \cup R''$ into two nonintersecting and nonempty parts R' and R'' ,

$$\text{there exist } r_1 \in R' \text{ and } r_2 \in R'' \text{ such that } g_{r_1 r_2} g_{r_2 r_1} \neq 0. \tag{2.7}$$

This means that for any pair of indices $r, s \in R_a$ there exists a finite sequence $\{r_k \in R_a \mid k = 1, \dots, n\}$ such that $r_1 = r$, $r_n = s$, and

$$\prod_{k=1}^{n-1} g_{r_k r_{k+1}} g_{r_{k+1} r_k} \neq 0. \tag{2.8}$$

Relations (2.7) and (2.8) may be represented graphically via a *connectivity condition* on an ordered graph whose vertices are labeled by the indices $r \in R_a$ and whose edges connect only those vertices $r_1 < r_2$ for which $q_{r_2 r_1} \neq 0$.

Furthermore, for $|I| \geq 2$ the set R is split into two sets S and T as follows: For any $R_a \subset R$, let

$$R_a := \begin{cases} S_a & \text{if there exist } r \in R_a \text{ and } i \in I \text{ with } g_{ir}g_{ri} \neq 0, \\ T_a & \text{otherwise.} \end{cases}$$

Suppose that the M_R sets R_a in (2.4) split into M_S sets S_a and M_T sets T_a . Then $M_R = M_S + M_T$. Number these sets as S_a for $a = 1, \dots, M_S$, and T_a for $a = 1, \dots, M_T$, and introduce

$$S := \bigcup_{a=1}^{M_S} S_a \quad \text{and} \quad T := \bigcup_{a=1}^{M_T} T_a.$$

The decomposition of the set S into subsets S_a has been used in the definition of the set S , and this will not be considered. However, the structure of the set T is crucial and needs further refinement.

For every $T_a \subset T$, let

$$T_a := \begin{cases} T_a^\bullet & \text{if } \exists i, j \in I \text{ with } i < j \text{ such that } T_a \subset \{i + 1, i + 2, \dots, j - 1\} \\ & \text{and } I \cap \{i + 1, i + 2, \dots, j - 1\} = \emptyset, \\ T_a^\circ, & \text{otherwise.} \end{cases}$$

In this way, we write

$$T = \{T_a^\bullet \mid a = 1, \dots, M_T^\bullet\} \cup \{T_a^\circ \mid a = 1, \dots, M_T^\circ\}, \quad \text{with } M_T = M_T^\bullet + M_T^\circ. \tag{2.9}$$

2.2.1. *List of diffusion algebras on N generators.* From now on, the expression “generators of a set I, S, T , or R ” means the generators indexed by elements from the corresponding set.

Definition 2.4 ([28, Definition 3.3]). A set of three generators $\{D_x, D_y, D_z\}$ with $x \in X, y \in Y$, and $z \in Z$, where X, Y , and Z are any of the sets I, R, S , and T , or any set in their decomposition, will be called a *triplet (of type) $\{X, Y, Z\}$* .

As can be seen, any triplet of type $\{I, I, I\}$ in a diffusion algebra of $N \geq 3$ generators gives rise to a diffusion algebra of type A_I or A_{II} ; any triple of type $\{I, I, R\}$ to a diffusion algebra of type $B^{(1)}, B^{(2)}, B^{(3)}$, or $B^{(4)}$; any triplet of type $\{I, R, R\}$ to a diffusion algebra of type $C^{(1)}$ or $C^{(2)}$; and any triplet of type $\{R, R, R\}$ to a diffusion algebra of type D .

Proposition 2.5 ([28, Lemma 3.4]). *For any diffusion algebra with $N \geq 3$ generators, the following statements hold:*

- (1) *If $|I| \geq 3$, then all subalgebras corresponding to triplets of type $\{I, I, I\}$ are of the same type, which is either A_I (that is, $g_{ij} = g$ for all $i, j \in I$) or A_{II} (that is, $g_{ji} = 0, g_{ij} = g_i - g_j, g_i \neq g_j$ for all $i < j$).*

- (2) If $|I| \geq 3$ and all subalgebras corresponding to triplets $\{I, I, I\}$ are of type A_I , then for any $s \in S$ and for all $i \in I$, we have that

$$g_{is} = g_{si} = g_s.$$

- (3) If $|I| \geq 3$ and all subalgebras corresponding to triplets $\{I, I, I\}$ are of type A_{II} , then $S = \emptyset$.
 (4) Let $|I| \geq 2$. For any $i \in I$ and every $t, t' \in T_a$ (T_a means both T_a° and T_a^\bullet) with $t < i$ and $t' > i$, the coefficients g_{ti} and $g_{it'}$ depend only on the index a of the set T_a and not on the individual indices t or t' . If $t, t' \in T_a^\circ$, one furthermore has $g_{ti} = -g_{it'}$.

For any $i < j$ and every $t, t' \in T_a$ such that $t < i$ and $t' > j$,

$$g_{ti} + \Lambda_{ij} = g_{tj} \quad \text{and} \quad g_{it'} = g_{jt'} + \Lambda_{ij}, \quad \text{where } \Lambda_{ij} := g_{ij} - g_{ji}.$$

- (5) Let $|I| = 1$. Denote the only index in I by \mathbf{i} , in order to stress that it is not a running index. For all r in R_a , one has

$$g_{ir} - g_{ri} = \Lambda_a. \tag{2.10}$$

Note that both the left- and right-hand sides of (2.10) depend only on the index a of the set R_a and not on the individual index r .

Diffusion algebras with N generators are listed as five families of algebras: A_I , A_{II} , B , C , and D . As in the case $N = 3$, the number of nonzero coefficients x_α , or equivalently, the cardinality of the set I , is used as a criterion for separating diffusion algebras into families of the types $A(|I| \geq 3)$, $B(|I| \geq 2)$, $C(|I| = 1)$, or $D(|I| = 0)$. Type A algebras are separated further into two families A_I and A_{II} , depending on the number of nonzero coefficients g_{ij} with indices i, j in the set I .

Next, we will see that different algebras in these families are obtained depending on the choice of the decomposition of the set $I_N = \{1, 2, \dots, N\}$ into ordered subsets I, S, T_a° ($a = 1, \dots, M_T^\circ$), T_b^\bullet ($b = 1, \dots, M_T^\bullet$) (or R_a , $a = 1, \dots, M_R$ for $N_I = 1$), as well as on the choice of coefficients in their defining relations. Next, we introduce notation for diffusion algebras in which the corresponding decomposition of the set I_N is given explicitly as an argument to the family symbol. The subscript indices a and b in our notation are treated as running indices, so that, e.g.

$$A_I(I, S, T_a^\circ, T_b^\bullet) \equiv A_I(I, S, T_a^\circ, \dots, T_{M_T^\circ}^\circ, T_b^\bullet, \dots, T_{M_T^\bullet}^\bullet),$$

where $I_N = I \cup S \cup \left(\bigcup_{a=1}^{M_T^\circ} T_a^\circ\right) \cup \left(\bigcup_{b=1}^{M_T^\bullet} T_b^\bullet\right)$, and I, S, T_a° , and T_b^\bullet are mutually nonintersecting ordered subsets in I_N . Notice that the values of the coefficients $g_{\alpha\beta}$ are not shown explicitly in this notation, so that it displays connective components in a variety of diffusion algebras rather than particular algebras.

Remark 2.6. For $N > 3$, the data in Proposition 2.5 may be viewed as a choice of an ordered decomposition of the index set I_N into subsets

$$I_N = I \cup S \cup \left(\bigcup_a T_a\right) \cup \left(\bigcup_a R_a\right),$$

together with parameters in the defining relations. Two such presentations define isomorphic diffusion algebras whenever they lie in the same orbit under the natural group of changes of generators generated by

- (i) $(\kappa_1, \dots, \kappa_N) \in (\mathbb{R} \setminus \{0\})^N$, $D_\alpha \mapsto \kappa_\alpha D_\alpha$, and
- (ii) $D_\alpha \mapsto D_{\sigma(\alpha)}$, where σ is a permutation that preserves the PBW ordering constraints and the admissible block structure.

This viewpoint explains why several coefficients in Proposition 2.5 depend only on the block label a rather than on the individual index: permuting indices inside a fixed block simply renames generators and hence yields an isomorphic algebra.

Next, relations in (2.11)–(2.15) below are to be complemented by relations (2.5), (2.6) for the elements of the subset R together with the conditions (2.7) or (2.8) on the coefficients involved.

Proposition 2.7 ([28, Theorem 3.5]). *The following list contains all possible diffusion algebras with N generators:*

- (1) *Diffusion algebras of type $A_I(I, S, T_a^\circ, T_b^\bullet)$ with $|I| \geq 3$:*

$$\begin{aligned}
 gD_iD_j - gD_jD_i &= x_jD_i - x_iD_j \quad \text{for all } i, j \in I, \\
 g_sD_sD_i - g_sD_iD_s &= x_iD_s \quad \text{for all } s \in S, i \in I, \\
 g_a^\circ(D_iD_t) &= -x_iD_t \quad \text{for all } a, t \in T_a^\circ, i \in I, \\
 g_b^+D_iD_t &= -x_iD_t \quad \text{for all } b, t \in T_b^\bullet \text{ and every } i < t, \\
 g_b^-D_tD_i &= x_iD_t \quad \text{for all } b, t \in T_b^\bullet \text{ and every } i > t,
 \end{aligned}
 \tag{2.11}$$

where $g, g_s, g_a^\circ, g_b^\pm \neq 0$.

- (2) *Diffusion algebras of type $A_{II}(I, T_a^\circ, T_b^\bullet)$ with $|I| \geq 3$:*

$$\begin{aligned}
 (g_i - g_j)D_iD_j &= x_jD_i - x_iD_j \quad \text{for all } i < j, \\
 (g_i + g_a^\circ)(D_iD_t) &= -x_iD_t \quad \text{for all } a, t \in T_a^\circ, i < t, \\
 (g_i + g_a^+)D_iD_t &= -x_iD_t \quad \text{for all } b, t \in T_b^\bullet, i < t, \\
 (g_b^- - g_i)(D_tD_i) &= x_iD_t \quad \text{for all } b, t \in T_b^\bullet, i > t,
 \end{aligned}
 \tag{2.12}$$

where $g_i \neq g_j$ for $i \neq j$ and $g_i \notin \{g_a^\circ, \pm g_b^\pm\}$.

- (3) *Diffusion algebras of type $B(I = \{\mathbf{i}, \mathbf{j}\}, S, T_a^\circ, T_b^\bullet)$: We use the notation \mathbf{i} and \mathbf{j} with $\mathbf{i} < \mathbf{j}$ for the two elements of the set I to emphasize that they are not running indices. Note that $\mathbf{i} < t < \mathbf{j}$ for all $t \in T_b^\bullet$ in this case:*

$$\begin{aligned}
 gD_iD_j - (g - \Lambda)D_jD_i &= x_jD_i - x_iD_j, \\
 g_sD_iD_s - (g_s - \Lambda)D_sD_i &= -x_iD_s \quad \text{for all } s \in S, \\
 g_sD_sD_j - (g_s - \Lambda)D_jD_s &= x_jD_s \quad \text{for all } s \in S, \\
 g_a^\circ(D_iD_t) &= -x_iD_t \quad \text{for all } t \in T_a^\circ, \\
 (g_a^\circ - \Lambda)(D_jD_t) &= -x_jD_t \quad \text{for all } t \in T_a^\circ, \\
 g_b^+D_iD_t &= -x_iD_t \quad \text{for all } t \in T_b^\bullet, \\
 g_b^-D_tD_j &= x_jD_t \quad \text{for all } t \in T_b^\bullet,
 \end{aligned}
 \tag{2.13}$$

where $g \neq 0$, $g_s \neq 0$ for all s , and $g_s \neq \Lambda$ for s such that either $s < \mathbf{i}$ or $s > \mathbf{j}$, $g_a^\circ \notin \{0, \Lambda\}$ and $g_b^\pm \neq 0$.

- (4) Diffusion algebras of type $C(I = \{\mathbf{i}\}, R_a)$: The only element of I is denoted by \mathbf{i} , and hence

$$g_r D_{\mathbf{i}} D_r - (g_r - \Lambda_a) D_r D_{\mathbf{i}} = -x_{\mathbf{i}} D_r \quad \text{for all } r \in R_a, \tag{2.14}$$

where $g_r \neq 0$ for $r < \mathbf{i}$, and $g_r \neq \Lambda_a$ for $r > \mathbf{i}$.

- (5) Diffusion algebras of type $D(R)$:

$$D_r D_s - q_{sr} D_s D_r = 0 \quad \text{for all } r, s \in R \text{ with } r < s. \tag{2.15}$$

Remark 2.8. (i) Hinchcliffe in his Ph.D. thesis [19, Definition 2.1.1] considered the following notation for diffusion algebras. Let R be the algebra generated by n indeterminates x_1, x_2, \dots, x_n over \mathbb{C} , subject to the relations

$$a_{ij} x_i x_j - b_{ij} x_j x_i = r_j x_i - r_i x_j$$

whenever $i < j$, for some parameters $a_{ij} \in \mathbb{C} \setminus 0$ and $b_{ij} \in \mathbb{C}$, for all $i < j$, and $r_i \in \mathbb{C}$, for all i . He defined the *standard monomials* to be those of the form $x_n^{i_n} x_{n-1}^{i_{n-1}} \cdots x_2^{i_2} x_1^{i_1}$. R is called a *diffusion algebra* if it admits a PBW basis of these standard monomials. In other words, R is a diffusion algebra if these standard monomials are a \mathbb{C} -vector space basis for R . If all the elements $q_{ij} := \frac{b_{ij}}{a_{ij}}$ are nonzero, then the diffusion algebras have a PBW basis in any order of the indeterminates [19, Remark 2.1.6]. From this definition, a diffusion algebra generated by n indeterminates has Gelfand–Kirillov dimension n , since because of the PBW basis, the vector subspace consisting of elements of total degree at most l is isomorphic to that of a commutative polynomial ring in n indeterminates (cf. [29, Theorems 4.14 and 4.18]). Notice that a diffusion algebra in one indeterminate is precisely a commutative polynomial ring in one indeterminate. A diffusion algebra with $x_t = 0$, for all $t = 1, \dots, n$, is a *multiparameter quantum affine n -space*.

(ii) Fajardo et al. [16, Section 2.4] studied ring-theoretical properties of a graded version of these algebras. The *diffusion algebras of type 2* are affine algebras \mathcal{D} generated by $2n$ variables $\{D_1, \dots, D_n, x_1, \dots, x_n\}$ over \mathbb{k} that admit a linear PBW basis of ordered monomials of the form $B_{\alpha_1}^{k_1} B_{\alpha_2}^{k_2} \cdots B_{\alpha_n}^{k_n}$, with $B_{\alpha_i} \in \{D_1, \dots, D_n, x_1, \dots, x_n\}$, for all $i \leq 2n$, $k_j \in \mathbb{N}$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, such that for all $1 \leq i < j \leq n$, there exist elements $\lambda_{ij} \in \mathbb{k}^*$ satisfying the relations

$$\lambda_{ij} D_i D_j - \lambda_{ji} D_j D_i = x_j D_i - x_i D_j. \tag{2.16}$$

Following Krebs and Sandow [25], the relations (2.16) are a consequence of subtracting (quadratic) operator relations of the type

$$\Gamma_{\gamma\delta}^{\alpha\beta} D_\alpha D_\beta = D_\gamma X_\delta - X_\gamma D_\delta \quad \text{for all } \gamma, \delta = 0, 1, \dots, n - 1,$$

where $\Gamma_{\gamma\delta}^{\alpha\beta} \in \mathbb{k}$ and D_i and X_j are operators on a particular vector space such that $[D_i, X_j] = 0$ does not necessarily hold [25, p. 3168].

3. DIFFERENTIAL SMOOTHNESS OF ALGEBRAS

We follow Brzeziński and Sitarz’s presentation on differential smoothness carried out in [12, Section 2] (cf. [5, 7]).

Definition 3.1 ([12, Section 2.1]). (i) A *differential graded algebra* is a non-negatively graded algebra Ω with the product denoted by \wedge , together with a degree-one linear map

$$d : \Omega^\bullet \rightarrow \Omega^{\bullet+1}$$

that satisfies the graded Leibniz rule and is such that $d \circ d = 0$.

(ii) A differential graded algebra (Ω, d) is a *calculus over an algebra* A if $\Omega^0 A = A$ and $\Omega^n A = A dA \wedge dA \wedge \dots \wedge dA$ (dA appears n times) for all $n \in \mathbb{N}$ (the latter is called the *density condition*). We write $(\Omega A, d)$ with

$$\Omega A = \bigoplus_{n \in \mathbb{N}} \Omega^n A.$$

By using the Leibniz rule, it follows that $\Omega^n A = dA \wedge dA \wedge \dots \wedge dA A$. A differential calculus ΩA is said to be *connected* if $\ker(d|_{\Omega^0 A}) = \mathbb{k}$.

(iii) A calculus $(\Omega A, d)$ is said to have *dimension* n if $\Omega^n A \neq 0$ and $\Omega^m A = 0$ for all $m > n$. An n -dimensional calculus ΩA admits a *volume form* if $\Omega^n A$ is isomorphic to A as a left and right A -module.

The existence of a right A -module isomorphism means that there is a free generator, say ω , of $\Omega^n A$ (as a right A -module), i.e., $\omega \in \Omega^n A$, such that all elements of $\Omega^n A$ can be uniquely expressed as ωa with $a \in A$. If ω is also a free generator of $\Omega^n A$ as a left A -module, this is said to be a *volume form* on ΩA .

The right A -module isomorphism $\Omega^n A \rightarrow A$ corresponding to a volume form ω is denoted by π_ω , i.e.,

$$\pi_\omega(\omega a) = a \quad \text{for all } a \in A. \tag{3.1}$$

By using that $\Omega^n A$ is also isomorphic to A as a left A -module, any free generator ω induces an algebra endomorphism ν_ω of A by the formula

$$a\omega = \omega\nu_\omega(a). \tag{3.2}$$

Note that if ω is a volume form, then ν_ω is an algebra automorphism.

Now, we proceed to recall the key ingredients of the integral calculus on A as dual to its differential calculus. For more details, see Brzeziński et al. [5, 10].

Let $(\Omega A, d)$ be a differential calculus on A . The space of n -forms $\Omega^n A$ is an A -bimodule. Consider $\mathcal{I}_n A$ the right dual of $\Omega^n A$, the space of all right A -linear maps $\Omega^n A \rightarrow A$, that is, $\mathcal{I}_n A := \text{Hom}_A(\Omega^n(A), A)$. Notice that each of the $\mathcal{I}_n A$ is an A -bimodule with the actions

$$(a \cdot \phi \cdot b)(\omega) = a\phi(b\omega) \quad \text{for all } \phi \in \mathcal{I}_n A, \omega \in \Omega^n A, \text{ and } a, b \in A.$$

The direct sum of all the $\mathcal{I}_n A$, that is, $\mathcal{I} A = \bigoplus_n \mathcal{I}_n A$, is a right ΩA -module with action given by

$$(\phi \cdot \omega)(\omega') = \phi(\omega \wedge \omega') \quad \text{for all } \phi \in \mathcal{I}_{n+m} A, \omega \in \Omega^n A, \text{ and } \omega' \in \Omega^m A. \tag{3.3}$$

Definition 3.2 ([5, Definition 2.1]). A *divergence* (also called *hom-connection*) on A is a linear map $\nabla : \mathcal{I}_1 A \rightarrow A$ such that

$$\nabla(\phi \cdot a) = \nabla(\phi)a + \phi(da) \quad \text{for all } \phi \in \mathcal{I}_1 A \text{ and } a \in A. \tag{3.4}$$

If A is an algebra of functions on the Euclidean space \mathbb{R}^n and $\Omega^1(A)$ is the standard module of one-forms, then we obtain the classical divergence from elementary vector calculus [6, p. 892].

Note that a divergence can be extended to the whole of $\mathcal{I}A$,

$$\nabla_n : \mathcal{I}_{n+1} A \rightarrow \mathcal{I}_n A,$$

by considering

$$\nabla_n(\phi)(\omega) = \nabla(\phi \cdot \omega) + (-1)^{n+1} \phi(d\omega) \quad \text{for all } \phi \in \mathcal{I}_{n+1}(A) \text{ and } \omega \in \Omega^n A. \tag{3.5}$$

By putting together (3.4) and (3.5), we get the Leibniz rule

$$\nabla_n(\phi \cdot \omega) = \nabla_{m+n}(\phi) \cdot \omega + (-1)^{m+n} \phi \cdot d\omega$$

for all elements $\phi \in \mathcal{I}_{m+n+1} A$ and $\omega \in \Omega^m A$ [5, Lemma 3.2]. In the case $n = 0$, if $\text{Hom}_A(A, M)$ is canonically identified with M , then ∇_0 reduces to the classical Leibniz rule.

Definition 3.3 ([5, Definition 3.4]). The right A -module map

$$F = \nabla_0 \circ \nabla_1 : \text{Hom}_A(\Omega^2 A, M) \rightarrow M$$

is called a *curvature* of a hom-connection (M, ∇_0) . (M, ∇_0) is said to be *flat* if its curvature is the zero map, that is, if $\nabla \circ \nabla_1 = 0$. This condition implies that $\nabla_n \circ \nabla_{n+1} = 0$ for all $n \in \mathbb{N}$.

$\mathcal{I}A$, together with ∇_n , form a chain complex called the *complex of integral forms* over A . The cokernel map of ∇ , that is,

$$\Lambda : A \rightarrow \text{Coker } \nabla = A / \text{Im } \nabla$$

is said to be the *integral on A associated to $\mathcal{I}A$* .

Given a left A -module X with action $a \cdot x$, for all $a \in A, x \in X$, and an algebra automorphism ν of A , the notation ${}^\nu X$ stands for X with the A -module structure twisted by ν , i.e., with the A -action $a \otimes x \mapsto \nu(a) \cdot x$.

The following definition of an *integrable differential calculus* seeks to portray a version of Hodge star isomorphisms between the complex of differential forms of a differentiable manifold and a complex of dual modules of it [8, p. 112].

Definition 3.4 ([12, Definition 2.1]). An n -dimensional differential calculus $(\Omega A, d)$ is said to be *integrable* if $(\Omega A, d)$ admits a complex of integral forms $(\mathcal{I}A, \nabla)$ for which there exist an algebra automorphism ν of A and A -bimodule isomorphisms

$$\Theta_k : \Omega^k A \rightarrow {}^\nu \mathcal{I}_{n-k} A, \quad k = 0, \dots, n,$$

rendering the following diagram commutative:

$$\begin{array}{ccccccccccc}
 A & \xrightarrow{d} & \Omega^1 A & \xrightarrow{d} & \Omega^2 A & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-1} A & \xrightarrow{d} & \Omega^n A \\
 \downarrow \Theta_0 & & \downarrow \Theta_1 & & \downarrow \Theta_2 & & & & \downarrow \Theta_{n-1} & & \downarrow \Theta_n \\
 {}^\nu \mathcal{I}_n A & \xrightarrow{\nabla_{n-1}} & {}^\nu \mathcal{I}_{n-1} A & \xrightarrow{\nabla_{n-2}} & {}^\nu \mathcal{I}_{n-2} A & \xrightarrow{\nabla_{n-3}} & \dots & \xrightarrow{\nabla_1} & {}^\nu \mathcal{I}_1 A & \xrightarrow{\nabla} & {}^\nu A
 \end{array}$$

The n -form $\omega := \Theta_n^{-1}(1) \in \Omega^n A$ is called an *integrating volume form*.

The algebra of complex matrices $M_n(\mathbb{C})$ with the n -dimensional calculus generated by derivations presented by Dubois-Violette et al. [14, 15], the quantum group $SU_q(2)$ with the three-dimensional left-covariant calculus developed by Woronowicz [43], and the quantum standard sphere with the restriction of the above calculus are examples of algebras admitting integrable calculi. For more details on the subject, see Brzeziński et al. [10].

The following proposition shows that the integrability of a differential calculus can be defined without explicit reference to integral forms. This allows us to guarantee the integrability by considering the existence of finitely many generating elements that allow us to determine left and right components of any homogeneous element of $\Omega(A)$.

Proposition 3.5 ([12, Theorem 2.2]). *Let $(\Omega A, d)$ be an n -dimensional differential calculus over an algebra A . The following assertions are equivalent:*

- (1) $(\Omega A, d)$ is an integrable differential calculus.
- (2) There exist an algebra automorphism ν of A and A -bimodule isomorphisms

$$\Theta_k : \Omega^k A \rightarrow {}^\nu \mathcal{I}_{n-k} A, \quad k = 0, \dots, n,$$

such that, for all $\omega' \in \Omega^k A$ and $\omega'' \in \Omega^m A$,

$$\Theta_{k+m}(\omega' \wedge \omega'') = (-1)^{(n-1)m} \Theta_k(\omega') \cdot \omega''.$$

- (3) There exist an algebra automorphism ν of A and an A -bimodule map $\vartheta : \Omega^n A \rightarrow {}^\nu A$ such that all left multiplication maps

$$\begin{aligned}
 \ell_{\vartheta}^k : \Omega^k A &\rightarrow \mathcal{I}_{n-k} A, \\
 \omega' &\mapsto \vartheta \cdot \omega', \quad k = 0, 1, \dots, n,
 \end{aligned}$$

where the actions \cdot are defined by (3.3), are bijective.

- (4) $(\Omega A, d)$ has a volume form ω such that all left multiplication maps

$$\begin{aligned}
 \ell_{\pi_\omega}^k : \Omega^k A &\rightarrow \mathcal{I}_{n-k} A, \\
 \omega' &\mapsto \pi_\omega \cdot \omega', \quad k = 0, 1, \dots, n-1,
 \end{aligned}$$

where π_ω is defined by (3.1), are bijective.

A volume form $\omega \in \Omega^n A$ is an *integrating form* if and only if it satisfies condition (4) of Proposition 3.5 [12, Remark 2.3].

The most interesting cases of differential calculi are those where $\Omega^k A$ are finitely generated and projective right or left (or both) A -modules [6].

Proposition 3.6. *The following statements hold:*

- (1) [12, Lemma 2.6] *Consider $(\Omega A, d)$ an integrable and n -dimensional calculus over A with integrating form ω . Then $\Omega^k A$ is a finitely generated projective right A -module if there exist a finite number of forms $\omega_i \in \Omega^k A$ and $\bar{\omega}_i \in \Omega^{n-k} A$ such that, for all $\omega' \in \Omega^k A$, we have*

$$\omega' = \sum_i \omega_i \pi_\omega(\bar{\omega}_i \wedge \omega').$$

- (2) [12, Lemma 2.7] *Let $(\Omega A, d)$ be an n -dimensional calculus over A admitting a volume form ω . Assume that for all $k = 1, \dots, n - 1$, there exists a finite number of forms $\omega_i^k, \bar{\omega}_i^k \in \Omega^k(A)$ such that for all $\omega' \in \Omega^k A$, we have that*

$$\omega' = \sum_i \omega_i^k \pi_\omega(\bar{\omega}_i^{n-k} \wedge \omega') = \sum_i \nu_\omega^{-1}(\pi_\omega(\omega' \wedge \omega_i^{n-k})) \bar{\omega}_i^k,$$

where π_ω and ν_ω are defined by (3.1) and (3.2), respectively. Then ω is an integral form and all the $\Omega^k A$ are finitely generated and projective as left and right A -modules.

Brzeziński and Sitarz [12, p. 421] point out that, in order to relate the dimension of an integrable calculus $(\Omega A, d)$ with the “size” of the underlying affine algebra A , a suitable notion of dimension is needed: its Gelfand–Kirillov dimension $\text{GKdim}(A)$. For more details about this dimension, see [24]. In the setting of diffusion algebras on N generators, we get that $\text{GKdim}(D) = N$ (see Remark 2.8).

After the preliminaries above, we arrive at the key notion of this paper.

Definition 3.7 ([12, Definition 2.4]). An affine algebra A with integer Gelfand–Kirillov dimension n is said to be *differentially smooth* if it admits an n -dimensional connected integrable differential calculus $(\Omega A, d)$.

From Definition 3.7 it follows that a differentially smooth algebra comes equipped with a well-behaved differential structure and with the precise concept of integration [11, p. 2414].

As we said in the Introduction, different authors have characterized the differential smoothness of several noncommutative algebras [8, 9, 10, 11, 12, 15, 21, 22, 31, 39, 35].

Remark 3.8. There are examples of algebras that are not differentially smooth. Consider the commutative algebra $A = \mathbb{C}[x, y]/\langle xy \rangle$. A proof by contradiction shows that for this algebra there are no one-dimensional connected integrable calculi over A , so it cannot be differentially smooth [12, Example 2.5].

4. DIFFERENTIAL SMOOTHNESS OF DIFFUSION ALGEBRAS

Throughout this section, \mathcal{D} denotes a diffusion algebra on $N \geq 3$ generators.

4.1. Diffusion algebras on four generators. The explicit four-generator description is given in Table 1. It is worth noting that several of these algebras appear to be isomorphic upon some index change. This task is beyond the scope of this paper, as for now we are only interested in describing possible combinations of algebras.

4.2. Diffusion algebras on five generators. As in Subsection 4.1, all possible diffusion algebras on five generators are shown. These can be found in Table 2.

4.3. Differential smoothness of diffusion algebras on N generators. Our aim in this section is to find an integral calculus of degree N that guarantees the differentiable smoothness of diffusion algebras.

Theorem 4.1. *Let \mathcal{D} be a diffusion algebra on $N \geq 3$ generators. If \mathcal{D} satisfies any one of the following properties:*

- (i) $|I| = L$ and $|S| = N - L$, for $3 \leq L \leq N$;
- (ii) $|I| = 1$, $|S| = N - 1$, and $g_s = \mathcal{G} \notin \{0, \Lambda_a\}$ for all $s \in S$;
- (iii) $|I| = 2$, $|S| = N - 2$, and $g_s = \mathcal{G} \notin \{0, \Lambda\}$ for all $s \in S$; or
- (iv) $|I| = 0$,

then \mathcal{D} is differentially smooth.

Proof. In view of Definition 3.7, to prove that D is differentially smooth it suffices to exhibit an N -dimensional connected integrable differential calculus $(\Omega D, d)$ over D . We now construct an explicit calculus adapted to the chosen set of generators of D .

To this end, we choose $\Omega^1(D)$ to be a free right D -module of rank N with basis $\{dD_a\}_{a=1}^N$, and we endow it with a left D -module structure,

$$p dD_a = dD_a \nu_{D_a}(p), \quad a \in I \cup S, p \in D,$$

where ν_{D_a} are algebra automorphisms of D .

(i) Consider the maps given by

$$\begin{aligned} \nu_{D_i}(D_j) &= D_j - g^{-1}x_j, \quad \text{for } i, j \in I, \\ \nu_{D_i}(D_s) &= D_s, \quad \text{for } i \in I, s \in S, \\ \nu_{D_s}(D_i) &= D_i - g_s^{-1}x_i, \quad \text{for } i \in I, s \in S, \quad \text{and} \\ \nu_{D_s}(D_l) &= D_l, \quad \text{for } s, l \in S. \end{aligned}$$

It can be seen that, for each $i \in I$ and $s \in S$, the maps ν_{D_i} and ν_{D_s} can be extended to algebra homomorphisms of \mathcal{D} . As a matter of fact, these maps respect the relations (2.11) for \mathcal{D} and commute with each other.

Consider $\Omega^1(\mathcal{D})$ a free right \mathcal{D} -module of rank N with generators dD_i, dD_s for all $i \in I$ and $s \in S$. For every element $p \in \mathcal{D}$, define a left \mathcal{D} -module structure by

$$p dD_a = dD_a \nu_{D_a}(p), \quad \text{with } a \in I \cup S. \quad (4.1)$$

The relations in $\Omega^1(\mathcal{D})$ are given by

$$\begin{aligned} D_i dD_j &= dD_j (D_i - g^{-1}x_i), \quad i, j \in I, \\ D_i dD_s &= dD_s (D_i - g_s^{-1}x_i), \quad i \in I, s \in S, \end{aligned}$$

$$\begin{aligned} D_s dD_i &= dD_i D_s, \quad i \in I, s \in S, \text{ and} \\ D_s dD_l &= dD_l D_s, \quad s, l \in S. \end{aligned} \tag{4.2}$$

We want to extend the correspondences

$$D_a \mapsto dD_a \quad \text{for every } a \in I \cup S$$

to a map $d : \mathcal{D} \rightarrow \Omega^1(\mathcal{D})$ satisfying the Leibniz rule. This is possible if it is compatible with the nontrivial relations (2.11), i.e., if the equalities

$$gdD_i D_j + gD_i dD_j - gdD_j D_i - gD_j dD_i = x_j dD_i - x_i dD_j \quad \text{for } i, j \in I,$$

and

$$g_s dD_s D_i + g_s D_s dD_i - g_s dD_i D_s - g_s D_i dD_s = x_i dD_s \quad \text{for } i \in I, s \in S,$$

hold.

Define \mathbb{k} -linear maps

$$\partial_{D_a} : \mathcal{D} \rightarrow \mathcal{D} \quad \text{for every } a \in I \cup S,$$

such that

$$d(p) = \sum_{a \in I \cup S} dD_a \partial_{D_a}(p) \quad \text{for all } p \in \mathcal{D}.$$

Since dD_a with $a \in I \cup S$ are free generators of the right \mathcal{D} -module $\Omega^1(\mathcal{D})$, these maps are well defined. Note that $d(p) = 0$ if and only if $\partial_{D_a}(p) = 0$ for each $a \in I \cup S$. By using the relations in (4.2) and the definitions of the maps ν_{D_a} with $a \in I \cup S$, we get that

$$\partial_{D_a}(D_1^{k_1} \cdots D_N^{k_N}) = k_a (D_1 - g_a^{-1} x_1)^{k_1} \cdots (D_{a-1} - g_a^{-1} x_{a-1})^{k_{a-1}} D_a^{k_a-1} \cdots D_N^{k_N},$$

where $g_a^{-1} = g^{-1}$ if $a \in I$, and $g_a^{-1} = g_s^{-1}$ if $r = s \in S$. Thus, $d(p) = 0$ if and only if p is a scalar multiple of the identity. This shows that $(\Omega\mathcal{D}, d)$ is connected with $\Omega(\mathcal{D}) = \bigoplus_{k=0}^{n-1} \Omega^k(\mathcal{D})$.

The universal extension of d to higher forms compatible with (4.1) gives the following rules for $\Omega^k(\mathcal{D})$ ($2 \leq k \leq N - 1$):

$$\bigwedge_{r=1}^k dD_{q(r)} = (-1)^\sharp \bigwedge_{r=1}^k dD_{p(r)},$$

where $q : \{1, \dots, k\} \rightarrow \{1, \dots, N\}$ is an injective map, $p : \{1, \dots, k\} \rightarrow \text{Im}(q)$ is an increasing injective map, and \sharp is the number of 2-permutations needed to transform q into p .

By using the fact that, for every $i \in I$ and $s \in S$, the automorphisms ν_{D_i} and ν_{D_s} commute with each other, there are no additional relations beyond the previous ones. In this way,

$$\begin{aligned} \Omega^{N-1}(\mathcal{D}) &= [dD_1 \wedge dD_2 \wedge \cdots \wedge dD_{N-1} \oplus dD_1 \wedge dD_3 \wedge \cdots \wedge dD_N \\ &\quad \oplus \cdots \oplus dD_2 \wedge \cdots \wedge dD_N] \mathcal{D}. \end{aligned}$$

Since $\Omega^N(\mathcal{D}) = \omega\mathcal{D} \cong \mathcal{D}$ as a right and left \mathcal{D} -module with

$$\omega = dD_1 \wedge \cdots \wedge dD_N \quad \text{and} \quad \nu_\omega = \nu_{D_1} \circ \cdots \circ \nu_{D_N},$$

we have that ω is a volume form of \mathcal{D} . From Proposition 3.6 (2) we obtain that ω is an integral form by setting

$$\begin{aligned} \omega_i^j &= \bigwedge_{k=1}^j dD_{p_{i,j}(k)} \quad \text{for } 1 \leq i \leq \binom{N}{j}, \quad \text{and} \\ \bar{\omega}_i^{N-j} &= (-1)^{\sharp_i} \bigwedge_{k=j+1}^n dD_{\bar{p}_{i,j}(k)} \quad \text{for } 1 \leq i \leq \binom{N}{j}, \end{aligned}$$

where

$$p_{i,j} : \{1, \dots, j\} \rightarrow \{1, \dots, N\} \quad \text{and} \quad \bar{p}_{i,j} : \{j+1, \dots, N\} \rightarrow (\text{Im}(p_{i,j}))^c$$

are increasing injective maps and $\sharp_{i,j}$ is the number of 2-permutations needed to transform $\{\bar{p}_{i,j}(j+1), \dots, \bar{p}_{i,j}(N), p_{i,j}(1), \dots, p_{i,j}(j)\}$ into $\{1, \dots, N\}$.

Let $\omega' \in \Omega^j(\mathcal{D})$. Then

$$\omega' = \sum_{i=1}^{\binom{N}{j}} \bigwedge_{k=1}^j dD_{p_{i,j}(k)} a_i, \quad \text{with } a_i \in \mathbb{k}.$$

This implies that we have the equalities given by

$$\begin{aligned} &\sum_{i=1}^{\binom{N}{j}} \omega_i^j \pi_\omega(\bar{\omega}_i^{N-j} \wedge \omega') \\ &= \sum_{i=1}^{\binom{N}{j}} \bigwedge_{k=1}^j dD_{p_{i,j}(k)} \pi_\omega \left(a_i (-1)^{\sharp_{i,j}} \bigwedge_{k=j+1}^N dD_{\bar{p}_{i,j}(k)} \wedge \bigwedge_{k=1}^j dD_{p_{i,j}(k)} \right) \\ &= \sum_{i=1}^{\binom{N}{j}} \bigwedge_{k=1}^j dx_{p_{i,j}(k)} a_i \\ &= \omega'. \end{aligned}$$

By Proposition 3.6 (2) it follows that \mathcal{D} is differentially smooth.

(ii) Consider the maps given by

$$\begin{aligned} \nu_{D_i}(D_i) &= D_i, \\ \nu_{D_i}(D_s) &= \mathcal{G}(\mathcal{G} - \Lambda_a)^{-1} D_s, \\ \nu_{D_s}(D_i) &= \mathcal{G}^{-1}((\mathcal{G} - \Lambda_a)D_i - x_i), \\ \nu_{D_s}(D_l) &= D_l \quad \text{for } s, l \in S. \end{aligned}$$

It is straightforward to show that the map ν_{D_i} and, for each $s \in S$, the map ν_{D_s} can be extended to algebra homomorphisms of \mathcal{D} and respect the relations (2.14) for \mathcal{D} . Again, these maps commute with each other.

Consider $\Omega^1(\mathcal{D})$ a free right \mathcal{D} -module of rank N with generators dD_i for every dD_s and $s \in S$. For all $p \in \mathcal{D}$ define a left \mathcal{D} -module structure by

$$pdD_a = dD_a \nu_{D_a}(p) \quad \text{for every } a \in \{\mathbf{i}\} \cup S. \tag{4.3}$$

The relations in $\Omega^1(\mathcal{D})$ are given by

$$\begin{aligned} D_i dD_s &= dD_s \mathcal{G}^{-1}((\mathcal{G} - \Lambda_a)D_i - x_i), \quad s \in S, \\ D_s dD_i &= dD_i \mathcal{G}(\mathcal{G} - \Lambda_a)^{-1}D_s, \quad s \in S, \text{ and} \\ D_s dD_l &= dD_l D_s, \quad s, l \in S. \end{aligned} \tag{4.4}$$

We want to extend the correspondences

$$D_a \mapsto dD_a, \quad \text{with } a \in \{\mathbf{i}\} \cup S$$

to a map $d : \mathcal{D} \rightarrow \Omega^1(\mathcal{D})$ satisfying the Leibniz rule. By using the relations (2.14), the equality

$$\mathcal{G}dD_i D_s + \mathcal{G}D_i dD_s - (\mathcal{G} - \Lambda_a)dD_s D_i - (\mathcal{G} - \Lambda_a)D_s dD_i = -x_i dD_s \quad \text{for all } s \in S$$

must be satisfied.

Define \mathbb{k} -linear maps

$$\partial_{D_a} : \mathcal{D} \rightarrow \mathcal{D} \quad \text{for each } a \in \{\mathbf{i}\} \cup S$$

such that

$$d(p) = \sum_{a \in \{\mathbf{i}\} \cup S} dD_a \partial_{D_a}(p) \quad \text{for all } p \in \mathcal{D}.$$

Since $dD_a, a \in \{\mathbf{i}\} \cup S$, are free generators of the right \mathcal{D} -module $\Omega^1(\mathcal{D})$, these maps are well defined, and $d(p) = 0$ if and only if $\partial_{D_a}(p) = 0, a \in \{\mathbf{i}\} \cup S$. By using the relations in (4.4) and the definitions of the maps $\nu_{D_a}, a \in \{\mathbf{i}\} \cup S$, we obtain

$$\partial_{D_i}(D_1^{k_1} \dots D_i^{k_i} \dots D_N^{k_N}) = \prod_{j=1}^{i-1} \mathcal{G}^{k_j} (\mathcal{G} - \Lambda_a)^{-k_j} k_i D_1^{k_1} \dots D_i^{k_i-1} \dots D_N^{k_N},$$

$$\partial_{D_s}(D_1^{k_1} \dots D_i^{k_i} \dots D_N^{k_N}) = k_s D_1^{k_1} \dots D_s^{k_s-1} \dots D_N^{k_N} \quad \text{for } s < \mathbf{i},$$

$$\partial_{D_s}(D_1^{k_1} \dots D_i^{k_i} \dots D_N^{k_N}) = k_s \mathcal{G}^{-k_i} D_1^{k_1} \dots ((\mathcal{G} - \Lambda_a)D_i - x_i)^{k_i} \dots D_s^{k_s-1} \dots D_N^{k_N} \quad \text{for } s > \mathbf{i},$$

Thus $d(p) = 0$ if and only if p is a scalar multiple of the identity, whence it follows that $(\Omega\mathcal{D}, d)$ is connected, where $\Omega(\mathcal{D}) = \bigoplus_{k=0}^{n-1} \Omega^k(\mathcal{D})$.

From this treatment, the rest of the proof is completely analogous to case (i). Thus, \mathcal{D} is differentially smooth.

(iii) For all elements $s, l \in S$, consider the following maps:

$$\begin{aligned} \nu_{D_i}(D_i) &= g^{-1}((g - \Lambda)D_i - x_i), \\ \nu_{D_i}(D_j) &= (g - \Lambda)^{-1}(gD_j - x_j), \\ \nu_{D_i}(D_s) &= (\mathcal{G} - \Lambda)^{-1}\mathcal{G}D_s, \\ \nu_{D_j}(D_i) &= g^{-1}((g - \Lambda)D_i - x_i), \\ \nu_{D_j}(D_j) &= g^{-1}((g - \Lambda)D_j - x_j), \\ \nu_{D_j}(D_s) &= (\mathcal{G} - \Lambda)^{-1}\mathcal{G}D_s, \\ \nu_{D_s}(D_i) &= \mathcal{G}^{-1}((\mathcal{G} - \Lambda)D_i - x_i), \end{aligned}$$

$$\begin{aligned} \nu_{D_s}(D_j) &= (\mathcal{G} - \Lambda)^{-1}(\mathcal{G}D_j - x_j), \quad \text{and} \\ \nu_{D_s}(D_l) &= D_l. \end{aligned}$$

Notice that these morphisms are similar to the corresponding ones in case (ii), so by using similar reasoning we can prove that all of them guarantee that \mathcal{D} is differentially smooth.

(iv) In this case, the algebra \mathcal{D} is precisely the *quantum affine N -space*, whence its differential smoothness follows by [22, Corollary 6 and Theorem 9] or [11, Corollary 4.9]. \square

Following an argument similar to the one presented by Brzeziński and Sitarz [12, Example 2.5], we obtain the following result.

Theorem 4.2. *If $|T| \neq 0$, then there are no one-dimensional connected integrable calculi on the diffusion algebra \mathcal{D} .*

Proof. Consider the sets T° and T^\bullet as the only connective components in the decomposition in expression (2.9) of the subset $T \in I_N$.

(i) Suppose that $|T^\circ| \geq 1$. We can write the relations relevant to T° in the unique form given by

$$\mathcal{G}(D_i D_t) = -x_i D_t, \quad i \in I, t \in T_a^\circ, \tag{4.5}$$

where $\mathcal{G} \in \{g_a^\circ, g_i + g_a^\circ, g_a^\circ - \Lambda\}$ depending on the case in Proposition 2.7.

By applying the differential d to the expression (4.5) we get that

$$\mathcal{G}d((D_i D_t)) = d(-x_i D_t), \quad i \in I, t \in T_a^\circ.$$

If $(D_i D_t) = D_i D_t$, using that d is \mathbb{k} -linear the Leibniz rule yields

$$\mathcal{G}dD_i D_t + \mathcal{G}D_i dD_t + x_i dD_t = 0, \quad i \in I, t \in T_a^\circ.$$

Since $\Omega^1(D)$ is a D -bimodule, and we work with a calculus in which $\Omega^1(D)$ is free as a right D -module with generators $\{dD_1, \dots, dD_N\}$, for each t there is an algebra automorphism $\nu_{D_t} : D \rightarrow D$ encoding the left action on the generator dD_t , namely

$$p dD_t = dD_t \nu_{D_t}(p) \quad \forall p \in D.$$

We emphasize that ν_{D_t} comes from the bimodule structure of $\Omega^1(D)$ and should not be confused with the automorphism ν_ω in (3.2), which is defined only after choosing a volume form $\omega \in \Omega^N(D)$. Then

$$\mathcal{G}dD_i D_t + \mathcal{G}dD_t \nu_{D_t}(D_i) + x_i dD_t = 0, \quad i \in I, t \in T_a^\circ.$$

In this case, we have

$$\mathcal{G}D_t = 0, \quad t \in T_a^\circ,$$

which occurs only if $\mathcal{G} = 0$. This contradicts all the cases in Proposition 2.7.

On the other hand, if $(D_i D_t) = D_t D_i$, by an argument similar to the previous one we obtain

$$\mathcal{G}\nu_{D_i}(D_t) = 0, \quad i \in I, t \in T_a^\circ.$$

If $\mathcal{G} = 0$, this again contradicts all the cases in Proposition 2.7. On the other hand, if $\nu_{D_i}(D_t) = 0$, this contradicts the fact that ν_{D_i} is an automorphism.

(ii) Suppose that $|T^\bullet| \geq 1$. We can write the relations relevant to T^\bullet in the following unique form:

$$\mathcal{G}(D_i D_t) = \operatorname{sgn}(i - t) x_i D_t, \quad i \in I, t \in T_b^\bullet,$$

where $\mathcal{G} \in \{g_b^+, g_b^-, g_i + g_b^+, g_b^- - g_i\}$, depending on the case in Proposition 2.7. The proof is exactly the same as in the expression (4.5); we only need to change the sign of $i - t$. \square

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TABLE 1. Diffusion algebras on four generators

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_1 - gD_1D_2 = x_2D_1 - x_1D_2, \quad gD_3D_1 - gD_1D_3 = x_3D_1 - x_1D_3,$ $gD_3D_2 - gD_2D_3 = x_3D_2 - x_2D_3, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad g_4D_4D_3 - g_4D_3D_4 = x_1D_4$	$I = \{1, 2, 3\}, S = \{4\}$	$g, g_4 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_3D_1 - x_1D_3, \quad gD_4D_1 - gD_1D_4 = x_4D_1 - x_1D_4,$ $gD_4D_3 - gD_3D_4 = x_4D_3 - x_3D_4, \quad g_2D_2D_1 - g_2D_1D_2 = x_1D_2,$ $g_2D_3D_2 - g_2D_2D_3 = x_3D_2, \quad g_2D_4D_2 - g_2D_2D_4 = x_4D_2$	$I = \{1, 3, 4\}, S = \{2\}$	$g, g_2 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_2D_1 - x_1D_2, \quad gD_4D_1 - gD_1D_4 = x_4D_1 - x_1D_4,$ $gD_4D_3 - gD_3D_4 = x_4D_3 - x_3D_4, \quad g_3D_3D_1 - g_3D_1D_3 = x_1D_3,$ $g_3D_3D_2 - g_3D_2D_3 = x_2D_3, \quad g_3D_3D_4 - g_3D_4D_3 = x_4D_3$	$I = \{1, 2, 4\}, S = \{3\}$	$g, g_3 \neq 0$
	$gD_3D_2 - gD_2D_3 = x_3D_2 - x_2D_3, \quad gD_4D_2 - gD_2D_4 = x_4D_2 - x_2D_4,$ $gD_4D_3 - gD_3D_4 = x_4D_3 - x_3D_4, \quad g_1D_2D_1 - g_1D_1D_2 = x_2D_1,$ $g_1D_3D_1 - g_1D_1D_3 = x_3D_1, \quad g_1D_4D_1 - g_1D_1D_4 = x_4D_1$	$I = \{2, 3, 4\}, S = \{1\}$	$g, g_1 \neq 0$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, \quad (g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3,$ $(g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4, \quad (g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3,$ $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4, \quad (g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4$	$I = \{1, 2, 3, 4\}$	$g_i \neq g_j$ for $i, j \in I, i \neq j$
A_{II}	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, \quad (g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3,$ $(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, \quad (g_1 + g_1^c)D_1D_4 = -x_1D_4,$ $(g_2 + g_2^c)D_2D_4 = -x_2D_4, \quad (g_3 + g_3^c)D_3D_4 = -x_3D_4$	$I = \{1, 2, 3\}, T_a^c = \{4\}$	$g_i \neq g_j$ for $i, j \in I, i \neq j, g_i \neq g_i^c$
	$(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, \quad (g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4,$ $(g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4, \quad (g_2 + g_2^c)D_1D_2 = -x_2D_1,$ $(g_3 + g_3^c)D_1D_3 = -x_3D_1, \quad (g_4 + g_4^c)D_1D_4 = -x_4D_1$	$I = \{2, 3, 4\}, T_a^c = \{1\}$	$g_i \neq g_j$ for $i, j \in I, i \neq j, g_i \neq g_i^c$
	$(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3, \quad (g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4,$ $(g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4, \quad (g_1 + g_1^c)D_1D_2 = -x_1D_2,$ $(g_3^c - g_4)D_3D_4 = x_3D_4, \quad (g_4^c - g_4)D_4D_3 = x_4D_3$	$I = \{1, 3, 4\}, T_a^c = \{2\}$	$g_i \neq g_j$ for $i, j \in I, i \neq j, g_i \neq g_i^c$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, \quad (g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4,$ $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4, \quad (g_1 + g_1^c)D_1D_3 = -x_1D_3,$ $(g_2 + g_2^c)D_2D_3 = -x_2D_3, \quad (g_3^c - g_4)D_3D_4 = x_4D_3$	$I = \{1, 2, 4\}, T_a^c = \{3\}$	$g_i \neq g_j$ for $i, j \in I, i \neq j, g_i \neq g_i^c$
	$gD_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4$	$I = \{1, 2\}, S = \{3, 4\}$	$g \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
B	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4$	$I = \{1, 3\}, S = \{2, 4\}$	$g \neq 0, g_2 \neq 0, g_4 \notin \{0, \Lambda\}$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2, \quad g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3,$ $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2$	$I = \{1, 4\}, S = \{2, 3\}$	$g \neq 0, g_s \neq 0, s \in S$
	$gD_2D_3 - (g - \Lambda)D_3D_2 = x_3D_2 - x_2D_3, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4$	$I = \{2, 3\}, S = \{1, 4\}$	$g \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3, \quad g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1,$ $g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3$	$I = \{2, 4\}, S = \{1, 3\}$	$g \neq 0, g_3 \neq 0, g_1 \notin \{0, \Lambda\}$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2, \quad g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1,$ $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2$	$I = \{3, 4\}, S = \{1, 2\}$	$g \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$gD_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3, \quad g_4^cD_1D_4 = -x_1D_4,$ $(g_4^c - \Lambda)D_2D_4 = -x_2D_4,$	$I = \{1, 2\}, S = \{3\}, T_a^c = \{4\}$	$g \neq 0, g_3, g_4^c \notin \{0, \Lambda\}$
	$gD_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4, \quad g_4^cD_1D_3 = -x_1D_3,$ $(g_4^c - \Lambda)D_2D_3 = -x_2D_3,$	$I = \{1, 2\}, S = \{4\}, T_a^c = \{3\}$	$g \neq 0, g_4, g_4^c \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2, \quad g_4^cD_1D_4 = -x_1D_4,$ $(g_4^c - \Lambda)D_3D_4 = -x_3D_4,$	$I = \{1, 3\}, S = \{2\}, T_a^c = \{4\}$	$g \neq 0, g_2 \neq 0, g_4^c \notin \{0, \Lambda\}$
	$gD_2D_3 - (g - \Lambda)D_3D_2 = x_3D_2 - x_2D_3, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_1D_2,$ $g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1, \quad g_4^cD_2D_4 = -x_2D_4,$ $(g_4^c - \Lambda)D_3D_4 = -x_3D_4,$	$I = \{2, 3\}, S = \{1\}, T_a^c = \{4\}$	$g \neq 0, g_1, g_4^c \notin \{0, \Lambda\}$

TABLE 1. Diffusion algebras on four generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions	
B	$gD_2D_3 - (g - \Lambda)D_3D_2 = x_3D_2 - x_2D_3, \quad g_4D_2D_4 - (g_4 - \Lambda)D_1D_2 = -x_2D_4,$ $g_4D_3D_4 - (g_4 - \Lambda)D_1D_3 = -x_3D_4, \quad g_2^{\circ}D_1D_2 = -x_2D_1,$ $(g_2^{\circ} - \Lambda)D_1D_3 = -x_3D_1,$	$I = \{2, 3\}, S = \{4\}, T_a^{\circ} = \{1\}$	$g \neq 0, g_4, g_4^{\circ} \notin \{0, \Lambda\}$	
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3, \quad g_2^{\circ}D_1D_2 = -x_2D_1,$ $(g_2^{\circ} - \Lambda)D_1D_4 = -x_4D_1,$	$I = \{2, 4\}, S = \{3\}, T_a^{\circ} = \{1\}$	$g \neq 0, g_3 \neq 0, g_2^{\circ} \notin \{0, \Lambda\}$	
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1, \quad g_2^{\circ}D_2D_3 = -x_3D_2,$ $(g_2^{\circ} - \Lambda)D_2D_4 = -x_4D_2,$	$I = \{3, 4\}, S = \{1\}, T_a^{\circ} = \{2\}$	$g \neq 0, g_1, g_1^{\circ} \notin \{0, \Lambda\}$	
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2, \quad g_1^{\circ}D_1D_3 = -x_3D_1,$ $(g_1^{\circ} - \Lambda)D_1D_4 = -x_4D_1,$	$I = \{3, 4\}, S = \{2\}, T_a^{\circ} = \{1\}$	$g \neq 0, g_2, g_2^{\circ} \notin \{0, \Lambda\}$	
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4, \quad g_1^{\circ}D_1D_2 = -x_1D_2,$ $g_6^-D_2D_3 = x_2D_3,$	$I = \{1, 3\}, S = \{4\}, T_b^{\circ} = \{2\}$	$g \neq 0, g_4 \notin \{0, \Lambda\}, g_6^{\pm} \neq 0$	
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_4D_2 - (g_2 - \Lambda)D_2D_4 = -x_4D_2, \quad g_1^{\circ}D_1D_3 = -x_1D_3,$ $g_6^-D_3D_4 = x_4D_3,$	$I = \{1, 4\}, S = \{2\}, T_b^{\circ} = \{3\}$	$g \neq 0, g_2 \neq 0, g_6^{\pm} \neq 0$	
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_4D_3 - (g_3 - \Lambda)D_3D_4 = -x_4D_3, \quad g_1^{\circ}D_1D_2 = -x_1D_2,$ $g_6^-D_2D_4 = x_4D_2,$	$I = \{1, 4\}, S = \{3\}, T_b^{\circ} = \{2\}$	$g \neq 0, g_3 \neq 0, g_6^{\pm} \neq 0$	
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_4D_1 - (g_1 - \Lambda)D_1D_4 = -x_4D_1, \quad g_2^{\circ}D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3,$	$I = \{2, 4\}, S = \{1\}, T_b^{\circ} = \{3\}$	$g \neq 0, g_1 \notin \{0, \Lambda\}, g_6^{\pm} \neq 0$	
	$gD_1D_2 - (g - \Lambda)D_2D_1 = x_1D_2 - x_2D_1, \quad g_2^{\circ}D_1D_3 = -x_1D_3,$ $g_4^{\circ}D_1D_4 = -x_1D_4, \quad (g_2^{\circ} - \Lambda)D_2D_3 = -x_2D_3,$ $(g_2^{\circ} - \Lambda)D_2D_4 = -x_2D_4,$	$I = \{1, 2\}, T_a^{\circ} = \{3, 4\}$	$g \neq 0, g_2^{\circ} \notin \{0, \Lambda\}$	
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_1D_3 - x_3D_1, \quad g_2^{\circ}D_1D_4 = -x_1D_4,$ $(g_2^{\circ} - \Lambda)D_3D_4 = -x_3D_4, \quad g_1^{\circ}D_1D_2 = -x_1D_2,$ $g_6^-D_2D_3 = x_3D_2,$	$I = \{1, 3\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{4\}$	$g \neq 0, g_2^{\circ} \notin \{0, \Lambda\}, g_6^{\pm} \neq 0$	
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_1D_4 - x_4D_1, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_4 = x_4D_2, \quad g_6^+D_1D_3 = -x_1D_3,$ $g_6^-D_3D_4 = x_4D_3,$	$I = \{1, 4\}, T_b^{\circ} = \{2, 3\}$	$g \neq 0, g_6^{\pm} \neq 0$	
	$gD_2D_3 - (g - \Lambda)D_3D_2 = x_2D_3 - x_3D_2, \quad g_2^{\circ}D_1D_2 = -x_1D_2,$ $g_2^{\circ}D_1D_3 = -x_1D_3, \quad (g_2^{\circ} - \Lambda)D_1D_3 = -x_3D_1,$ $(g_2^{\circ} - \Lambda)D_3D_4 = -x_3D_4,$	$I = \{2, 3\}, T_a^{\circ} = \{1, 4\}$	$g \neq 0, g_2^{\circ} \notin \{0, \Lambda\}$	
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_2D_4 - x_4D_2, \quad g_2^{\circ}D_1D_2 = -x_2D_1,$ $(g_2^{\circ} - \Lambda)D_1D_4 = -x_4D_1, \quad g_1^{\circ}D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3,$	$I = \{2, 4\}, T_a^{\circ} = \{1\}, T_b^{\circ} = \{3\}$	$g \neq 0, g_2^{\circ} \notin \{0, \Lambda\}, g_6^{\pm} \neq 0$	
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_3D_4 - x_4D_3, \quad g_2^{\circ}D_1D_3 = -x_3D_1,$ $g_2^{\circ}D_2D_3 = -x_3D_2, \quad (g_2^{\circ} - \Lambda)D_1D_4 = -x_4D_1,$ $(g_2^{\circ} - \Lambda)D_2D_4 = -x_4D_2,$	$I = \{3, 4\}, T_a^{\circ} = \{1, 2\}$	$g \neq 0, g_2^{\circ} \notin \{0, \Lambda\}$	
	C	$g_2D_1D_2 - (g_2 - \Lambda_a)D_2D_1 = -x_1D_2, \quad g_3D_1D_3 - (g_3 - \Lambda_a)D_3D_1 = -x_1D_3,$ $g_4D_1D_4 - (g_4 - \Lambda_a)D_4D_1 = -x_1D_4,$	$I = \{1\}, R_a = \{2, 3, 4\}$	$g_2, g_3, g_4 \neq \Lambda_a$
		$g_1D_2D_1 - (g_1 - \Lambda_a)D_1D_2 = -x_2D_1, \quad g_3D_2D_3 - (g_3 - \Lambda_a)D_3D_2 = -x_2D_3,$ $g_4D_2D_4 - (g_4 - \Lambda_a)D_4D_2 = -x_2D_4,$	$I = \{2\}, R_a = \{1, 3, 4\}$	$g_1 \neq 0, g_3, g_4 \neq \Lambda_a$
$g_1D_3D_1 - (g_1 - \Lambda_a)D_1D_3 = -x_3D_1, \quad g_2D_3D_2 - (g_2 - \Lambda_a)D_2D_3 = -x_3D_2,$ $g_4D_3D_4 - (g_4 - \Lambda_a)D_4D_3 = -x_3D_4,$		$I = \{3\}, R_a = \{1, 2, 4\}$	$g_1, g_2 \neq 0, g_4 \neq \Lambda_a$	
$g_1D_4D_1 - (g_1 - \Lambda_a)D_1D_4 = -x_4D_1, \quad g_2D_4D_2 - (g_2 - \Lambda_a)D_2D_4 = -x_4D_2,$ $g_3D_4D_3 - (g_3 - \Lambda_a)D_3D_4 = -x_4D_3,$		$I = \{4\}, R_a = \{1, 2, 3\}$	$g_1, g_2, g_3 \neq 0$	
D	$D_1D_2 - q_{21}D_2D_1 = 0, \quad D_1D_3 - q_{31}D_3D_1 = 0,$ $D_1D_4 - q_{41}D_4D_1 = 0, \quad D_2D_3 - q_{32}D_3D_2 = 0,$ $D_2D_4 - q_{42}D_4D_2 = 0, \quad D_3D_4 - q_{43}D_4D_3 = 0$	$R = \{1, 2, 3, 4\}$	$q_{21}, q_{31}, q_{41}, q_{32}, q_{42}, q_{43} \neq 0$	

TABLE 2. Diffusion algebras on five generators

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1,$ $gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3,$ $g_5D_5D_3 - g_5D_3D_5 = x_3D_5, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5$	$I = \{1, 2, 3, 4\}, S = \{5\}$	$g, g_5 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1,$ $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3,$ $g_4D_4D_3 - g_4D_3D_4 = x_3D_4, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4$	$I = \{1, 2, 3, 5\}, S = \{4\}$	$g, g_4 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1, \quad g_3D_3D_1 - g_3D_1D_3 = x_1D_3,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2,$ $g_3D_3D_2 - g_3D_2D_3 = x_2D_3, \quad gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4,$ $g_3D_3D_4 - g_3D_4D_3 = x_4D_3, \quad g_3D_3D_5 - g_3D_5D_3 = x_5D_3$	$I = \{1, 2, 4, 5\}, S = \{3\}$	$g, g_3 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1, \quad g_2D_2D_1 - g_2D_1D_2 = x_1D_2,$ $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3, \quad gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3,$ $g_2D_2D_3 - g_2D_3D_2 = x_3D_2, \quad gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4,$ $g_2D_2D_4 - g_2D_4D_2 = x_4D_2, \quad g_2D_2D_5 - g_2D_5D_2 = x_5D_2$	$I = \{1, 3, 4, 5\}, S = \{2\}$	$g, g_2 \neq 0$
	$gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2,$ $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2, \quad g_1D_1D_2 - g_1D_2D_1 = x_2D_1,$ $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3, \quad gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3,$ $g_1D_1D_3 - g_1D_3D_1 = x_3D_1, \quad gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4,$ $g_1D_1D_4 - g_1D_4D_1 = x_4D_1, \quad g_1D_1D_5 - g_1D_5D_1 = x_5D_1$	$I = \{2, 3, 4, 5\}, S = \{1\}$	$g, g_1 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad g_4D_4D_3 - g_4D_3D_4 = x_3D_4,$ $g_a^{\circ}D_5D_1 = -x_1D_5, \quad g_a^{\circ}D_5D_2 = -x_2D_5,$ $g_a^{\circ}D_5D_3 = -x_3D_5$	$I = \{1, 2, 3\}, S = \{4\}, T_a^{\circ} = \{5\}$	$g, g_4, g_a^{\circ} \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad g_5D_5D_3 - g_5D_3D_5 = x_3D_5,$ $g_a^{\circ}D_4D_1 = -x_1D_4, \quad g_a^{\circ}D_4D_2 = -x_2D_4,$ $g_a^{\circ}D_4D_3 = -x_3D_4$	$I = \{1, 2, 3\}, S = \{5\}, T_a^{\circ} = \{4\}$	$g, g_5, g_a^{\circ} \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad g_5D_5D_3 - g_5D_3D_5 = x_3D_5,$ $g_4D_4D_1 - g_4D_1D_4 = x_1D_4, \quad g_4D_4D_2 - g_4D_2D_4 = x_2D_4,$ $g_4D_4D_3 - g_4D_3D_4 = x_3D_4$	$I = \{1, 2, 3\}, S = \{4, 5\}$	$g, g_4, g_5 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad g_3D_3D_1 - g_3D_1D_3 = x_1D_3,$ $g_3D_3D_2 - g_3D_2D_3 = x_2D_3, \quad g_3D_3D_4 - g_3D_4D_3 = x_4D_3,$ $g_a^{\circ}D_5D_1 = -x_1D_5, \quad g_a^{\circ}D_5D_2 = -x_2D_5,$ $g_a^{\circ}D_5D_4 = -x_4D_5$	$I = \{1, 2, 4\}, S = \{3\}, T_a^{\circ} = \{5\}$	$g, g_3, g_a^{\circ} \neq 0$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5,$ $g_6^+D_1D_3 = -x_1D_3, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3$	$I = \{1, 2, 4\}, S = \{5\}, T_6^* = \{3\}$	$g, g_5, g_6^\pm \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5,$ $g_3D_3D_1 - g_3D_1D_3 = x_1D_3, \quad g_3D_3D_2 - g_3D_2D_3 = x_2D_3,$ $g_3D_3D_4 - g_3D_4D_3 = x_4D_3$	$I = \{1, 2, 4\}, S = \{3, 5\}$	$g, g_3, g_5 \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2, \quad g_3D_3D_1 - g_3D_1D_3 = x_1D_3,$ $g_3D_3D_2 - g_3D_2D_3 = x_2D_3, \quad g_3D_3D_5 - g_3D_5D_3 = x_5D_3,$ $g_6^+D_1D_4 = -x_1D_4, \quad g_6^+D_2D_4 = -x_2D_4,$ $g_6^-D_4D_5 = x_5D_4$	$I = \{1, 2, 5\}, S = \{3\}, T_6^* = \{4\}$	$g, g_3, g_6^\pm \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4,$ $g_6^+D_1D_3 = -x_1D_3, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_5 = x_5D_3$	$I = \{1, 2, 5\}, S = \{4\}, T_6^* = \{3\}$	$g, g_4, g_6^\pm \neq 0$
	$gD_2D_1 - gD_1D_2 = x_1D_2 - x_2D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_2 - gD_2D_5 = x_2D_5 - x_5D_2, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4,$ $g_3D_3D_1 - g_3D_1D_3 = x_1D_3, \quad g_3D_3D_2 - g_3D_2D_3 = x_2D_3,$ $g_3D_3D_5 - g_3D_5D_3 = x_5D_3$	$I = \{1, 2, 5\}, S = \{3, 4\}$	$g, g_3, g_4 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3, \quad g_2D_2D_1 - g_2D_1D_2 = x_1D_2,$ $g_2D_2D_3 - g_2D_3D_2 = x_3D_2, \quad g_2D_2D_4 - g_2D_4D_2 = x_4D_2,$ $g_a^cD_5D_1 = -x_1D_5, \quad g_a^cD_5D_3 = -x_3D_5,$ $g_a^cD_5D_4 = -x_4D_5$	$I = \{1, 3, 4\}, S = \{2\}, T_a^c = \{5\}$	$g, g_2, g_a^c \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $g_5D_5D_3 - g_5D_3D_5 = x_3D_5, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5,$ $g_6^+D_1D_2 = -x_1D_2, \quad g_6^-D_2D_3 = x_3D_2,$ $g_6^-D_2D_4 = x_4D_2$	$I = \{1, 3, 4\}, S = \{5\}, T_6^* = \{2\}$	$g, g_5, g_6^\pm \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1,$ $gD_4D_3 - gD_3D_4 = x_3D_4 - x_4D_3, \quad g_5D_5D_1 - g_5D_1D_5 = x_1D_5,$ $g_5D_5D_3 - g_5D_3D_5 = x_3D_5, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5,$ $g_2D_2D_1 - g_2D_1D_2 = x_1D_2, \quad g_2D_2D_3 - g_2D_3D_2 = x_3D_2,$ $g_2D_2D_4 - g_2D_4D_2 = x_4D_2$	$I = \{1, 3, 4\}, S = \{2, 5\}$	$g, g_2, g_5 \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3, \quad g_2D_2D_1 - g_2D_1D_2 = x_1D_2,$ $g_2D_2D_3 - g_2D_3D_2 = x_3D_2, \quad g_2D_2D_5 - g_2D_5D_2 = x_5D_2,$ $g_6^+D_1D_4 = -x_1D_4, \quad g_6^+D_3D_4 = -x_3D_4,$ $g_6^-D_4D_5 = x_5D_4$	$I = \{1, 3, 5\}, S = \{2\}, T_6^* = \{4\}$	$g, g_2, g_6^\pm \neq 0$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $g_4D_4D_3 - g_4D_3D_4 = x_3D_4, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4,$ $g_6^+D_1D_2 = -x_1D_2, \quad g_6^-D_2D_3 = x_3D_2,$ $g_6^-D_2D_5 = x_5D_2$	$I = \{1, 3, 5\}, S = \{4\}, T_6^\bullet = \{2\}$	$g, g_4, g_6^\pm \neq 0$
	$gD_3D_1 - gD_1D_3 = x_1D_3 - x_3D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_3 - gD_3D_5 = x_3D_5 - x_5D_3, \quad g_4D_4D_1 - g_4D_1D_4 = x_1D_4,$ $g_4D_4D_3 - g_4D_3D_4 = x_3D_4, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4,$ $g_2D_2D_1 - g_2D_1D_2 = x_1D_2, \quad g_2D_2D_3 - g_2D_3D_2 = x_3D_2,$ $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$	$I = \{1, 3, 5\}, S = \{2, 4\}$	$g, g_2, g_4 \neq 0$
	$gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4, \quad g_2D_2D_1 - g_2D_1D_2 = x_1D_2,$ $g_2D_2D_4 - g_2D_4D_2 = x_4D_2, \quad g_2D_2D_5 - g_2D_5D_2 = x_5D_2,$ $g_6^+D_1D_3 = -x_1D_3, \quad g_6^-D_3D_4 = x_4D_3,$ $g_6^-D_3D_5 = x_5D_3$	$I = \{1, 4, 5\}, S = \{2\}, T_6^\bullet = \{3\}$	$g, g_2, g_6^\pm \neq 0$
	$gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4, \quad g_3D_3D_1 - g_3D_1D_3 = x_1D_3,$ $g_3D_3D_4 - g_3D_4D_3 = x_4D_3, \quad g_3D_3D_5 - g_3D_5D_3 = x_5D_3,$ $g_6^+D_1D_2 = -x_1D_2, \quad g_6^-D_2D_4 = x_4D_2,$ $g_6^-D_2D_5 = x_5D_2$	$I = \{1, 4, 5\}, S = \{3\}, T_6^\bullet = \{2\}$	$g, g_3, g_6^\pm \neq 0$
	$gD_4D_1 - gD_1D_4 = x_1D_4 - x_4D_1, \quad gD_5D_1 - gD_1D_5 = x_1D_5 - x_5D_1,$ $gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4, \quad g_3D_3D_1 - g_3D_1D_3 = x_1D_3,$ $g_3D_3D_4 - g_3D_4D_3 = x_4D_3, \quad g_3D_3D_5 - g_3D_5D_3 = x_5D_3,$ $g_2D_2D_1 - g_2D_1D_2 = x_1D_2, \quad g_2D_2D_4 - g_2D_4D_2 = x_4D_2,$ $g_2D_2D_5 - g_2D_5D_2 = x_5D_2$	$I = \{1, 4, 5\}, S = \{2, 3\}$	$g, g_2, g_3 \neq 0$
	$gD_2D_4 - gD_4D_2 = x_4D_2 - x_2D_4, \quad gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_1D_1D_4 - g_1D_4D_1 = x_4D_1,$ $g_1D_1D_2 - g_1D_2D_1 = x_2D_1, \quad g_1D_1D_3 - g_1D_3D_1 = x_3D_1,$ $g_a^\circ D_4D_5 = -x_4D_5, \quad g_a^\circ D_2D_5 = -x_2D_5,$ $g_a^\circ D_3D_5 = -x_3D_5$	$I = \{2, 3, 4\}, S = \{1\}, T_a^\circ = \{5\}$	$g, g_1, g_a^\circ \neq 0$
	$gD_2D_4 - gD_4D_2 = x_4D_2 - x_2D_4, \quad gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad g_5D_5D_3 - g_5D_3D_5 = x_3D_5,$ $g_a^\circ D_1D_4 = -x_4D_1, \quad g_a^\circ D_1D_2 = -x_2D_1,$ $g_a^\circ D_1D_3 = -x_3D_1$	$I = \{2, 3, 4\}, S = \{5\}, T_a^\circ = \{1\}$	$g, g_5, g_a^\circ \neq 0$
	$gD_2D_4 - gD_4D_2 = x_4D_2 - x_2D_4, \quad gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_5D_5D_4 - g_5D_4D_5 = x_4D_5,$ $g_5D_5D_2 - g_5D_2D_5 = x_2D_5, \quad g_5D_5D_3 - g_5D_3D_5 = x_3D_5,$ $g_1D_1D_4 - g_1D_4D_1 = x_4D_1, \quad g_1D_1D_2 - g_1D_2D_1 = x_2D_1,$ $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$	$I = \{2, 3, 4\}, S = \{1, 5\}$	$g, g_1, g_5 \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5, \quad gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_1D_1D_5 - g_1D_5D_1 = x_5D_1,$ $g_1D_1D_2 - g_1D_2D_1 = x_2D_1, \quad g_1D_1D_3 - g_1D_3D_1 = x_3D_1,$ $g_6^+D_2D_4 = -x_2D_4, \quad g_6^+D_3D_4 = -x_3D_4,$ $g_6^-D_4D_5 = x_5D_4$	$I = \{2, 3, 5\}, S = \{1\}, T_6^\bullet = \{4\}$	$g, g_1, g_6^\pm \neq 0$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
A_I	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5, \quad gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad g_4D_4D_3 - g_4D_3D_4 = x_3D_4,$ $g_6^+D_1D_5 = -x_5D_1, \quad g_6^+D_1D_2 = -x_2D_1,$ $g_6^+D_1D_3 = -x_3D_1$	$I = \{2, 3, 5\}, S = \{4\}, T_a^\circ = \{1\}$	$g, g_4, g_6^+ \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5, \quad gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5,$ $gD_3D_2 - gD_2D_3 = x_2D_3 - x_3D_2, \quad g_4D_4D_5 - g_4D_5D_4 = x_5D_4,$ $g_4D_4D_2 - g_4D_2D_4 = x_2D_4, \quad g_4D_4D_3 - g_4D_3D_4 = x_3D_4,$ $g_1D_1D_5 - g_1D_5D_1 = x_5D_1, \quad g_1D_1D_2 - g_1D_2D_1 = x_2D_1,$ $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$	$I = \{2, 3, 5\}, S = \{1, 4\}$	$g, g_1, g_5 \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5, \quad gD_4D_5 - gD_5D_4 = x_5D_4 - x_4D_5,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad g_1D_1D_5 - g_1D_5D_1 = x_5D_1,$ $g_1D_1D_2 - g_1D_2D_1 = x_2D_1, \quad g_1D_1D_4 - g_1D_4D_1 = x_4D_1,$ $g_6^+D_2D_3 = -x_2D_3, \quad g_6^-D_3D_4 = x_4D_3,$ $g_6^-D_3D_5 = x_5D_3$	$I = \{2, 4, 5\}, S = \{1\}, T_b^\bullet = \{3\}$	$g, g_1, g_6^\pm \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5, \quad gD_4D_5 - gD_5D_4 = x_5D_4 - x_4D_5,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad g_3D_3D_5 - g_3D_5D_3 = x_5D_3,$ $g_3D_3D_2 - g_3D_2D_3 = x_2D_3, \quad g_3D_3D_4 - g_3D_4D_3 = x_4D_3,$ $g_6^+D_1D_5 = -x_5D_1, \quad g_6^+D_1D_2 = -x_2D_1,$ $g_6^+D_1D_4 = -x_4D_1$	$I = \{2, 4, 5\}, S = \{3\}, T_a^\circ = \{1\}$	$g, g_5, g_6^+ \neq 0$
	$gD_2D_5 - gD_5D_2 = x_5D_2 - x_2D_5, \quad gD_4D_5 - gD_5D_4 = x_5D_4 - x_4D_5,$ $gD_4D_2 - gD_2D_4 = x_2D_4 - x_4D_2, \quad g_3D_3D_5 - g_3D_5D_3 = x_5D_3,$ $g_3D_3D_2 - g_3D_2D_3 = x_2D_3, \quad g_3D_3D_4 - g_3D_4D_3 = x_4D_3,$ $g_1D_1D_5 - g_1D_5D_1 = x_5D_1, \quad g_1D_1D_2 - g_1D_2D_1 = x_2D_1,$ $g_1D_1D_4 - g_1D_4D_1 = x_4D_1$	$I = \{2, 4, 5\}, S = \{1, 3\}$	$g, g_1, g_5 \neq 0$
	$gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4, \quad gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4,$ $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5, \quad g_1D_1D_4 - g_1D_4D_1 = x_4D_1,$ $g_1D_1D_5 - g_1D_5D_1 = x_5D_1, \quad g_1D_1D_3 - g_1D_3D_1 = x_3D_1,$ $g_6^+D_2D_4 = -x_4D_2, \quad g_6^+D_2D_5 = -x_5D_2,$ $g_6^+D_2D_3 = -x_3D_2$	$I = \{3, 4, 5\}, S = \{1\}, T_a^\circ = \{2\}$	$g, g_1, g_6^+ \neq 0$
	$gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4, \quad gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4,$ $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5, \quad g_2D_2D_4 - g_2D_4D_2 = x_4D_2,$ $g_2D_2D_5 - g_2D_5D_2 = x_5D_2, \quad g_2D_2D_3 - g_2D_3D_2 = x_3D_2,$ $g_6^+D_1D_4 = -x_4D_1, \quad g_6^+D_1D_5 = -x_5D_1,$ $g_6^+D_1D_3 = -x_3D_1$	$I = \{3, 4, 5\}, S = \{2\}, T_a^\circ = \{1\}$	$g, g_2, g_6^+ \neq 0$
	$gD_5D_4 - gD_4D_5 = x_4D_5 - x_5D_4, \quad gD_3D_4 - gD_4D_3 = x_4D_3 - x_3D_4,$ $gD_3D_5 - gD_5D_3 = x_5D_3 - x_3D_5, \quad g_2D_2D_4 - g_2D_4D_2 = x_4D_2,$ $g_2D_2D_5 - g_2D_5D_2 = x_5D_2, \quad g_2D_2D_3 - g_2D_3D_2 = x_3D_2,$ $g_1D_1D_4 - g_1D_4D_1 = x_4D_1, \quad g_1D_1D_5 - g_1D_5D_1 = x_5D_1,$ $g_1D_1D_3 - g_1D_3D_1 = x_3D_1$	$I = \{3, 4, 5\}, S = \{1, 2\}$	$g, g_1, g_2 \neq 0$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
A_{II}	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3,$ $(g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4, (g_1 - g_5)D_1D_5 = x_5D_1 - x_1D_5,$ $(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, (g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4,$ $(g_2 - g_5)D_2D_5 = x_5D_2 - x_2D_5, (g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4,$ $(g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5, (g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5$	$I = \{1, 2, 3, 4, 5\}$	$g_i \neq g_j$ for $i, j \in I, i \neq j$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3,$ $(g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4, (g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3,$ $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4, (g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4,$ $(g_1 + g_6^+)D_1D_5 = -x_1D_5, (g_2 + g_6^+)D_2D_5 = -x_2D_5,$ $(g_3 + g_6^+)D_3D_5 = -x_3D_5, (g_4 + g_6^+)D_4D_5 = -x_4D_5$	$I = \{1, 2, 3, 4\}, T_a^{\circ} = \{5\}$	$g_i \neq g_j, i \neq j, g_i \neq g_6^+,$ for $i, j \in I$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3,$ $(g_1 - g_5)D_1D_5 = x_5D_1 - x_1D_5, (g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3,$ $(g_2 - g_5)D_2D_5 = x_5D_2 - x_2D_5, (g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5,$ $(g_1 + g_6^+)D_1D_4 = -x_1D_4, (g_2 + g_6^+)D_2D_4 = -x_2D_4,$ $(g_3 + g_6^+)D_3D_4 = -x_3D_4, (g_6^- - g_5)D_4D_5 = x_5D_4$	$I = \{1, 2, 3, 5\}, T_b^{\bullet} = \{4\}$	$g_i \neq g_j, i \neq j, g_i \notin \{\mp g_6^{\pm}\},$ for $i, j \in I$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4,$ $(g_1 - g_5)D_1D_5 = x_5D_1 - x_1D_5, (g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4,$ $(g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5, (g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5,$ $(g_1 + g_6^+)D_1D_3 = -x_1D_3, (g_2 + g_6^+)D_2D_3 = -x_2D_3,$ $(g_6^- - g_4)D_3D_4 = x_4D_3, (g_6^- - g_5)D_3D_5 = x_5D_3$	$I = \{1, 2, 4, 5\}, T_b^{\bullet} = \{3\}$	$g_i \neq g_j, i \neq j, g_i \notin \{\mp g_6^{\pm}\},$ for $i, j \in I$
	$(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3, (g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4,$ $(g_1 - g_5)D_1D_5 = x_5D_1 - x_1D_5, (g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4,$ $(g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5, (g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5,$ $(g_1 + g_6^+)D_1D_2 = -x_1D_2, (g_6^- - g_3)D_2D_3 = x_3D_2,$ $(g_6^- - g_4)D_2D_4 = x_4D_2, (g_6^- - g_5)D_2D_5 = x_5D_2$	$I = \{1, 3, 4, 5\}, T_a^{\bullet} = \{2\}$	$g_i \neq g_j, i \neq j, g_i \notin \{\mp g_6^{\pm}\},$ for $i, j \in I$
	$(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, (g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4,$ $(g_2 - g_5)D_2D_5 = x_5D_2 - x_2D_5, (g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4,$ $(g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5, (g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5,$ $(g_2 + g_6^+)D_1D_2 = -x_2D_1, (g_3 + g_6^+)D_1D_3 = -x_3D_1,$ $(g_4 + g_6^+)D_1D_4 = -x_4D_1, (g_5 + g_6^+)D_1D_5 = -x_5D_1$	$I = \{2, 3, 4, 5\}, T_a^{\circ} = \{1\}$	$g_i \neq g_j, i \neq j, g_i \neq g_6^+,$ for $i, j \in I$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3,$ $(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, (g_1 + g_6^+)D_1D_4 = -x_1D_4,$ $(g_2 + g_6^+)D_2D_4 = -x_2D_4, (g_3 + g_6^+)D_3D_4 = -x_3D_4,$ $(g_1 + g_6^+)D_1D_5 = -x_1D_5, (g_2 + g_6^+)D_2D_5 = -x_2D_5,$ $(g_3 + g_6^+)D_3D_5 = -x_3D_5$	$I = \{1, 2, 3\}, T_a^{\circ} = \{4, 5\}$	$g_i \neq g_j, i \neq j, g_i \neq g_6^+,$ for $i, j \in I$
	$(g_4 - g_2)D_4D_2 = x_2D_4 - x_4D_2, (g_4 - g_3)D_4D_3 = x_3D_4 - x_4D_3,$ $(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, (g_4 + g_6^+)D_1D_4 = -x_4D_1,$ $(g_2 + g_6^+)D_1D_2 = -x_2D_1, (g_3 + g_6^+)D_1D_3 = -x_3D_1,$ $(g_4 + g_6^+)D_4D_5 = -x_4D_5, (g_2 + g_6^+)D_2D_5 = -x_2D_5,$ $(g_3 + g_6^+)D_3D_5 = -x_3D_5$	$I = \{2, 3, 4\}, T_a^{\circ} = \{1, 5\}$	$g_i \neq g_j, i \neq j, g_i \neq g_6^+,$ for $i, j \in I$
	$(g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5, (g_4 - g_3)D_4D_3 = x_3D_4 - x_4D_3,$ $(g_5 - g_3)D_5D_3 = x_3D_5 - x_5D_3, (g_4 + g_6^+)D_1D_4 = -x_4D_1,$ $(g_5 + g_6^+)D_1D_5 = -x_5D_1, (g_3 + g_6^+)D_1D_3 = -x_3D_1,$ $(g_4 + g_6^+)D_2D_4 = -x_4D_2, (g_5 + g_6^+)D_2D_5 = -x_5D_2,$ $(g_3 + g_6^+)D_2D_3 = -x_3D_2$	$I = \{3, 4, 5\}, T_a^{\circ} = \{1, 2\}$	$g_i \neq g_j, i \neq j, g_i \neq g_6^+,$ for $i, j \in I$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4,$ $(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4, (g_1 + g_6^+)D_1D_3 = -x_1D_3,$ $(g_2 + g_6^+)D_2D_3 = -x_2D_3, (g_6^- - g_4)D_3D_4 = x_4D_3,$ $(g_1 + g_6^+)D_1D_5 = -x_1D_5, (g_2 + g_6^+)D_2D_5 = -x_2D_5,$ $(g_4 + g_6^+)D_4D_5 = -x_4D_5$	$I = \{1, 2, 4\}, T_b^{\bullet} = \{3\}, T_a^{\circ} = \{5\}$	$g_i \neq g_j, i \neq j, g_i \notin \{g_6^{\circ}, \mp g_6^{\pm}\},$ for $i, j \in I$
	$(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3, (g_1 - g_4)D_1D_4 = x_4D_1 - x_1D_4,$ $(g_3 - g_4)D_3D_4 = x_4D_3 - x_3D_4, (g_1 + g_6^+)D_1D_2 = -x_1D_2,$ $(g_6^- - g_3)D_2D_3 = x_3D_2, (g_6^- - g_4)D_2D_4 = x_4D_2,$ $(g_1 + g_6^+)D_1D_5 = -x_1D_5, (g_3 + g_6^+)D_3D_5 = -x_3D_5,$ $(g_4 + g_6^+)D_4D_5 = -x_4D_5$	$I = \{1, 2, 4\}, T_b^{\bullet} = \{3\}, T_a^{\circ} = \{5\}$	$g_i \neq g_j, i \neq j, g_i \notin \{g_6^{\circ}, \mp g_6^{\pm}\},$ for $i, j \in I$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
A_{II}	$(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, (g_2 - g_5)D_2D_5 = x_5D_2 - x_2D_5,$ $(g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5, (g_2 + g_3^+)D_2D_4 = -x_2D_4,$ $(g_3 + g_3^+)D_3D_4 = -x_3D_4, (g_5^- - g_3)D_4D_5 = x_5D_4,$ $(g_2 + g_2^+)D_1D_2 = -x_2D_1, (g_3 + g_2^+)D_1D_3 = -x_3D_1,$ $(g_5 + g_2^+)D_1D_5 = -x_1D_5$	$I = \{2, 3, 5\}, T_5^* = \{4\}, T_a^* = \{1\}$	$g_i \neq g_j, i \neq j, g_i \notin \{g_3^+, \mp g_5^+\},$ for $i, j \in I$
	$(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4, (g_2 - g_5)D_2D_5 = x_5D_2 - x_2D_5,$ $(g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5, (g_2 + g_4^+)D_2D_3 = -x_2D_3,$ $(g_5^- - g_4)D_3D_4 = x_4D_3, (g_5^- - g_5)D_3D_5 = x_5D_3,$ $(g_2 + g_2^+)D_1D_2 = -x_2D_1, (g_4 + g_2^+)D_1D_4 = -x_4D_1,$ $(g_5 + g_2^+)D_1D_5 = -x_1D_5$	$I = \{2, 4, 5\}, T_5^* = \{3\}, T_a^* = \{1\}$	$g_i \neq g_j, i \neq j, g_i \notin \{g_3^+, \mp g_5^+\},$ for $i, j \in I$
	$(g_2 - g_4)D_2D_4 = x_4D_2 - x_2D_4, (g_2 - g_5)D_2D_5 = x_5D_2 - x_2D_5,$ $(g_4 - g_5)D_4D_5 = x_5D_4 - x_4D_5, (g_2 + g_4^+)D_2D_3 = -x_2D_3,$ $(g_5^- - g_4)D_3D_4 = x_4D_3, (g_5^- - g_5)D_3D_5 = x_5D_3,$ $(g_2 + g_2^+)D_1D_2 = -x_2D_1, (g_4 + g_2^+)D_1D_4 = -x_4D_1,$ $(g_5 + g_2^+)D_1D_5 = -x_1D_5$	$I = \{2, 4, 5\}, T_5^* = \{3\}, T_a^* = \{1\}$	$g_i \neq g_j, i \neq j, g_i \notin \{g_3^+, \mp g_5^+\},$ for $i, j \in I$
	$(g_1 - g_2)D_1D_2 = x_2D_1 - x_1D_2, (g_1 - g_5)D_1D_5 = x_5D_1 - x_1D_5,$ $(g_2 - g_3)D_2D_3 = x_3D_2 - x_2D_3, (g_1 + g_1^+)D_1D_3 = -x_1D_3,$ $(g_2 + g_2^+)D_2D_3 = -x_2D_3, (g_5^- - g_3)D_3D_5 = x_5D_3,$ $(g_1 + g_1^+)D_1D_4 = -x_1D_4, (g_2 + g_1^+)D_2D_4 = -x_2D_4,$ $(g_5^- - g_3)D_4D_5 = x_5D_4$	$I = \{1, 2, 5\}, T_5^* = \{3, 4\}$	$g_i \neq g_j, i \neq j, g_i \notin \{\mp g_3^+\},$ for $i, j \in I$
	$(g_1 - g_3)D_1D_3 = x_3D_1 - x_1D_3, (g_1 - g_5)D_1D_5 = x_5D_1 - x_1D_5,$ $(g_3 - g_5)D_3D_5 = x_5D_3 - x_3D_5, (g_1 + g_1^+)D_1D_2 = -x_1D_2,$ $(g_5^- - g_3)D_2D_3 = x_2D_3, (g_5^- - g_5)D_2D_5 = x_5D_2,$ $(g_1 + g_1^+)D_1D_4 = -x_1D_4, (g_3 + g_1^+)D_3D_4 = -x_3D_4,$ $(g_5^- - g_3)D_4D_5 = x_5D_4$	$I = \{1, 3, 5\}, T_5^* = \{2, 4\}$	$g_i \neq g_j, i \neq j, g_i \notin \{\mp g_3^+\},$ for $i, j \in I$
B	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_2 - (g_3 - \Lambda)D_2D_3 = x_2D_3, g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5,$	$I = \{1, 2\}, S = \{3, 4, 5\}$	$g \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_2 - (g_3 - \Lambda)D_2D_3 = x_2D_3, g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{1, 2\}, S = \{3, 4\}, T_a^* = \{5\}$	$g \neq 0, g_s, g_5^+ \notin \{0, \Lambda\}, s \in S$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_2 - (g_3 - \Lambda)D_2D_3 = x_2D_3, g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, g_5^+D_1D_4 = -x_1D_4,$ $(g_5^+ - \Lambda)D_2D_4 = -x_2D_4,$	$I = \{1, 2\}, S = \{3, 5\}, T_a^* = \{4\}$	$g \neq 0, g_s, g_5^+ \notin \{0, \Lambda\}, s \in S$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, g_5^+D_1D_3 = -x_1D_3,$ $(g_5^+ - \Lambda)D_2D_3 = -x_2D_3,$	$I = \{1, 2\}, S = \{4, 5\}, T_a^* = \{3\}$	$g \neq 0, g_s, g_5^+ \notin \{0, \Lambda\}, s \in S$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_2 - (g_3 - \Lambda)D_2D_3 = x_2D_3, g_5^+D_1D_4 = -x_1D_4,$ $(g_5^+ - \Lambda)D_2D_4 = -x_2D_4, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{1, 2\}, S = \{3\}, T_a^* = \{4, 5\}$	$g \neq 0, g_s, g_5^+ \notin \{0, \Lambda\}$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, g_5^+D_1D_5 = -x_1D_5,$ $g_5^+D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, g_5^+D_1D_3 = -x_1D_3,$ $(g_5^+ - \Lambda)D_2D_3 = -x_2D_3,$	$I = \{1, 2\}, S = \{4, 5\}, T_a^* = \{3\}$	$g \neq 0, g_s, g_5^+ \notin \{0, \Lambda\}, s \in S$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_2 - (g_3 - \Lambda)D_2D_3 = x_2D_3, g_5^+D_1D_4 = -x_1D_4,$ $(g_5^+ - \Lambda)D_2D_4 = -x_2D_4, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{1, 2\}, S = \{3\}, T_a^* = \{4, 5\}$	$g \neq 0, g_s, g_5^+ \notin \{0, \Lambda\}$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_3 = -x_2D_3, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{1, 2\}, S = \{4\}, T_a^* = \{3, 5\}$	$g \neq 0, g_4, g_5^+ \notin \{0, \Lambda\}$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, g_5^+D_1D_4 = -x_1D_4,$ $(g_5^+ - \Lambda)D_2D_3 = -x_2D_3, g_5^+D_1D_4 = -x_1D_4,$ $(g_5^+ - \Lambda)D_2D_4 = -x_2D_4,$	$I = \{1, 2\}, S = \{5\}, T_a^* = \{3, 4\}$	$g \neq 0, g_5, g_5^+ \notin \{0, \Lambda\}$
	$g_3D_1D_2 - (g - \Lambda)D_2D_1 = x_2D_1 - x_1D_2, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_3 = -x_2D_3, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_4 = -x_2D_4, g_5^+D_1D_5 = -x_1D_5,$ $(g_5^+ - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{1, 2\}, T_a^* = \{3, 4, 5\}$	$g \neq 0, g_5^+ \notin \{0, \Lambda\}$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_3 - (g_5 - \Lambda)D_3D_5 = x_3D_5.$	$I = \{1, 3\}, S = \{2, 4, 5\}$	$g, g_2 \neq 0, g_4, g_5 \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_5^{\circ}D_1D_5 = -x_1D_5,$ $(g_5^{\circ} - \Lambda)D_3D_5 = -x_3D_5.$	$I = \{1, 3\}, S = \{2, 4\}, T_a^{\circ} = \{5\}$	$g, g_2 \neq 0, g_4, g_5^{\circ} \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_3 - (g_5 - \Lambda)D_3D_5 = x_3D_5, \quad g_6^{\circ}D_1D_4 = -x_1D_4,$ $(g_6^{\circ} - \Lambda)D_3D_4 = -x_3D_4.$	$I = \{1, 3\}, S = \{2, 5\}, T_a^{\circ} = \{4\}$	$g, g_2 \neq 0, g_5, g_6^{\circ} \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_3 - (g_5 - \Lambda)D_3D_5 = x_3D_5, \quad g_6^{\circ}D_1D_2 = -x_1D_2,$ $g_6^{\circ}D_2D_3 = x_3D_2.$	$I = \{1, 3\}, S = \{4, 5\}, T_b^{\circ} = \{2\}$	$g, g_6^{\circ} \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_6^{\circ}D_1D_4 = -x_1D_4,$ $(g_6^{\circ} - \Lambda)D_3D_4 = -x_3D_4, \quad g_6^{\circ}D_1D_5 = -x_1D_5,$ $(g_6^{\circ} - \Lambda)D_3D_5 = -x_3D_5.$	$I = \{1, 3\}, S = \{2\}, T_a^{\circ} = \{4, 5\}$	$g, g_2 \neq 0, g_6^{\circ} \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_5^{\circ}D_1D_2 = -x_1D_2,$ $g_5^{\circ}D_2D_3 = x_3D_2, \quad g_6^{\circ}D_1D_5 = -x_1D_5,$ $(g_6^{\circ} - \Lambda)D_3D_5 = -x_3D_5.$	$I = \{1, 3\}, S = \{4\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{5\}$	$g, g_6^{\circ} \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_3 - (g_5 - \Lambda)D_3D_5 = x_3D_5, \quad g_6^{\circ}D_1D_2 = -x_1D_2,$ $g_6^{\circ}D_2D_3 = x_3D_2, \quad g_6^{\circ}D_1D_4 = -x_1D_4,$ $(g_6^{\circ} - \Lambda)D_3D_4 = -x_3D_4.$	$I = \{1, 3\}, S = \{5\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{4\}$	$g, g_6^{\circ} \neq 0, g_5, g_6^{\circ} \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_6^{\circ}D_1D_2 = -x_1D_2,$ $(g_6^{\circ} - \Lambda)D_3D_5 = -x_3D_5, \quad g_6^{\circ}D_1D_4 = -x_1D_4,$ $g_6^{\circ}D_2D_3 = x_3D_2, \quad g_6^{\circ}D_1D_5 = -x_1D_5,$ $(g_6^{\circ} - \Lambda)D_3D_4 = -x_3D_4.$	$I = \{1, 3\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{4, 5\}$	$g, g_6^{\circ} \neq 0, g_s \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_5^{\circ}D_1D_2 = -x_1D_2,$ $g_5^{\circ}D_2D_3 = x_3D_2, \quad g_6^{\circ}D_1D_5 = -x_1D_5,$ $(g_6^{\circ} - \Lambda)D_3D_5 = -x_3D_5.$	$I = \{1, 3\}, S = \{4\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{5\}$	$g, g_6^{\circ} \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_3 - (g_5 - \Lambda)D_3D_5 = x_3D_5, \quad g_6^{\circ}D_1D_2 = -x_1D_2,$ $g_6^{\circ}D_2D_3 = x_3D_2, \quad g_6^{\circ}D_1D_4 = -x_1D_4,$ $(g_6^{\circ} - \Lambda)D_3D_4 = -x_3D_4.$	$I = \{1, 3\}, S = \{5\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{4\}$	$g, g_6^{\circ} \neq 0, g_5, g_6^{\circ} \notin \{0, \Lambda\}$
	$gD_1D_3 - (g - \Lambda)D_3D_1 = x_3D_1 - x_1D_3, \quad g_6^{\circ}D_1D_5 = -x_1D_5,$ $(g_6^{\circ} - \Lambda)D_3D_5 = -x_3D_5, \quad g_6^{\circ}D_1D_2 = -x_1D_2,$ $g_6^{\circ}D_2D_3 = x_3D_2, \quad g_6^{\circ}D_1D_4 = -x_1D_4,$ $(g_6^{\circ} - \Lambda)D_3D_4 = -x_3D_4.$	$I = \{1, 3\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{4, 5\}$	$g, g_6^{\circ} \neq 0, g_s \notin \{0, \Lambda\}$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5.$	$I = \{1, 4\}, S = \{2, 3, 5\}$	$g, g_2, g_3 \neq 0, g_5 \notin \{0, \Lambda\}$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5^{\circ}D_1D_5 = -x_1D_5,$ $(g_5^{\circ} - \Lambda)D_4D_5 = -x_4D_5.$	$I = \{1, 4\}, S = \{2, 3\}, T_a^{\circ} = \{5\}$	$g, g_s \neq 0, g_5^{\circ} \notin \{0, \Lambda\}, s \in S$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^{\circ}D_1D_3 = -x_1D_3,$ $g_6^{\circ}D_3D_4 = x_4D_3.$	$I = \{1, 4\}, S = \{2, 5\}, T_b^{\circ} = \{3\}$	$g, g_2 \neq 0, g_5, g_6^{\circ} \notin \{0, \Lambda\}$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^{\circ}D_1D_2 = -x_1D_2,$ $g_6^{\circ}D_2D_4 = x_4D_2.$	$I = \{1, 4\}, S = \{3, 5\}, T_b^{\circ} = \{2\}$	$g, g_3, g_5, g_6^{\circ} \neq 0$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5^{\circ}D_1D_5 = -x_1D_5,$ $(g_5^{\circ} - \Lambda)D_4D_5 = -x_4D_5.$	$I = \{1, 4\}, S = \{2\}, T_b^{\circ} = \{3\}, T_a^{\circ} = \{5\}$	$g, g_2, g_6^{\circ} \neq 0, g_5^{\circ} \notin \{0, \Lambda\}$
$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5^{\circ}D_1D_2 = -x_1D_2,$ $g_5^{\circ}D_2D_4 = x_4D_2, \quad g_6^{\circ}D_1D_5 = -x_1D_5,$ $(g_6^{\circ} - \Lambda)D_4D_5 = -x_4D_5.$	$I = \{1, 4\}, S = \{3\}, T_b^{\circ} = \{2\}, T_a^{\circ} = \{5\}$	$g, g_3, g_5^{\circ} \neq 0, g_6^{\circ} \notin \{0, \Lambda\}$	

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_5D_1D_5 - (g_5 - \Lambda)D_5D_1 = -x_1D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_4 = x_4D_2, \quad g_6^+D_1D_3 = -x_1D_3,$ $g_6^-D_3D_4 = x_4D_3,$	$I = \{1, 4\}, S = \{5\}, T_b^* = \{2, 3\}$	$g, g_6^\pm \neq 0, g_5 \notin \{0, \Lambda\}$
	$gD_1D_4 - (g - \Lambda)D_4D_1 = x_4D_1 - x_1D_4, \quad g_6^+D_1D_5 = -x_1D_5,$ $(g_6^+ - \Lambda)D_4D_5 = -x_4D_5, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_4 = x_4D_2, \quad g_6^+D_1D_3 = -x_1D_3,$ $g_6^+D_3D_4 = x_4D_3,$	$I = \{1, 4\}, T_b^* = \{2, 3\}, T_a^* = \{5\}$	$g, g_6^\pm \neq 0, g_6^+ \notin \{0, \Lambda\}$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_5 - (g_2 - \Lambda)D_5D_2 = x_5D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_5D_3, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_5D_4,$	$I = \{1, 5\}, S = \{2, 3, 4\}$	$g, g_s \neq 0, s \in S$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_5 - (g_2 - \Lambda)D_5D_2 = x_5D_2, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_5D_3, \quad g_6^+D_1D_4 = -x_1D_4,$ $g_6^-D_4D_5 = x_5D_4,$	$I = \{1, 5\}, S = \{2, 3\}, T_b^* = \{4\}$	$g, g_s \neq 0, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_5 - (g_2 - \Lambda)D_5D_2 = x_5D_2, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_5D_4, \quad g_6^+D_1D_3 = -x_1D_3,$ $g_6^-D_3D_5 = x_5D_3,$	$I = \{1, 5\}, S = \{2, 4\}, T_b^* = \{3\}$	$g, g_s \neq 0, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_5D_3, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_5D_4, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_5 = x_5D_2,$	$I = \{1, 5\}, S = \{3, 4\}, T_b^* = \{2\}$	$g, g_s \neq 0, g_6^\pm \neq 0, s \in S$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_2D_1D_2 - (g_2 - \Lambda)D_2D_1 = -x_1D_2,$ $g_2D_2D_5 - (g_2 - \Lambda)D_5D_2 = x_5D_2, \quad g_6^+D_1D_4 = -x_1D_4,$ $g_6^-D_4D_5 = x_5D_4,$	$I = \{1, 5\}, S = \{2\}, T_b^* = \{3, 4\}$	$g, g_2, g_6^\pm \neq 0$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_3D_1D_3 - (g_3 - \Lambda)D_3D_1 = -x_1D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_5D_3, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_5 = x_5D_2, \quad g_6^+D_1D_4 = -x_1D_4,$ $g_6^-D_4D_5 = x_5D_4,$	$I = \{1, 5\}, S = \{3\}, T_b^* = \{2, 4\}$	$g, g_3, g_6^\pm \neq 0$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_4D_1D_4 - (g_4 - \Lambda)D_4D_1 = -x_1D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_5D_4, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_5 = x_5D_2, \quad g_6^+D_1D_3 = -x_1D_3,$ $g_6^-D_3D_5 = x_5D_3,$	$I = \{1, 5\}, S = \{4\}, T_b^* = \{2, 3\}$	$g, g_4, g_6^\pm \neq 0$
	$gD_1D_5 - (g - \Lambda)D_5D_1 = x_5D_1 - x_1D_5, \quad g_6^+D_1D_4 = -x_1D_4,$ $g_6^-D_4D_5 = x_5D_4, \quad g_6^+D_1D_2 = -x_1D_2,$ $g_6^-D_2D_5 = x_5D_2, \quad g_6^+D_1D_3 = -x_1D_3,$ $g_6^-D_3D_5 = x_5D_3,$	$I = \{1, 5\}, T_b^* = \{2, 3, 4\}$	$g, g_6^\pm \neq 0$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_2 - (g_1 - \Lambda)D_2D_1 = x_2D_1, \quad g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5,$	$I = \{2, 3\}, S = \{1, 4, 5\}$	$g \neq 0, g_s \notin \{0, \Lambda\}, s \in S$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_2 - (g_1 - \Lambda)D_2D_1 = x_2D_1, \quad g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, \quad g_6^+D_3D_5 = -x_3D_5,$ $(g_6^+ - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{2, 3\}, S = \{1, 4\}, T_a^* = \{5\}$	$g \neq 0, g_s, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_2 - (g_1 - \Lambda)D_2D_1 = x_2D_1, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, \quad g_6^+D_3D_4 = -x_3D_4,$ $(g_6^+ - \Lambda)D_2D_4 = -x_2D_4,$	$I = \{2, 3\}, S = \{1, 5\}, T_a^* = \{4\}$	$g \neq 0, g_s, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, \quad g_6^+D_1D_3 = -x_3D_1,$ $(g_6^+ - \Lambda)D_1D_2 = -x_2D_1,$	$I = \{2, 3\}, S = \{4, 5\}, T_a^* = \{1\}$	$g \neq 0, g_s, g_6^\pm \notin \{0, \Lambda\}, s \in S$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_2 - (g_1 - \Lambda)D_2D_1 = x_2D_1, \quad g_1^cD_3D_4 = -x_3D_4,$ $(g_1^c - \Lambda)D_2D_4 = -x_2D_4, \quad g_1^cD_3D_5 = -x_3D_5,$ $(g_1^c - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{2, 3\}, S = \{1\}, T_a^c = \{4, 5\}$	$g \neq 0, g_1, g_1^c \notin \{0, \Lambda\}$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_4D_3D_4 - (g_4 - \Lambda)D_4D_3 = -x_3D_4,$ $g_4D_4D_2 - (g_4 - \Lambda)D_2D_4 = x_2D_4, \quad g_4^cD_1D_3 = -x_3D_1,$ $(g_4^c - \Lambda)D_1D_2 = -x_2D_1, \quad g_4^cD_3D_5 = -x_3D_5,$ $(g_4^c - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{2, 3\}, S = \{4\}, T_a^c = \{1, 5\}$	$g \neq 0, g_4, g_4^c \notin \{0, \Lambda\}$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_2 - (g_5 - \Lambda)D_2D_5 = x_2D_5, \quad g_5^cD_1D_3 = -x_3D_1,$ $(g_5^c - \Lambda)D_1D_2 = -x_2D_1, \quad g_5^cD_3D_4 = -x_3D_4,$ $(g_5^c - \Lambda)D_2D_4 = -x_2D_4,$	$I = \{2, 3\}, S = \{5\}, T_a^c = \{1, 4\}$	$g \neq 0, g_5, g_5^c \notin \{0, \Lambda\}$
	$gD_3D_2 - (g - \Lambda)D_2D_3 = x_2D_3 - x_3D_2, \quad g_6^cD_1D_3 = -x_3D_1,$ $(g_6^c - \Lambda)D_1D_2 = -x_2D_1, \quad g_6^cD_3D_4 = -x_3D_4,$ $(g_6^c - \Lambda)D_2D_4 = -x_2D_4, \quad g_6^cD_3D_5 = -x_3D_5,$ $(g_6^c - \Lambda)D_2D_5 = -x_2D_5,$	$I = \{2, 3\}, T_a^c = \{1, 4, 5\}$	$g \neq 0, g_6^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5D_2D_5 - (g_5 - \Lambda)D_5D_2 = -x_2D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5,$	$I = \{2, 4\}, S = \{1, 3, 5\}$	$g, g_3 \neq 0, g_1, g_5 \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_5D_2D_5 - (g_5 - \Lambda)D_5D_2 = -x_2D_5,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_3^cD_2D_5 = -x_2D_5,$ $(g_3^c - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{2, 4\}, S = \{1, 3\}, T_a^c = \{5\}$	$g, g_3 \neq 0, g_1, g_5^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_5D_2D_5 - (g_5 - \Lambda)D_5D_2 = -x_2D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3,$	$I = \{2, 4\}, S = \{1, 5\}, T_b^* = \{3\}$	$g, g_6^\pm \neq 0, g_5 \notin \{0, \Lambda\}, s \in S$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5D_2D_5 - (g_5 - \Lambda)D_5D_2 = -x_2D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^cD_1D_2 = -x_2D_1,$ $(g_6^c - \Lambda)D_1D_4 = -x_4D_1,$	$I = \{2, 4\}, S = \{3, 5\}, T_a^c = \{1\}$	$g, g_3 \neq 0, g_5, g_6^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3, \quad g_6^cD_2D_5 = -x_2D_5,$ $(g_6^c - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{2, 4\}, S = \{1\}, T_b^* = \{3\}, T_a^c = \{5\}$	$g, g_6^\pm \neq 0, g_1, g_6^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_5^cD_1D_2 = -x_2D_1,$ $(g_5^c - \Lambda)D_1D_4 = -x_4D_1, \quad g_5^cD_2D_5 = -x_2D_5,$ $(g_5^c - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{2, 4\}, S = \{3\}, T_a^c = \{1, 5\}$	$g, g_3 \neq 0, g_5^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_5D_2D_5 - (g_5 - \Lambda)D_5D_2 = -x_2D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3, \quad g_6^cD_1D_1 = -x_2D_1,$ $(g_6^c - \Lambda)D_1D_4 = -x_4D_1,$	$I = \{2, 4\}, S = \{5\}, T_b^* = \{3\}, T_a^c = \{1\}$	$g, g_6^\pm \neq 0, g_5, g_6^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_2 = x_4D_2 - x_2D_4, \quad g_6^cD_2D_5 = -x_2D_5,$ $(g_6^c - \Lambda)D_4D_5 = -x_4D_5, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_4 = x_4D_3, \quad g_6^cD_2D_5 = -x_2D_5,$ $(g_6^c - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{2, 4\}, T_b^* = \{3\}, T_a^c = \{1, 5\}$	$g, g_6^\pm \neq 0, g_6^c \notin \{0, \Lambda\}$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_5 - (g_1 - \Lambda)D_5D_1 = x_5D_1, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_5D_3, \quad g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_5D_4,$	$I = \{2, 5\}, S = \{1, 3, 4\}$	$g, g_3, g_4 \neq 0, g_1 \notin \{0, \Lambda\}$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_5 - (g_1 - \Lambda)D_5D_1 = x_5D_1, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_5D_3, \quad g_6^+D_2D_4 = -x_2D_4,$ $g_6^-D_4D_5 = x_5D_4,$	$I = \{2, 5\}, S = \{1, 3\}, T_b^* = \{4\}$	$g, g_3, g_6^\pm \neq 0, g_1 \notin \{0, \Lambda\}$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_5 - (g_1 - \Lambda)D_5D_1 = x_5D_1, \quad g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_5D_4, \quad g_6^cD_2D_3 = -x_2D_3,$ $g_6^-D_3D_5 = x_5D_3,$	$I = \{2, 5\}, S = \{1, 4\}, T_b^* = \{3\}$	$g, g_4, g_6^\pm \neq 0, g_1 \notin \{0, \Lambda\}$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_3D_3, \quad g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_3D_4, \quad g_5^+D_1D_2 = -x_2D_1,$ $(g_5 - \Lambda)D_1D_5 = -x_3D_1,$	$I = \{2, 5\}, S = \{3, 4\}, T_a^\circ = \{1\}$	$g, g_3 \neq 0, g_5^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_1D_2D_1 - (g_1 - \Lambda)D_1D_2 = -x_2D_1,$ $g_1D_1D_5 - (g_1 - \Lambda)D_5D_1 = x_3D_1, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_5 = x_3D_3, \quad g_6^+D_2D_4 = -x_2D_4,$ $g_6^-D_4D_5 = x_3D_4,$	$I = \{2, 5\}, S = \{1\}, T_b^\circ = \{3, 4\}$	$g, g_6^\pm \neq 0, g_1 \notin \{0, \Lambda\}$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_3D_2D_3 - (g_3 - \Lambda)D_3D_2 = -x_2D_3,$ $g_3D_3D_5 - (g_3 - \Lambda)D_5D_3 = x_3D_3, \quad g_6^+D_1D_2 = -x_2D_1,$ $(g_6^- - \Lambda)D_1D_5 = -x_3D_1, \quad g_6^+D_2D_4 = -x_2D_4,$ $g_6^-D_4D_5 = x_3D_4,$	$I = \{2, 5\}, S = \{3\}, T_a^\circ = \{1\}, T_b^\circ = \{4\}$	$g, g_3, g_6^\pm \neq 0, g_6^\pm \notin \{0, \Lambda\}$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_4D_2D_4 - (g_4 - \Lambda)D_4D_2 = -x_2D_4,$ $g_4D_4D_5 - (g_4 - \Lambda)D_5D_4 = x_3D_4, \quad g_6^+D_1D_2 = -x_2D_1,$ $(g_6^- - \Lambda)D_1D_5 = -x_3D_1, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_5 = x_3D_3,$	$I = \{2, 5\}, S = \{4\}, T_a^\circ = \{1\}, T_b^\circ = \{3\}$	$g, g_4, g_6^\pm \neq 0, g_6^\pm \notin \{0, \Lambda\}$
	$gD_2D_5 - (g - \Lambda)D_5D_2 = x_5D_2 - x_2D_5, \quad g_6^+D_2D_4 = -x_2D_4,$ $g_6^-D_4D_5 = x_3D_4, \quad g_6^+D_2D_3 = -x_2D_3,$ $g_6^-D_3D_5 = x_3D_3, \quad g_6^+D_1D_2 = -x_2D_1,$ $(g_6^- - \Lambda)D_1D_5 = -x_3D_1,$	$I = \{2, 5\}, T_b^\circ = \{3, 4\}, T_a^\circ = \{1\}$	$g, g_6^\pm \neq 0, g_6^\pm \notin \{0, \Lambda\}$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5,$	$I = \{3, 4\}, S = \{1, 2, 5\}$	$g \neq 0, g_4 \notin \{0, \Lambda\}, s \in S$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_6^+D_3D_5 = -x_3D_5,$ $(g_6^- - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{3, 4\}, S = \{1, 2\}, T_a^\circ = \{5\}$	$g \neq 0, g_4, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^+D_3D_2 = -x_3D_2,$ $(g_6^- - \Lambda)D_4D_2 = -x_4D_2,$	$I = \{3, 4\}, S = \{1, 5\}, T_a^\circ = \{2\}$	$g \neq 0, g_4, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^+D_1D_3 = -x_3D_1,$ $(g_6^- - \Lambda)D_1D_4 = -x_4D_1,$	$I = \{3, 4\}, S = \{2, 5\}, T_a^\circ = \{1\}$	$g \neq 0, g_4, g_6^\pm \notin \{0, \Lambda\}, s \in S$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_1D_3D_1 - (g_1 - \Lambda)D_1D_3 = -x_3D_1,$ $g_1D_1D_4 - (g_1 - \Lambda)D_4D_1 = x_4D_1, \quad g_6^+D_3D_2 = -x_3D_2,$ $(g_6^- - \Lambda)D_4D_2 = -x_4D_2, \quad g_6^+D_3D_5 = -x_3D_5,$ $(g_6^- - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{3, 4\}, S = \{1\}, T_a^\circ = \{2, 5\}$	$g \neq 0, g_4, g_6^\pm \notin \{0, \Lambda\}$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_6^+D_1D_3 = -x_3D_1,$ $(g_6^- - \Lambda)D_1D_4 = -x_4D_1, \quad g_6^+D_3D_5 = -x_3D_5,$ $(g_6^- - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{3, 4\}, S = \{2\}, T_a^\circ = \{1, 5\}$	$g \neq 0, g_2, g_6^\pm \notin \{0, \Lambda\}$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_5D_3D_5 - (g_5 - \Lambda)D_5D_3 = -x_3D_5,$ $g_5D_5D_4 - (g_5 - \Lambda)D_4D_5 = x_4D_5, \quad g_6^+D_1D_3 = -x_3D_1,$ $(g_6^- - \Lambda)D_1D_4 = -x_4D_1, \quad g_6^+D_3D_2 = -x_3D_2,$ $(g_6^- - \Lambda)D_4D_2 = -x_4D_2,$	$I = \{3, 4\}, S = \{5\}, T_a^\circ = \{1, 2\}$	$g \neq 0, g_5, g_6^\pm \notin \{0, \Lambda\}$
	$gD_3D_4 - (g - \Lambda)D_4D_3 = x_4D_3 - x_3D_4, \quad g_6^+D_1D_3 = -x_3D_1,$ $(g_6^- - \Lambda)D_1D_4 = -x_4D_1, \quad g_6^+D_3D_2 = -x_3D_2,$ $(g_6^- - \Lambda)D_4D_2 = -x_4D_2, \quad g_6^+D_3D_5 = -x_3D_5,$ $(g_6^- - \Lambda)D_4D_5 = -x_4D_5,$	$I = \{3, 4\}, T_a^\circ = \{1, 2, 5\}$	$g \neq 0, g_6^\pm \notin \{0, \Lambda\}$
	$gD_3D_5 - (g - \Lambda)D_5D_3 = x_3D_5 - x_5D_3, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_4D_5D_4 - (g_4 - \Lambda)D_4D_5 = -x_5D_4,$ $g_4D_3D_5 - (g_4 - \Lambda)D_5D_3 = x_3D_1, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1,$	$I = \{3, 5\}, S = \{1, 2, 4\}$	$g, g_4 \neq 0, g_1, g_2 \notin \{0, \Lambda\}$
	$gD_3D_5 - (g - \Lambda)D_5D_3 = x_3D_5 - x_5D_3, \quad g_2D_3D_2 - (g_2 - \Lambda)D_2D_3 = -x_3D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_6^+D_3D_4 = -x_3D_4,$ $g_6^-D_4D_5 = x_3D_4,$	$I = \{3, 5\}, S = \{1, 2\}, T_b^\circ = \{4\}$	$g \neq 0, g_4, g_6^\pm \notin \{0, \Lambda\}, s \in S$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
B	$gD_2D_3 - (g - \Lambda)D_3D_5 = x_3D_5 - x_5D_3, \quad g_4D_5D_4 - (g_4 - \Lambda)D_4D_5 = -x_5D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_2^cD_2D_5 = -x_5D_2,$ $(g_2^c - \Lambda)D_2D_3 = -x_3D_2,$	$I = \{3, 5\}, S = \{1, 4\}, T_a^c = \{2\}$	$g, g_4 \neq 0, g_1, g_2^c \notin \{0, \Lambda\}$
	$gD_2D_3 - (g - \Lambda)D_3D_5 = x_3D_5 - x_5D_3, \quad g_4D_5D_4 - (g_4 - \Lambda)D_4D_5 = -x_5D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_2^cD_2D_2 - (g_2 - \Lambda)D_2D_5 = -x_5D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_1^cD_1D_5 = -x_5D_1,$ $(g_1^c - \Lambda)D_1D_3 = -x_3D_1,$	$I = \{3, 5\}, S = \{2, 4\}, T_a^c = \{1\}$	$g, g_4 \neq 0, g_2, g_1^c \notin \{0, \Lambda\}$
	$gD_2D_3 - (g - \Lambda)D_3D_5 = x_3D_5 - x_5D_3, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_2^cD_2D_3 = -x_3D_2,$ $(g_2^c - \Lambda)D_2D_5 = -x_5D_2, \quad g_4^cD_3D_4 = -x_3D_4,$ $g_6^-D_4D_5 = x_5D_4,$	$I = \{3, 5\}, S = \{1\}, T_b^* = \{4\}, T_a^c = \{2\}$	$g, g_6^+ \neq 0, g_1, g_4^c \notin \{0, \Lambda\}$
	$gD_2D_3 - (g - \Lambda)D_3D_5 = x_3D_5 - x_5D_3, \quad g_2D_5D_2 - (g_2 - \Lambda)D_2D_5 = -x_5D_2,$ $g_2D_2D_3 - (g_2 - \Lambda)D_3D_2 = x_3D_2, \quad g_2^cD_1D_3 = -x_3D_1,$ $(g_2^c - \Lambda)D_1D_5 = -x_5D_1, \quad g_1^cD_3D_4 = -x_3D_4,$ $g_6^-D_4D_5 = x_5D_4,$	$I = \{3, 5\}, S = \{2\}, T_b^* = \{4\}, T_a^c = \{1\}$	$g, g_6^+ \neq 0, g_2, g_1^c \notin \{0, \Lambda\}$
	$gD_2D_3 - (g - \Lambda)D_3D_5 = x_3D_5 - x_5D_3, \quad g_4D_5D_4 - (g_4 - \Lambda)D_4D_5 = -x_5D_4,$ $g_4D_4D_3 - (g_4 - \Lambda)D_3D_4 = x_3D_4, \quad g_2^cD_1D_3 = -x_3D_1,$ $(g_2^c - \Lambda)D_1D_5 = -x_5D_1, \quad g_2^cD_2D_3 = -x_3D_2,$ $(g_2^c - \Lambda)D_2D_5 = -x_5D_2,$	$I = \{3, 5\}, S = \{4\}, T_a^c = \{1, 2\}$	$g, g_4 \neq 0, g_2^c \notin \{0, \Lambda\}$
	$gD_2D_3 - (g - \Lambda)D_3D_5 = x_3D_5 - x_5D_3, \quad g_6^+D_3D_4 = -x_3D_4,$ $g_6^-D_4D_5 = x_5D_4, \quad g_2^cD_1D_3 = -x_3D_1,$ $(g_2^c - \Lambda)D_1D_5 = -x_5D_1, \quad g_2^cD_2D_3 = -x_3D_2,$ $(g_2^c - \Lambda)D_2D_5 = -x_5D_2,$	$I = \{3, 5\}, T_b^* = \{4\}, T_a^c = \{1, 2\}$	$g, g_6^+ \neq 0, g_2^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_2D_5D_2 - (g_2 - \Lambda)D_2D_5 = -x_5D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_3D_5D_3 - (g_3 - \Lambda)D_3D_5 = -x_5D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3,$	$I = \{4, 5\}, S = \{1, 2, 3\}$	$g \neq 0, g_s, g_s^c \notin \{0, \Lambda\}, s \in S$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_2D_5D_2 - (g_2 - \Lambda)D_2D_5 = -x_5D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_2^cD_5D_3 = -x_5D_3,$ $(g_2^c - \Lambda)D_4D_3 = -x_4D_3,$	$I = \{4, 5\}, S = \{1, 2\}, T_a^c = \{3\}$	$g \neq 0, g_s, g_s^c \notin \{0, \Lambda\}, s \in S$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_3D_5D_3 - (g_3 - \Lambda)D_3D_5 = -x_5D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_2^cD_5D_2 = -x_5D_2,$ $(g_2^c - \Lambda)D_4D_2 = -x_4D_2,$	$I = \{4, 5\}, S = \{1, 3\}, T_a^c = \{2\}$	$g \neq 0, g_s, g_s^c \notin \{0, \Lambda\}, s \in S$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_2D_5D_2 - (g_2 - \Lambda)D_2D_5 = -x_5D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_3D_5D_3 - (g_3 - \Lambda)D_3D_5 = -x_5D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_2^cD_1D_5 = -x_5D_1,$ $(g_2^c - \Lambda)D_1D_4 = -x_4D_1,$	$I = \{4, 5\}, S = \{2, 3\}, T_a^c = \{1\}$	$g \neq 0, g_s, g_s^c \notin \{0, \Lambda\}, s \in S$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_1D_5D_1 - (g_1 - \Lambda)D_1D_5 = -x_5D_1,$ $g_1D_1D_3 - (g_1 - \Lambda)D_3D_1 = x_3D_1, \quad g_2^cD_5D_2 = -x_5D_2,$ $(g_2^c - \Lambda)D_4D_2 = -x_4D_2, \quad g_2^cD_5D_3 = -x_5D_3,$ $(g_2^c - \Lambda)D_4D_3 = -x_4D_3,$	$I = \{4, 5\}, S = \{1\}, T_a^c = \{2, 3\}$	$g \neq 0, g_1, g_2^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_2D_5D_2 - (g_2 - \Lambda)D_2D_5 = -x_5D_2,$ $g_2D_2D_4 - (g_2 - \Lambda)D_4D_2 = x_4D_2, \quad g_2^cD_1D_5 = -x_5D_1,$ $(g_2^c - \Lambda)D_1D_4 = -x_4D_1, \quad g_2^cD_5D_3 = -x_5D_3,$ $(g_2^c - \Lambda)D_4D_3 = -x_4D_3,$	$I = \{4, 5\}, S = \{2\}, T_b^c = \{1, 3\}$	$g \neq 0, g_2, g_2^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_3D_5D_3 - (g_3 - \Lambda)D_3D_5 = -x_5D_3,$ $g_3D_3D_4 - (g_3 - \Lambda)D_4D_3 = x_4D_3, \quad g_2^cD_1D_5 = -x_5D_1,$ $(g_2^c - \Lambda)D_1D_4 = -x_4D_1, \quad g_2^cD_5D_2 = -x_5D_2,$ $(g_2^c - \Lambda)D_4D_2 = -x_4D_2,$	$I = \{4, 5\}, S = \{3\}, T_a^c = \{1, 2\}$	$g \neq 0, g_3, g_2^c \notin \{0, \Lambda\}$
	$gD_2D_4 - (g - \Lambda)D_4D_5 = x_4D_5 - x_5D_4, \quad g_2^cD_1D_5 = -x_5D_1,$ $(g_2^c - \Lambda)D_1D_4 = -x_4D_1, \quad g_2^cD_5D_2 = -x_5D_2,$ $(g_2^c - \Lambda)D_4D_2 = -x_4D_2, \quad g_2^cD_5D_3 = -x_5D_3,$ $(g_2^c - \Lambda)D_4D_3 = -x_4D_3,$	$I = \{4, 5\}, T_a^c = \{1, 2, 3\}$	$g \neq 0, g_2^c \notin \{0, \Lambda\}$

TABLE 2. Diffusion algebras on five generators (continued)

Diffusion algebra	Relations	Conditions	Restrictions
C	$g_2D_1D_2 - (g_2 - \Lambda_a)D_2D_1 = -x_1D_2,$ $g_4D_1D_4 - (g_4 - \Lambda_a)D_4D_1 = -x_1D_4,$ $g_3D_1D_3 - (g_3 - \Lambda_a)D_3D_1 = -x_1D_3,$ $g_5D_1D_5 - (g_5 - \Lambda_a)D_5D_1 = -x_1D_5,$	$I = \{1\}, R_a = \{2, 3, 4, 5\}$	$g_2, g_3, g_4, g_5 \neq \Lambda_a$
	$g_1D_2D_1 - (g_1 - \Lambda_a)D_1D_2 = -x_2D_1,$ $g_3D_2D_3 - (g_3 - \Lambda_a)D_3D_2 = -x_2D_3,$ $g_4D_2D_4 - (g_4 - \Lambda_a)D_4D_2 = -x_2D_4,$ $g_5D_2D_5 - (g_5 - \Lambda_a)D_5D_2 = -x_2D_5,$	$I = \{2\}, R_a = \{1, 3, 4, 5\}$	$g_1 \neq 0, g_3, g_4, g_5 \neq \Lambda_a$
	$g_1D_3D_1 - (g_1 - \Lambda_a)D_1D_3 = -x_3D_1,$ $g_2D_3D_2 - (g_2 - \Lambda_a)D_2D_3 = -x_3D_2,$ $g_4D_3D_4 - (g_4 - \Lambda_a)D_4D_3 = -x_3D_4,$ $g_5D_3D_5 - (g_5 - \Lambda_a)D_5D_3 = -x_3D_5,$	$I = \{3\}, R_a = \{1, 2, 4, 5\}$	$g_1, g_2 \neq 0, g_4, g_5 \neq \Lambda_a$
	$g_1D_4D_1 - (g_1 - \Lambda_a)D_1D_4 = -x_4D_1,$ $g_2D_4D_2 - (g_2 - \Lambda_a)D_2D_4 = -x_4D_2,$ $g_3D_4D_3 - (g_3 - \Lambda_a)D_3D_4 = -x_4D_3,$ $g_5D_4D_5 - (g_5 - \Lambda_a)D_5D_4 = -x_4D_5,$	$I = \{4\}, R_a = \{1, 2, 3, 5\}$	$g_1, g_2, g_3 \neq 0, g_5 \neq \Lambda$
	$g_1D_5D_1 - (g_1 - \Lambda_a)D_1D_5 = -x_5D_1,$ $g_2D_5D_2 - (g_2 - \Lambda_a)D_2D_5 = -x_5D_2,$ $g_3D_5D_3 - (g_3 - \Lambda_a)D_3D_5 = -x_5D_3,$ $g_4D_5D_4 - (g_4 - \Lambda_a)D_4D_5 = -x_5D_4,$	$I = \{5\}, R_a = \{1, 2, 3, 4\}$	$g_1, g_2, g_3, g_4 \neq 0$
	$D_1D_2 - q_{21}D_2D_1 = 0,$ $D_1D_3 - q_{31}D_3D_1 = 0,$ $D_1D_4 - q_{41}D_4D_1 = 0,$ $D_1D_5 - q_{51}D_5D_1 = 0,$	$R = \{1, 2, 3, 4, 5\}$	$q_{sr} \neq 0, 1 \leq r < s \leq 5$
$D_2D_3 - q_{32}D_3D_2 = 0,$ $D_2D_4 - q_{42}D_4D_2 = 0,$ $D_2D_5 - q_{52}D_5D_2 = 0,$ $D_3D_4 - q_{43}D_4D_3 = 0,$ $D_3D_5 - q_{53}D_5D_3 = 0,$ $D_4D_5 - q_{54}D_5D_4 = 0$			

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