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SOME RESULTS ON MODULES WHOSE DIRECT COMPLEMENTS ARE ALMOST (ESSENTIALLY) UNIQUE

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Abstract. Direct complements in a module M are said to be almost (essentially) unique if, whenever $M = A \oplus B = A \oplus C$, $(B + C)/B$ is small in M/B ($B \cap C$ is essential in B). The module M is said to be a DCAU-module (DCEU-module) if direct complements of M are almost (essentially) unique. We determine the structure of both DCAU- and DCEU-modules over discrete valuation rings. When R is a non-local Dedekind domain, we describe the structure of the torsion part of a DCAU-R-module M (and of a DCEU-R-module N) which turns out to be a direct summand of M (of N). Moreover, we investigate the class of rings R for which every right DCAU-R-module is DCEU. A ring of this type will be called a right AE-ring. Analogous to this class of rings, we shed some light on right EA-rings (i.e., rings R for which every right DCEU-R-module is DCAU). Among other results, we show that every right AE-ring is right Bass and every commutative EA-ring is perfect. We provide examples to delineate the concepts and results.

1. Introduction

Throughout this paper, rings R are associative with unity and modules considered are unitary right R-modules unless stated otherwise. Let M be an R-module. The notation $N \subseteq M$ means that N is a subset of M; $N \leq M$ means that N is a submodule of M; $N \leq^{\text{ess}} M$ means that N is an essential submodule of M (i.e., $N \cap L \neq 0$ for every nonzero submodule L of M); and dually, $N \ll M$ means that N is a small submodule of M (i.e., $N + L \neq M$ for every proper submodule L of M). We use $\text{Rad}(M)$, $\text{Soc}(M)$, $\text{Z}(M)$, $E(M)$ and $\text{End}_R(M)$ to denote the Jacobson radical, the socle, the singular submodule, the injective hull and the endomorphism ring of M, respectively. For two R-modules X and Y, the set of R-homomorphisms from X to Y is denoted by $\text{Hom}_R(X, Y)$. If $M = R$, we write $\text{J}(R) = \text{Rad}(R)$. We denote the annihilator of M by $\text{Ann}_R(M)$, i.e., $\text{Ann}_R(M) = \{r \in R \mid mr = 0 \text{ for all } m \in M\}$. For each element x of a right R-module M, the annihilator of x in R is the set $\text{Ann}_R(x) = \{r \in R \mid xr = 0\}$.

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Recall that a module M is called *summand-square-free* (an *SSF-module*, for short) if it contains no nonzero isomorphic direct summands A and B such that $A \cap B = 0$. Dually, according to [10], a module M is called *summand-dual-square-free* (or an *SDSF-module*) if M has no proper direct summands A and B with $M = A + B$ and $M/A \cong M/B$. The notion of "essentially unique direct complements" was introduced by Mohamed and Müller in 1987 [27]. *Direct complements of a module M* are said to be *essentially unique* (or M is a *DCEU-module*) if whenever $M = A \oplus B = A \oplus C$, $B \cap C$ is essential in B (and hence also in C, by symmetry). In 1997 [29], the same authors characterized DCEU-modules as exactly the class of modules M for which idempotents in $\mathrm{End}_R(M)$ are central modulo $\triangle(M) = \{\varphi \in \mathrm{End}_R(M) \mid \mathrm{Ker}\,\varphi$ is essential in $M\}$. Among other results, Mazurek et al. showed in [26] that a module M is DCEU if and only if all idempotents of $\mathrm{End}_R(M)$ commute modulo $\triangle(M)$. Moreover, they proved that the class of SSF-modules properly contains the class of DCEU-modules. In 2023, Ibrahim and Yousif [19] provided a new characterization of DCEU-modules. In fact, they proved that a module M is DCEU if and only if whenever $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism, $\mathrm{Ker}\, f$ is essential in A. In the same paper, the authors dualized the notion of DCEU-modules and introduced the notion of "almost unique direct complements". Recall that *direct complements of a module M* are said to be *almost unique* (or M is a *DCAU-module*, for short) if whenever $M = A \oplus B = A \oplus C$, $(B + C)/B \ll M/B$ (also $(B + C)/C \ll M/C$, by symmetry). It turned out that many results on DCEU-modules have corresponding duals for DCAU-modules. For example, it was shown in [19, Proposition 4.3] that a module M is DCAU if and only if idempotents in $\mathrm{End}_R(M)$ commute modulo $\nabla(M)$ if and only if whenever $M = A \oplus B$ and $f : A \to B$ is an R-homomorphism, $\mathrm{Im}\, f$ is small in B, where $\nabla(M) = \{\varphi \in \mathrm{End}_R(M) \mid \mathrm{Im}\,\varphi \ll M\}$. Also, the class of DCAU-modules is properly contained in the class of SDSF-modules (see [19, Proposition 4.10 and Example 4.12]).

In this article, we continue the study of DCAU- and DCEU-modules, establishing several new results on the subject.

In Section 2, we provide several examples to delineate some properties of the concepts studied. Some of these examples will be useful to our work in the next sections.

In Section 3, we compile some basic facts of DCAU- and DCEU-modules which will be used throughout the paper to obtain some structure results of these notions over Dedekind domains. Among others, we show that given an abelian module N and a simple module S, the module $N \oplus S$ is DCAU (resp., DCEU) if and only if $\mathrm{Hom}_R(N, S) = 0$ (resp., $\mathrm{Hom}_R(S, N) = 0$). Moreover, we prove that if M is a semisimple module which is a direct sum of pairwise non-isomorphic simple submodules, then every submodule of its injective hull $E(M)$ is a DCEU-module (Corollary 3.18). Also, we characterize when a module having an essential socle is a DCEU-module (Proposition 3.21).

The investigations in Section 4 focus on the study of DCAU-R-modules and DCEU-R-modules when R is a Dedekind domain. It is not our purpose to fully describe the structure of these classes of modules, but we determine the torsion part of a DCAU-R-module and that of a DCEU-R-module (see Theorem 4.3 and Corollary 4.4). We also characterize when an R-module M which has a direct summand isomorphic to R is DCAU (or DCEU) (Proposition 4.7 and Corollary 4.8). We conclude this section by describing the structure of R-modules for which every submodule is DCAU (Proposition 4.11).

In Section 5, R denotes a discrete valuation ring. We determine the structure of both DCAU- and DCEU-R-modules in terms of torsion-free reduced abelian R-modules (Theorems 5.6 and 5.10). Then we apply these results to the case when the ring R is complete.

We begin Section 6 by investigating when the right R-module R_R is a DCAU-module (or a DCEU-module). Then we deal with the class of rings called right AE-rings (resp., right EA-rings), that is, the class of rings R for which every right DCAU-R-module is DCEU (resp., every right DCEU-R-module is DCAU). It is shown that every commutative EA-ring is a perfect ring and every right AE-ring is a right Bass ring (Theorem 6.14 and Proposition 6.19).

2. Some examples of DCAU- and DCEU-modules

We shall say in what follows that a module M is a *DCEU-module* (*DCAU-module*) if direct complements of M are essentially (almost) unique.

Remark 2.1. Many equivalent formulations of being DCAU or DCEU are given in [19, Propositions 3.3 and 4.3]. In this paper we will use frequently the fact that a module M is DCAU (resp., DCEU) if and only if for every decomposition $M = A \oplus B$ and any R-homomorphism $f : A \to B$, $\operatorname{Im} f$ is small in B (resp., $\operatorname{Ker} f$ is essential in A).

This section will be devoted to the exhibition of some examples of such modules. First recall that a submodule N of a module M is said to be *fully invariant* in M if $f(N) \subseteq N$ for every $f \in \operatorname{End}_R(M)$. Recall that a ring R is called *abelian* if all its idempotents are central. A module M is called *abelian* if its endomorphism ring $\operatorname{End}_R(M)$ is an abelian ring. It is well known that a module M is abelian if and only if all direct summands of M are fully invariant if and only if direct complements of M are unique if and only if $\operatorname{Hom}_R(A, B) = 0$ for every decomposition $M = A \oplus B$ of M (see, for example, [6, Theorem 4.4], [19, Lemma 3.17] and [31, Lemma 1]). In [32], abelian modules are called *weak duo* modules. It is clear that every abelian module is both a DCAU-module and a DCEU-module.

Example 2.2. For any prime p, let J_p and $\mathbb{Q}_p^*$ denote the group of p-adic integers and the ring of p-adic integers, respectively. Let $\mathbf{P}$ be a set of different primes. It is well known that for any positive integers k_p ($p \in \mathbf{P}$), the rings

$$\operatorname{End}_{\mathbb{Z}}\left(\bigoplus_{p\in\mathbf{P}} \mathbb{Z}/p^{k_p}\mathbb{Z}\right) \cong \operatorname{End}_{\mathbb{Z}}\left(\prod_{p\in\mathbf{P}} \mathbb{Z}/p^{k_p}\mathbb{Z}\right) \cong \prod_{p\in\mathbf{P}} \mathbb{Z}/p^{k_p}\mathbb{Z}$$

and

$$\mathrm{End}_{\mathbb{Z}}\left(\bigoplus_{p\in\mathbf{P}} J_p\right) \cong \mathrm{End}_{\mathbb{Z}}\left(\prod_{p\in\mathbf{P}} J_p\right) \cong \prod_{p\in\mathbf{P}} \mathbb{Q}_p^*$$

are commutative. It follows that the $\mathbb{Z}$-modules $\bigoplus_{p\in\mathbf{P}} \mathbb{Z}/p^{k_p}\mathbb{Z}$, $\prod_{p\in\mathbf{P}} \mathbb{Z}/p^{k_p}\mathbb{Z}$, $\bigoplus_{p\in\mathbf{P}} J_p$, and $\prod_{p\in\mathbf{P}} J_p$ are abelian $\mathbb{Z}$-modules and hence they are both DCAU and DCEU.

According to [26], a module M is called *summand-square-free* (an *SSF-module* for short) if it contains no nonzero isomorphic direct summands A and B with $A\cap B=0$. The dual notion to SSF-modules was introduced in 2017 [10]. A module M is called *summand-dual-square-free* (an *SDSF-module* for short) if M has no proper direct summands A and B with $M=A+B$ and $M/A\cong M/B$.

Example 2.3. It is easily seen that the following conditions are equivalent for a semisimple module M:

(i) M is a DCEU-module.
(ii) M is a DCAU-module.
(iii) M is an SSF-module.
(iv) M is an SDSF-module.
(v) M is an abelian module.
(vi) $M=\bigoplus_{i\in I} S_i$ is a direct sum of pairwise non-isomorphic simple submodules S_i $(i\in I)$.

According to [17, Proposition 2.4], [33, 4.8] and [34, 3.3], a ring R is said to be *right small* if R satisfies the following equivalent conditions:

(i) R_R is a small submodule in its injective hull $E(R_R)$.
(ii) $\mathrm{Rad}(E(R_R))=E(R_R)$.
(iii) $\mathrm{Rad}(E)=E$ for every injective right R-module E.

Note that every commutative domain is a small ring by [33, Folgerung 5.3] (see also [17, Proposition 2.4 and Corollary 2.5]).

Recall that a nonzero module M is called *hollow* if every proper submodule of M is small in M.

Example 2.4. Let an R-module $M=M_1\oplus M_2$ be a direct sum of two nonzero indecomposable direct summands M_1 and M_2 such that the decomposition $M=M_1\oplus M_2$ complements direct summands (e.g., we can assume that each endomorphism ring $\mathrm{End}_R(M_i)$ is local). Note that for any non-trivial decomposition $M=A\oplus B$ of M, we must have $A\cong M_1$ and $B\cong M_2$ or $A\cong M_2$ and $B\cong M_1$. This implies that M is a DCAU-module if and only if for every R-homomorphism $f: M_i\to M_j$ where $i\neq j\in\{1,2\}$, $\mathrm{Im}\, f\ll M_j$. To construct the following explicit examples, we will need the fact that the endomorphism ring of every indecomposable injective module is local (see [2, Lemma 25.4]).

(i) Assume that $M_1\cong S_1$ and $M_2\cong E(S_2)$ for some simple R-modules S_1 and S_2. Then M is a DCAU-module if and only if $\mathrm{Hom}_R(E(S_2),S_1)=0$. In fact, the necessity is clear. To show the sufficiency, suppose that there exists a nonzero R-homomorphism $f : M_1 \to M_2$. Since M_1 is simple, $f(M_1) \subseteq \mathrm{Soc}(M_2)$. But $\mathrm{Soc}(M_2)$ is a simple module which is isomorphic to S_2. Then $S_1 \cong M_1 \cong \mathrm{Soc}(M_2) \cong S_2$ and $f(M_1) = \mathrm{Soc}(M_2)$. It follows that S_2 is not injective as $\mathrm{Hom}_R(E(S_2), S_1) = 0$. Therefore $\mathrm{Soc}(M_2) \ll M_2$ and so $f(M_1) \ll M_2$.

(ii) Assume that R is a right hereditary ring and let S_1 and S_2 be two simple R-modules such that S_1 is not injective. Applying (i), we conclude that the module $S_1 \oplus E(S_2)$ is a DCAU-module. Indeed, we have $\mathrm{Hom}_R(E(S_2), S_1) = 0$, since otherwise $S_1 \cong E(S_2)/L$ for some submodule $L \leq E(S_2)$. This implies that S_1 is injective as R is right hereditary, a contradiction. In particular, the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}(q^\infty)$ is a DCAU-module for all prime numbers p and q.

(iii) Let R be a right small ring and let S be a simple R-module. Let E be an indecomposable injective R-module. Then $\mathrm{Rad}(E) = E$ and hence $\mathrm{Im}\, f \ll E$ for every $f \in \mathrm{Hom}_R(S, E)$. Moreover, $\mathrm{Hom}_R(E, S) = 0$. Hence $S \oplus E$ is a DCAU-module.

(iv) Let R be a commutative noetherian domain with quotient field $Q \neq R$. Let $M = M_1 \oplus M_2$, where M_1 is a nonzero indecomposable injective R-module (e.g., $M_1 \cong Q$) and M_2 is a nonzero finitely generated R-module such that $\mathrm{End}_R(M_2)$ is local (e.g., $M_2 \cong R/\mathfrak{m}^n$ for some maximal ideal $\mathfrak{m}$ of R and some positive integer n). Note that $\mathrm{Rad}(M_1) = M_1$. Since R is noetherian, every submodule of M_2 is finitely generated and hence $\mathrm{Hom}_R(M_1, M_2) = 0$. In addition, for any $f \in \mathrm{Hom}_R(M_2, M_1)$, $\mathrm{Im}\, f$ is finitely generated and so $\mathrm{Im}\, f \ll M_1$. Thus M is a DCAU-module.

(v) Let an R-module $M = M_1 \oplus M_2$ be a direct sum of two nonzero hollow submodules M_1 and M_2 where each endomorphism ring $\mathrm{End}_R(M_i)$ is local. Assume also that M_j is not a homomorphic image of M_i for every $i \neq j \in \{1, 2\}$. Clearly, M is a DCAU-module. For example, let R be a commutative local ring which is not a valuation ring (e.g., we can take R to be the ring $k[x^2, x^3]/(x^4)$, where k is a field (see [3, Example on p. 91])). Then R contains two ideals I_1 and I_2 which are not comparable. Therefore $R/I_1 \oplus R/I_2$ is a DCAU-R-module.

We exhibit in the following example some specific DCEU-modules.

Example 2.5. Let an R-module $M = M_1 \oplus M_2$ be a direct sum of two uniform submodules M_1 and M_2 such that each $\mathrm{End}_R(M_i)$ $(i = 1, 2)$ is a local ring. Note that any non-trivial decomposition of M is an indecomposable decomposition and it is equivalent to $M = M_1 \oplus M_2$. It follows that M is a DCEU-module if and only if M_i is not isomorphic to a submodule of M_j for all $i \neq j \in \{1, 2\}$.

(i) Assume that $M = M_1 \oplus M_2$ such that M_1 is simple and $M_2 \cong E(S_2)$ for some simple R-module S_2. It is easily seen that the module M is a DCEU-module if and only if the modules M_1 and S_2 are not isomorphic.

(ii) Assume that $M = M_1 \oplus M_2$ is a direct sum of non-isomorphic submodules M_1 and M_2 such that each M_i is indecomposable and injective. It is clear that M_i is not isomorphic to a submodule of M_j for all $i \neq j \in \{1, 2\}$. Therefore M is a DCEU-module.

(iii) Let R be a commutative noetherian ring and let P_1 and P_2 be two different prime ideals of R. Let an R-module $M = M_1 \oplus M_2$ such that $M_i \cong E(R/P_i)$ for $i = 1, 2$. Note that M_1 and M_2 are not isomorphic by [36, Lemma 2.31 Corollary].

Moreover, each M_i is indecomposable by [36, Lemma 2.29]. Hence M is a DCEU-module by (ii). Note that if each P_i is a maximal ideal of R, then $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j \in \{1, 2\}$ by [36, Proposition 4.21]. So, in this case M turns out to be an abelian module.

The following example shows that the notions of DCAU-modules and DCEU-modules are independent of each other.

Example 2.6. (i) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}(p^\infty)$, where p is a prime number. Using Example 2.4 (ii), we see that M is a DCAU-module. On the other hand, M is not a DCEU-module by Example 2.5 (i).

(ii) Let R be a discrete valuation ring with quotient field Q and consider the R-module $M = Q \oplus Q/R$. Then M is a DCEU-module by Example 2.5 (ii). However, it is clear that M is not a DCAU-module.

The previous example shows also that a direct sum of two DCAU-modules (resp., DCEU-modules) need not be a DCAU-module (resp., DCEU-module). In the same vein, note that for every indecomposable module M, M is both a DCAU- and a DCEU-module but $M \oplus M$ is neither DCAU nor DCEU. On the other hand, the next result, which is taken from [19, Propositions 3.7 and 4.14], shows that being DCAU or DCEU is preserved by taking direct summands.

Proposition 2.7. *Every direct summand of a DCAU-module (resp., DCEU-module) is again a DCAU-module (resp., DCEU-module).*

The next example shows that the DCAU and the DCEU properties do not always transfer from a module to each of its factor modules. But first, recall from [19, Proposition 4.10] and [26, Theorem 5.6] that every DCAU-module (resp., DCEU-module) is SDSF (resp., SSF).

Example 2.8. Let M be an indecomposable module that contains a proper submodule N such that M/N is neither SDSF nor SSF. Note that M is both DCAU and DCEU. However, M/N is neither a DCAU-module nor a DCEU-module. To construct an explicit example, set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and let $R = \mathbb{Z}_2 \propto (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ be the trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. It is clear that R is a local ring and so R has only one simple right R-module up to isomorphism, say S. It was shown in [8, Example 1.12] and [7, Example 4, p. 4765] that $E(S)/S \cong S \oplus S$. It suffices to take $M = E(S)$ and $N = S$.

In the next example we show that a submodule of a DCAU-module (or a DCEU-module) may not inherit the property.

Example 2.9. (i) Let p be a prime number. Then the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ is a DCAU-module (see Example 2.4 (ii)). On the other hand, note that M has a submodule N which is isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. It is clear that N is not a DCAU-module.

(ii) Consider the ring $R = \mathbb{Z}_2 \propto (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ given in Example 2.8. By [8, Example 1.12], $\mathrm{Soc}(R) \cong S \oplus S$. Hence $\mathrm{Soc}(R)$ is not a DCEU-module. On the other hand, it is clear that R is a DCEU-R-module, since R is a local ring.

The next remark should be contrasted with Example 2.9 (ii).

Remark 2.10. Let M be a DCEU-module over a right V-ring R. Then M is SSF by [26, Theorem 5.6]. Therefore $\operatorname{Soc}(M)$ is SSF (and so it is DCEU), since otherwise there would exist two nonzero semisimple isomorphic submodules A and B of M such that $A \cap B = 0$. Therefore A contains a simple submodule S_a and B contains a simple submodule S_b such that $S_a \cong S_b$. Note that $S_a \cap S_b = 0$ and the simple R-modules S_a and S_b are injective as R is a right V-ring. It follows that S_a and S_b are direct summands of M. This contradicts the fact that M is SSF.

3. Some properties of DCAU- and DCEU-modules

Several interesting characterizations of DCAU- and DCEU-modules were provided in [19]. In this section, we establish more properties of these types of modules that will be used throughout the paper. We begin by giving an exposition of some facts about DCAU-modules.

It is shown in [19, Proposition 3.20] that every non-singular module which is a DCEU-module is an abelian module. Next, we dualize this result. First recall that according to [38] a module M is called *non-cosingular* if $\overline{Z}(M) = M$, where $\overline{Z}(M) = \bigcap\{U \leq M \mid M/U \text{ is small in } E(M/U)\}$. Note that every injective module over a right hereditary ring and every module over a right V-ring is non-cosingular by [38, Propositions 2.5 and 2.7].

Proposition 3.1. *Let M be a non-cosingular module. Then M is a DCAU-module if and only if M is abelian.*

Proof. The sufficiency is clear. Conversely, assume that M is a DCAU-module. Let $M = A \oplus B$ be a decomposition of M and let $f : A \to B$ be a homomorphism. Then $f(A) \ll B$. Since $f(A) \cong A/\operatorname{Ker} f$, $A/\operatorname{Ker} f \ll E(A/\operatorname{Ker} f)$. This implies that $\overline{Z}(A) \subseteq \operatorname{Ker} f$. But $\overline{Z}(A) = A$ by [38, Proposition 2.4], so $f = 0$. This shows that M is abelian. □

Proposition 3.2. *Let M be an R-module whose direct summands have projective covers (e.g., we can assume that R is a right perfect ring). Assume that $p : P \to M$ is a projective cover of M. If P is a DCAU-module, then so is M.*

Proof. Suppose that P is DCAU. To prove that M is DCAU, take a decomposition $M = M_1 \oplus M_2$ of M and let $f \in \operatorname{Hom}_R(M_1, M_2)$. By [2, Lemma 27.2], P has a decomposition $P = P_1 \oplus P_2$ such that $f_i = p_{|P_i} : P_i \to M_i$ is a projective cover of M_i for $i = 1, 2$. Therefore there exists an R-homomorphism $\alpha : P_1 \to P_2$ such that $f_2\alpha = ff_1$. Since P is a DCAU-module, it follows that $\alpha(P_1) \ll P_2$. Hence $f(M_1) = ff_1(P_1) = f_2\alpha(P_1) \ll M_2$ by [28, Lemma 4.2], and the proof is complete. □

A module M is said to have the *(finite) exchange property* if for any (finite) index set I, whenever $M \oplus N = \bigoplus_{i\in I} A_i$ for modules N and A_i, $M \oplus N = M \oplus (\bigoplus_{i\in I} B_i)$ for submodules $B_i \leq A_i$. In the next proposition we consider a specific case where the direct sum of DCAU-modules is again a DCAU-module.

Proposition 3.3. *Let R be any ring and let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 satisfying the following conditions:*

(a) *M_1 is a DCAU-module having the finite internal exchange property;*
(b) *M_2 is an abelian module;*
(c) $\operatorname{Hom}_R(M_2, M_1) = 0$*; and*
(d) $\operatorname{Im} f \ll M_2$ *for every R-homomorphism $f : M_1 \to M_2$.*

Then M is a DCAU-module.

Proof. Let $M = A \oplus B$ be a decomposition of M. Since M_1 has the finite internal exchange property, $M = M_1 \oplus A_2 \oplus B_2$ for some submodules $A_2 \leq A$ and $B_2 \leq B$. By modularity, there exist submodules $A_1 \leq A$ and $B_1 \leq B$ such that $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Clearly, $A_2 \oplus B_2 \cong M_2$ is abelian and $A_1 \oplus B_1 \cong M_1$ is DCAU. Now to show that M is DCAU, let $f : A \to B$ be an R-homomorphism. Since $\operatorname{Hom}_R(M_2, M_1) = 0$, we have $\operatorname{Hom}_R(A_2, B_1) = 0$. Thus $f_2(A_2) \subseteq B_2$, where f_2 denotes the restriction of f to A_2. Hence $\operatorname{Im} f_2 = 0$ as $A_2 \oplus B_2$ is abelian. Let f_1 denote the restriction of f to A_1. For each $i = 1, 2$, let $\pi_i : B \to B_i$ be the projection map with respect to the decomposition $B = B_1 \oplus B_2$. Since $A_1 \oplus B_1$ is DCAU, it follows that $\operatorname{Im} \pi_1 f_1 \ll B_1$. In addition, from condition (d), we obtain $\operatorname{Im} \pi_2 f_1 \ll B_2$. Note that $f_1(A_1) \subseteq \operatorname{Im} \pi_1 f_1 + \operatorname{Im} \pi_2 f_1$. Then $f_1(A_1) \ll B$ and so $\operatorname{Im} f \ll B$. In the same manner we can see that $\operatorname{Im} g \ll A$ for every $g \in \operatorname{Hom}_R(B, A)$. This proves the proposition. □

As an application of the previous proposition, we obtain the following corollary, which provides more examples of DCAU-modules.

Corollary 3.4. *Let an R-module $M = N \oplus S$ be the direct sum of an abelian submodule N and a simple submodule S. Then M is a DCAU-module if and only if* $\operatorname{Hom}_R(N, S) = 0$.

Proof. The necessity is clear. To prove the sufficiency, we only need to show that $\operatorname{Im} f \ll N$ for every $f \in \operatorname{Hom}_R(S, N)$ (see Proposition 3.3). Assume, to the contrary, that there exists a nonzero R-homomorphism $f : S \to N$ such that $\operatorname{Im} f$ is not small in N. Therefore $L + \operatorname{Im} f = N$ for some proper submodule L of N. It is easily seen that $\operatorname{Im} f$ is a simple module which is isomorphic to S. Thus $L \oplus \operatorname{Im} f = N$ and hence $\operatorname{Hom}_R(N, S) \neq 0$, a contradiction. □

From Corollary 3.4, we conclude that for any abelian module M (e.g., M is indecomposable) with $\operatorname{Rad}(M) = M$ and any simple module S, the module $M \oplus S$ is a DCAU-module. The next proposition is an extension of this fact.

Proposition 3.5. *Let R be any ring and let M be an R-module such that $M =$* $\operatorname{Rad}(M) \oplus N$ *for some cyclic submodule $N \leq M$. Then the following statements are equivalent:*

(i) *M is a DCAU-R-module.*
(ii) $\operatorname{Rad}(M)$ *is a DCAU-R-module and N is an abelian module.*

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 2.7 and [19, Proposition 4.18].

(ii) $\Rightarrow$ (i) To show that M is a DCAU-R-module, take a decomposition $M = A \oplus B$ of M. Then $\operatorname{Rad}(M) = \operatorname{Rad}(A) \oplus \operatorname{Rad}(B)$ is a direct summand of M. So $A = \operatorname{Rad}(A) \oplus A'$ and $B = \operatorname{Rad}(B) \oplus B'$ for some submodules $A' \leq A$ and $B' \leq B$. Since $M = \operatorname{Rad}(M) \oplus A' \oplus B'$, $A' \oplus B' \cong N$. Let $f : A \to B$ be an R-homomorphism and let g denote the restriction of f to $\operatorname{Rad}(A)$. Clearly, $g(\operatorname{Rad}(A)) \subseteq \operatorname{Rad}(B)$. Hence $f(\operatorname{Rad}(A)) = \operatorname{Im} g \ll \operatorname{Rad}(B)$, since $\operatorname{Rad}(M)$ is DCAU. Moreover, note that $\operatorname{Hom}_R(A', B') = 0$, since N is abelian. Then $h(A') \subseteq \operatorname{Rad}(B)$, where h is the restriction of f to A'. Note that A' is cyclic. Then $h(A')$ is also cyclic, since $A'/\operatorname{Ker} h \cong h(A')$. Therefore $f(A') = h(A') \ll B$. It follows that $f(A) = f(\operatorname{Rad}(A)) + f(A') \ll B$, and the proof is complete. □

The following result will be of interest.

Proposition 3.6. *Let M be a module over a ring R.*

(i) *If $R \oplus M$ is a DCAU-module, then M is a DCAU-module with* $\operatorname{Rad}(M) = M$*. The converse holds when R is an abelian ring with* $\operatorname{J}(R) = 0$.

(ii) *If $R \oplus M$ is a DCAU-module and R is a right hereditary ring, then* $\operatorname{Hom}_R(M, R) = 0$.

Proof. (i) Assume that $R \oplus M$ is DCAU. Let $a \in M$ and consider the R-homomorphism $\varphi_a : R \to M$ defined by $\varphi_a(r) = ar$ for all $r \in R$. Then $\operatorname{Im} \varphi_a = aR \ll M$ and so $a \in \operatorname{Rad}(M)$. Hence $\operatorname{Rad}(M) = M$. The converse follows from Proposition 3.5.

(ii) Assume that R is right hereditary and $R \oplus M$ is DCAU. Let $f : M \to R$ be an R-homomorphism. Then $M/\operatorname{Ker} f \cong \operatorname{Im} f$ is a projective module. It follows that $\operatorname{Ker} f$ is a direct summand of M. So there exists a projective submodule P of M such that $M = \operatorname{Ker} f \oplus P$. Using (i), we obtain $\operatorname{Rad}(M) = M$ and hence $\operatorname{Rad}(P) = P$. Therefore $P = 0$, since every nonzero projective module contains a maximal submodule (see [2, Proposition 17.14]). Thus $f = 0$ and the result follows. □

Next, we investigate the class of modules for which every submodule is a DCAU-module.

Proposition 3.7. *Let M be a module. Then the following are equivalent:*

(i) *Every submodule of M is a DCAU-module.*

(ii) *Every submodule of M is abelian.*

Proof. (i) $\Rightarrow$ (ii) Without loss of generality it is sufficient to prove that M is abelian. Let $M = A \oplus B$ be a decomposition of M. Suppose that there exists a nonzero R-homomorphism $f : A \to B$. Then there exists $0 \neq a \in A$ such that $b = f(a) \neq 0$. Set $I_1 = \operatorname{Ann}_R(a)$ and $I_2 = \operatorname{Ann}_R(b)$. It is clear that $aR \oplus bR \cong R/I_1 \oplus R/I_2$. So, by assumption, $R/I_1 \oplus R/I_2$ is a DCAU-module. Note that $I_1 \subseteq I_2$. Let $g : R/I_1 \to R/I_2$ be the canonical epimorphism. Then $\operatorname{Im} g = R/I_2 \ll R/I_2$ and so $I_2 = R$. This yields $b = 0$, a contradiction. Therefore M is abelian.

(ii) $\Rightarrow$ (i) This is immediate. □

Corollary 3.8. *Let M be a nonzero R-module. Assume that one of the following two conditions is satisfied:*

(a) $\mathrm{Ann}_R(x) = 0$ *for every* $0 \neq x \in M$.
(b) *R is a right uniserial ring (that is, its right ideals are linearly ordered by inclusion).*

Then the following are equivalent:

(i) *Every submodule of M is a DCAU-module.*
(ii) *M is a uniform module.*

Proof. (i) ⇒ (ii) Let $0 \neq a \in M$ and $b \in M$ such that $aR \cap bR = 0$. By hypothesis, $I_1 = \mathrm{Ann}_R(a)$ and $I_2 = \mathrm{Ann}_R(b)$ are comparable. So without loss of generality we can assume that $I_1 \subseteq I_2$. Clearly, there exists an R-epimorphism $f : aR \to bR$. Since $aR \oplus bR$ is abelian by Proposition 3.7, it follows that $b = 0$. This implies that M is uniform.

(ii) ⇒ (i) This follows from the fact that every submodule of M is indecomposable. □

Corollary 3.9. *Let R be a commutative domain. Let M be a nonzero R-module which is not torsion. Then the following are equivalent:*

(i) *Every submodule of M is a DCAU-module.*
(ii) *M is uniform.*

Proof. (i) ⇒ (ii) By assumption, M contains an element x_0 such that $\mathrm{Ann}_R(x_0) = 0$. Then $x_0R \cong R$ is a uniform R-module. We claim that x_0R is essential in M. Suppose that $x_0R \cap xR = 0$ for some $0 \neq x \in M$. Note that $xR \cong R/I$, where $I = \mathrm{Ann}_R(x)$. Hence, there exists a nonzero R-epimorphism $f : x_0R \to xR$. Since $x_0R \oplus xR$ is abelian by Proposition 3.7, we infer that $x = 0$, a contradiction. This proves the claim. It follows that M is a uniform module.

(ii) ⇒ (i) This is clear. □

Compare Corollaries 3.8 and 3.9 with the following example.

Example 3.10. Let $\mathbf{P}$ be a set of prime numbers of cardinality at least 2. Let $M = \bigoplus_{p \in \mathbf{P}} M_p$ such that for each $p \in \mathbf{P}$, either $M_p = \mathbb{Z}(p^\infty)$ or $M_p = \mathbb{Z}/p^{n_p}\mathbb{Z}$ for some positive integer n_p. It is easily seen that every submodule of M has the same form as M. From [32, Theorem 3.10], it follows that every submodule of M is abelian. Hence every submodule of M is DCAU. On the other hand, M need not be a uniform module.

Now we move to the second part of this section, which deals with DCEU-modules. We begin with the following easy lemma.

Lemma 3.11. *Let M be a nonzero module. Then:*

(i) *For any non-essential submodule N of M, the module $M \oplus M/N$ is not a DCEU-module.*
(ii) *For any nonzero submodule L of M, the module $M \oplus L$ is not a DCEU-module.*

In the next three results we consider specific cases where the direct sum of two DCEU-modules is again a DCEU-module. The first one is a dualization of Proposition 3.3.

Proposition 3.12. *Let R be any ring and let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 satisfying the following conditions:*

(a) *M_1 is a DCEU-module having the finite internal exchange property;*
(b) *M_2 is an abelian module;*
(c) $\operatorname{Hom}_R(M_1, M_2) = 0$*; and*
(d) $\operatorname{Ker} f \leq^{\text{ess}} M_2$ *for every* $f \in \operatorname{Hom}_R(M_2, M_1)$.

Then M is a DCEU-module.

Proof. Let $M = A \oplus B$ be a decomposition of M. As in the proof of Proposition 3.3, there exist submodules $A_1 \leq A$ and $B_1 \leq B$ such that $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, $A_2 \oplus B_2 \cong M_2$ is abelian and $A_1 \oplus B_1 \cong M_1$ is DCEU. Let $f : A \to B$ be an R-homomorphism. Since $\operatorname{Hom}_R(M_1, M_2) = 0$, we have $\operatorname{Hom}_R(A_1, B_2) = 0$. Thus $f_1(A_1) \subseteq B_1$, where f_1 denotes the restriction of f to A_1. Hence $\operatorname{Ker} f_1 \leq^{\text{ess}} A_1$ as $A_1 \oplus B_1$ is DCEU. Let f_2 denote the restriction of f to A_2. Since $A_2 \oplus B_2$ is abelian, it follows that $f_2(A_2) \subseteq B_1$. Consider now the R-homomorphism $h_2 : A_2 \oplus B_2 \to A_1 \oplus B_1$ defined by $h_2(a_2) = f_2(a_2)$ and $h_2(b_2) = 0$ for all $a_2 \in A_2$ and $b_2 \in B_2$. Then $\operatorname{Ker} h_2 = \operatorname{Ker} f_2 \oplus B_2 \leq^{\text{ess}} A_2 \oplus B_2$ by (d). Hence $\operatorname{Ker} f_2 \leq^{\text{ess}} A_2$ by [2, Proposition 5.20 (2)]. Therefore $\operatorname{Ker} f_1 \oplus \operatorname{Ker} f_2 \leq^{\text{ess}} A_1 \oplus A_2 = A$. But $\operatorname{Ker} f_1 \oplus \operatorname{Ker} f_2 \subseteq \operatorname{Ker} f \subseteq A$, so $\operatorname{Ker} f \leq^{\text{ess}} A$. Similarly, $\operatorname{Ker} g \leq^{\text{ess}} B$ for every R-homomorphism $g : B \to A$. This completes the proof. □

The next corollary is an analogue of Corollary 3.4. It provides examples of DCEU-modules.

Corollary 3.13. *Let an R-module $M = N \oplus S$ be the direct sum of an abelian submodule N and a simple submodule S. Then M is a DCEU-module if and only if* $\operatorname{Hom}_R(S, N) = 0$.

Proof. The necessity is clear. Conversely, in view of Proposition 3.12, it suffices to prove that $\operatorname{Ker} f \leq^{\text{ess}} N$ for every $f \in \operatorname{Hom}_R(N, S)$. Assume, to the contrary, that there exists $0 \neq f \in \operatorname{Hom}_R(N, S)$ such that $\operatorname{Ker} f$ is not essential in N. This implies that $\operatorname{Ker} f \cap L = 0$ for some nonzero submodule L of N. Since $N/\operatorname{Ker} f \cong S$, $\operatorname{Ker} f$ is a maximal submodule of N. Thus $N = \operatorname{Ker} f \oplus L$ and so $L \cong S$. This contradicts the fact that $\operatorname{Hom}_R(S, N) = 0$. □

Proposition 3.14. *Let M be an R-module such that $M = Z(M) \oplus N$ for some non-singular submodule $N \leq M$. Then the following statements are equivalent:*

(i) *M is a DCEU-module.*
(ii) *$Z(M)$ is a DCEU-module and N is an abelian module.*

Proof. (i) ⇒ (ii) This follows from Proposition 2.7 and [19, Proposition 3.20].

(ii) ⇒ (i) Let $M = A \oplus B$ be a decomposition of M. Then $Z(M) = Z(A) \oplus Z(B)$ is a direct summand of M. Hence $Z(A)$ is a direct summand of A and $Z(B)$ is a direct summand of B. Let $A' \leq A$ and $B' \leq B$ such that $A = Z(A) \oplus A'$

and $B = Z(B) \oplus B'$. Now take an R-homomorphism $f : A \to B$ and let g denote the restriction of f to $Z(A)$. Note that $g(Z(A)) \subseteq Z(B)$ by [14, Proposition 1.22 (b)]. Hence $\operatorname{Ker} g \leq^{\text{ess}} Z(A)$, since $Z(M)$ is DCEU. Moreover, note that $\operatorname{Hom}_R(A', B') = 0$, since $A' \oplus B' \cong N$ is abelian. It follows that $h(A') \subseteq Z(B)$, where h denotes the restriction of f to A'. Therefore $A'/\operatorname{Ker} h \cong \operatorname{Im} h$ is singular. Since A' is non-singular, it follows from [14, Proposition 1.21] that $\operatorname{Ker} h \leq^{\text{ess}} A'$. It is easily seen that $\operatorname{Ker} g \oplus \operatorname{Ker} h \subseteq \operatorname{Ker} f$. Hence $\operatorname{Ker} f \leq^{\text{ess}} A$ (see [2, Proposition 5.20 (2)]), and the proof is complete. □

The next result yields information about the question: When is a module M with a direct summand isomorphic to R_R a DCEU-module?

Proposition 3.15. *Let M be a module over a ring R. If $R \oplus M$ is a DCEU-module, then M is a singular DCEU-module. The converse holds when R is an abelian right non-singular ring. In particular, if R is a commutative domain, then $R \oplus M$ is a DCEU-module if and only if M is a torsion DCEU-module.*

Proof. Suppose that $R \oplus M$ is a DCEU-module. Take $x \in M$ and consider the R-homomorphism $f : R \to M$ defined by $f(r) = xr$ for all $r \in R$. Hence $\operatorname{Ker} f = \operatorname{Ann}_R(x) \leq^{\text{ess}} R_R$. It follows that M is a singular module. Moreover, M is a DCEU-module by Proposition 2.7. The converse follows from Proposition 3.14 and the last assertion is immediate. □

Next, we will be concerned with injective DCEU-modules. The following result should be compared with Example 2.9 (ii).

Proposition 3.16. *If E is an injective DCEU-module, then every submodule of E is DCEU.*

Proof. Let E be an injective DCEU-module and let M be a submodule of E. Then $E(M)$ is a direct summand of E. Therefore $E(M)$ is DCEU by Proposition 2.7. We want to prove that M is DCEU. So without loss of generality we can assume that $E = E(M)$. Take a decomposition $M = M_1 \oplus M_2$ of M and let $f : M_1 \to M_2$ be an R-homomorphism. Clearly, $E = E_1 \oplus E_2$, where $E_i = E(M_i)$ $(i = 1, 2)$. Since E_2 is injective, there exists an R-homomorphism $g : E_1 \to E_2$ such that $g(x_1) = f(x_1)$ for all $x_1 \in M_1$. By hypothesis, $\operatorname{Ker} g \leq^{\text{ess}} E_1$. But $\operatorname{Ker} g \cap M_1 \subseteq \operatorname{Ker} f$, hence $\operatorname{Ker} f \leq^{\text{ess}} M_1$. The result follows. □

Recall that a module M is called *quasi-continuous* if it satisfies the following two conditions:

- (C_1) Every submodule of M is essential in a direct summand of M.
- (C_3) Whenever N and L are direct summands of M with $N \cap L = 0$, $N + L$ is a direct summand of M.

It is well known that every quasi-injective module is quasi-continuous. The next result will be useful to characterize when an injective module that has an essential socle is DCEU.

Proposition 3.17. *Let M be a quasi-continuous module and let E be an injective hull of M. Then M is a DCEU-module if and only if so is E.*

Proof. The sufficiency follows from Proposition 3.16. Conversely, suppose that M is DCEU. Let $E = E_1 \oplus E_2$ be a decomposition of E and let $f : E_1 \to E_2$ be an R-homomorphism. By [28, Theorem 2.8], we have $M = M_1 \oplus M_2$, where $M_i = M \cap E_i$ for each $i \in \{1, 2\}$. Consider the R-homomorphism $g : E \to E$ defined by $g(x_1+x_2) = f(x_1)+x_2$ for all $x_1 \in E_1$ and $x_2 \in E_2$. It is easy to check that g is an idempotent endomorphism of $E(M)$. Using again [28, Theorem 2.8], we conclude that $g(M) \subseteq M$. Hence $g(M_1) \subseteq M_2$. Now consider the R-homomorphism $h : M_1 \to M_2$ defined by $h(a_1) = g(a_1)$ for all $a_1 \in M_1$. Since M is a DCEU-module, it follows that $\operatorname{Ker} h \leq^{\text{ess}} M_1$. But $M_1 \leq^{\text{ess}} E_1$, so $\operatorname{Ker} h \leq^{\text{ess}} E_1$. Thus $\operatorname{Ker} f \leq^{\text{ess}} E_1$ as $\operatorname{Ker} h \subseteq \operatorname{Ker} f$. This shows that E is DCEU. □

Combining Example 2.3 and Propositions 3.16 and 3.17, we obtain the following two corollaries. The first one provides more examples of DCEU-modules.

Corollary 3.18. *Let M be a semisimple module which is a direct sum of pairwise non-isomorphic simple submodules. Then every submodule of $E(M)$ is a DCEU-module.*

Corollary 3.19. *Let M be an injective module having an essential socle. Then M is a DCEU-module if and only if so is* $\operatorname{Soc}(M)$ *(i.e.,* $\operatorname{Soc}(M)$ *is SSF).*

Example 3.20. Let R be a ring that has a unique simple right R-module S (up to isomorphism) (e.g., R can be a local ring). Let M be an injective R-module with essential socle. Then M is a DCEU-module if and only if $M \cong E(S)$. If, moreover, R is a right semiartinian ring, then $E(S)$ is the only injective DCEU-R-module (up to isomorphism).

Let E and M be two R-modules. Recall that E is said to be a *simple M-injective* module if for any submodule X of M and any R-homomorphism $f : X \to E$ such that $\operatorname{Im} f$ is simple, there exists an R-homomorphism $\overline{f} : M \to E$ such that $\overline{f}_{|X} = f$ (see [30, p. 156]). The next result gives a characterization for a module that has an essential socle to be a DCEU-module.

Proposition 3.21. *Let M be a module with essential socle. Then the following are equivalent:*

(i) *M is a DCEU-module.*
(ii) *Given any non-trivial decomposition $M = A \oplus B$ of M, if S is a simple submodule of A, then no R-monomorphism $f : S \to B$ can be lifted to an R-homomorphism $g : A \to B$ (in particular, B is not simple A-injective).*
(iii) *For every non-trivial decomposition $M = A \oplus B$ of M and every R-homomorphism $f : A \to B$, $f(\operatorname{Soc}(A)) = 0$.*

Proof. (i) ⇒ (ii) Let $M = A \oplus B$ with $A \neq 0$ and $B \neq 0$. Assume that there exist a simple submodule S of A and an R-monomorphism $f : S \to B$ such that f can be lifted to an R-homomorphism $g : A \to B$. Then $g(S) = f(S) \neq 0$ and hence S is not contained in $\operatorname{Ker} g$. This implies that $\operatorname{Ker} g$ is not essential in A. Therefore M is not a DCEU-module, a contradiction.

(ii) ⇒ (i) Let $M = A \oplus B$ be a non-trivial decomposition of M. From (ii), it follows that for any R-homomorphism $h : A \to B$, we have $h(S) = 0$ for every simple submodule S of A. This yields $\mathrm{Soc}(A) \subseteq \operatorname{Ker} h$. Note that $\mathrm{Soc}(A) \leq^{\mathrm{ess}} A$, since $\mathrm{Soc}(M) \leq^{\mathrm{ess}} M$. Hence $\operatorname{Ker} h \leq^{\mathrm{ess}} A$. Thus M is a DCEU-module.

(i) $\Leftrightarrow$ (iii) This follows from the definition of a DCEU-module and the fact that $\mathrm{Soc}(A) \leq^{\mathrm{ess}} A$ for every direct summand A of M. □

4. Modules over Dedekind domains

In this section, we shed some light on the structure of DCAU- and DCEU-modules over Dedekind domains. Recall that a commutative domain R is called *h-local* if every nonzero element of R is an element of only finitely many maximal ideals of R and every nonzero prime ideal of R is contained in only one maximal ideal of R. Examples of h-local domains include commutative local domains and Dedekind domains. Let M be a module over a commutative h-local domain R. The set $T(M) = \{x \in M \mid \mathrm{Ann}_R(x) \neq 0\}$ is a submodule of M which is called the torsion submodule of M. If $T(M) = M$, then M is said to be a torsion module; and M is torsion-free precisely when $T(M) = 0$. For every maximal ideal $\mathfrak{m}$ of R, we will denote by $T_{\mathfrak{m}}(M)$ the set $\{x \in M \mid x = 0$ or the only maximal ideal over $\mathrm{Ann}_R(x)$ is $\mathfrak{m}\}$. Note that if $M = T_{\mathfrak{m}}(M)$ for some maximal ideal $\mathfrak{m}$ of R, then M can be regarded as a module over the localization $R_{\mathfrak{m}}$. Moreover, in this case, the submodules of M are the same whether M is regarded as an R-module or as an $R_{\mathfrak{m}}$-module (see [5, Corollary 2.7]).

Proposition 4.1. *Let R be a commutative h-local domain and let M be a torsion R-module. Then the following statements are equivalent:*

(i) *M is a DCAU-(DCEU-)R-module.*
(ii) *$T_{\mathfrak{m}}(M)$ is a DCAU-(DCEU-)R-module for every maximal ideal $\mathfrak{m}$ of R.*
(iii) *$T_{\mathfrak{m}}(M)$ is a DCAU-(DCEU-)$R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let Ω denote the set of all maximal ideals of R. Note that $M = \bigoplus_{\mathfrak{m} \in \Omega} T_{\mathfrak{m}}(M)$ by [5, Theorem 2.6].

(i) $\Rightarrow$ (ii) This follows from Proposition 2.7.

(ii) $\Rightarrow$ (i) It is not difficult to see that $\mathrm{Hom}_R(T_{\mathfrak{m}_1}(M), T_{\mathfrak{m}_2}(M)) = 0$ for every $\mathfrak{m}_1 \neq \mathfrak{m}_2 \in \Omega$. Now this implication follows from [19, Propositions 3.8 and 4.15].

(ii) $\Leftrightarrow$ (iii) This is obvious. □

In the remainder of this section, R denotes a Dedekind domain which is not a field, with quotient field Q. Let $\mathbf{P}$ denote the set of all nonzero prime ideals of R and take $\mathfrak{p} \in \mathbf{P}$. Let M be an R-module. The set $T_{\mathfrak{p}}(M) = \{x \in M \mid \mathfrak{p}^n x = 0$ for some integer $n \geq 0\}$ is a submodule of M which is called the $\mathfrak{p}$-primary component of M. Note that $T(M) = \bigoplus_{\mathfrak{p} \in \mathbf{P}} T_{\mathfrak{p}}(M)$. We will denote by $R(\mathfrak{p}^\infty)$ the $\mathfrak{p}$-primary component of the torsion R-module Q/R. By [18, Lemma 2.4], $R(\mathfrak{p}^\infty)$ is a hollow module. For an R-module M, it is well known that M is injective if and only if $\mathrm{Rad}(M) = M$ if and only if M is a direct sum of copies of Q and $R(\mathfrak{p}^\infty)$ for various nonzero prime ideals $\mathfrak{p}$ (see [18, Lemma 2.1]). For every $\mathfrak{p} \in \mathbf{P}$, we define $\mathfrak{p}^0 = R$. The next lemma is just a slight reformulation of [20, Theorem 9].

Lemma 4.2. *Let R be a Dedekind domain and let $\mathfrak{p}$ be a nonzero prime ideal of R. If M is an R-module such that $T_{\mathfrak{p}}(M) \neq 0$, then M possesses a direct summand which is isomorphic either to $R(\mathfrak{p}^\infty)$ or to $R/\mathfrak{p}^n$ for some positive integer n.*

Proof. This follows by the same method as in [20, Theorem 9]. □

Theorem 4.3. *Let R be a Dedekind domain and let M be a DCAU-(DCEU-)R-module. Then:*

(i) *For any nonzero prime ideal $\mathfrak{p}$ of R for which $T_{\mathfrak{p}}(M) \neq 0$, $T_{\mathfrak{p}}(M)$ is a direct summand of M and $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)^a \oplus (R/\mathfrak{p}^n)^b$ with $a, b \in \{0,1\}$ ($T_{\mathfrak{p}}(M) \cong R/\mathfrak{p}^n$ or $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)$) for some positive integer n.*
(ii) *$T(M)$ is a DCAU-(DCEU-)R-module.*

Proof. (i) Let $\mathfrak{p}$ be a nonzero prime ideal of R such that $T_{\mathfrak{p}}(M) \neq 0$. Then, by Lemma 4.2, M has a direct summand A such that $A \subseteq T_{\mathfrak{p}}(M)$ and $A \cong R(\mathfrak{p}^\infty)$ or $A \cong R/\mathfrak{p}^n$ for some positive integer n. Let B be a submodule of M such that $M = A \oplus B$. Assume that $T_{\mathfrak{p}}(B) \neq 0$. So, using again Lemma 4.2, there exists a direct summand C of B such that $C \cong R(\mathfrak{p}^\infty)$ or $C \cong R/\mathfrak{p}^k$ for some positive integer k. Note that $B = C \oplus D$ for some submodule $D \leq B$. Then $M = A \oplus C \oplus D$. It follows that $A \oplus C$ is a DCAU-(DCEU-)R-module (see Proposition 2.7). Moreover, $A \oplus C$ is isomorphic to one of the following modules:

$$\begin{aligned} X_1 &= R/\mathfrak{p}^n \oplus R/\mathfrak{p}^k, \\ X_2 &= R/\mathfrak{p}^n \oplus R(\mathfrak{p}^\infty), \\ X_3 &= R/\mathfrak{p}^k \oplus R(\mathfrak{p}^\infty), \quad \text{or} \\ X_4 &= R(\mathfrak{p}^\infty) \oplus R(\mathfrak{p}^\infty). \end{aligned}$$

Without loss of generality we can assume that $n \leq k$. Therefore $R/\mathfrak{p}^n$ embeds in $R/\mathfrak{p}^k$ and $R/\mathfrak{p}^n$ is a homomorphic image of $R/\mathfrak{p}^k$. So X_1 is neither DCAU nor DCEU. Moreover, it is clear that X_4 is neither DCAU nor DCEU. In addition, X_2 and X_3 are not DCEU, since $R/\mathfrak{p}^l$ embeds in $R(\mathfrak{p}^\infty)$ for any positive integer l.

In conclusion, if M is DCEU, then $T_{\mathfrak{p}}(B) = 0$ and hence $T_{\mathfrak{p}}(M) = A$. On the other hand, if M is DCAU, then by an argument similar to the one provided above, we obtain that $T_{\mathfrak{p}}(D) = 0$ and we have the following two cases:

Case 1: $T_{\mathfrak{p}}(B) \neq 0$. In this case, $T_{\mathfrak{p}}(M) = A \oplus C \cong R/\mathfrak{p}^l \oplus R(\mathfrak{p}^\infty)$ for some positive integer l.
Case 2: $T_{\mathfrak{p}}(B) = 0$. In this case, we have $T_{\mathfrak{p}}(M) = A$.

Thus, we get the desired result.

(ii) From the statement (i) and Proposition 2.7, it follows that each $T_{\mathfrak{p}}(M)$ is a DCAU-(DCEU-)R-module. Therefore, applying Proposition 4.1, we conclude that $T(M)$ is a DCAU-(DCEU-)R-module. □

Corollary 4.4. *Let R be a Dedekind domain and let M be a torsion R-module. Then the following conditions are equivalent:*

(i) *M is a DCAU-(DCEU-)R-module.*
(ii) *For every nonzero prime ideal $\mathfrak{p}$ of R, $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)^a \oplus (R/\mathfrak{p}^n)^b$ with $a, b \in \{0,1\}$ ($T_{\mathfrak{p}}(M) \cong R/\mathfrak{p}^n$ or $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)$) for some non-negative integer n.*

Proof. (i) $\Rightarrow$ (ii) This follows from Theorem 4.3.

(ii) $\Rightarrow$ (i) Let $\mathfrak{p}$ be a nonzero prime ideal of R and let n be a positive integer. Note that every indecomposable module is both DCAU and DCEU. So $R(\mathfrak{p}^\infty)$ and $R/\mathfrak{p}^n$ are both DCAU and DCEU. Moreover, we have $\mathrm{Hom}_R(R(\mathfrak{p}^\infty), R/\mathfrak{p}^n) = 0$ and every proper submodule of $R(\mathfrak{p}^\infty)$ is small. Hence $R(\mathfrak{p}^\infty) \oplus R/\mathfrak{p}^n$ is a DCAU-module by Example 2.4. Now the implication follows from Proposition 4.1. □

The next result is an immediate consequence of Proposition 3.14.

Proposition 4.5. *Let R be a Dedekind domain and let M be an R-module such that $T(M)$ is a direct summand of M. Then the following statements are equivalent:*

(i) *M is a DCEU-R-module.*
(ii) *$T(M)$ is a DCEU-R-module and $M/T(M)$ is an abelian module.*

Remark 4.6. Let R be a Dedekind domain.

(i) Note that the torsion submodule $T(M)$ of a DCAU- or a DCEU-R-module M need not be a direct summand of M. For example, the torsion submodule of the $\mathbb{Z}$-module $M = \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$ is $T(M) = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$, which is clearly not a direct summand of M. On the other hand, M is both a DCAU- and a DCEU-module (see Example 2.2).

(ii) The dual of Proposition 4.5 need not be true in general. To see this, consider the R-module $M = M_1 \oplus M_2$ such that $M_1 \cong R$ and $M_2 \cong R/\mathfrak{p}^n$, where $\mathfrak{p}$ is a nonzero prime ideal of R and n is a positive integer. Since $R/\mathfrak{p}^n$ is a homomorphic image of R, it follows that M is not a DCAU-module. On the other hand, it is clear that R is an abelian R-module and $T(M) = M_2 \cong R/\mathfrak{p}^n$ is DCAU.

Proposition 4.7. *Let R be a Dedekind domain with quotient field Q and let $M = M_1 \oplus M_2$ be an R-module such that $M_1 \cong R$. Then the following are equivalent:*

(i) *M is a DCAU-R-module.*
(ii) *M_2 is injective such that either M_2 is torsion-free and $M_2 \cong Q$ or M_2 is a torsion module such that for each nonzero prime ideal $\mathfrak{p}$ of R, $T_{\mathfrak{p}}(M_2) = 0$ or $T_{\mathfrak{p}}(M_2) \cong R(\mathfrak{p}^\infty)$.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 3.6 (i), we have $\mathrm{Rad}(M_2) = M_2$. Using [18, Lemma 2.1], it follows that M_2 is isomorphic to a direct sum of copies of Q and $R(\mathfrak{p}^\infty)$ for various nonzero prime ideals $\mathfrak{p}$ of R. Since $R(\mathfrak{p}^\infty)$ is a homomorphic image of Q and M is a DCAU-module, we infer that M_2 cannot be mixed (i.e., M_2 is either torsion or torsion-free). It is clear that if M_2 is nonzero and torsion-free, then $M_2 \cong Q$. The torsion case follows from Corollary 4.4.

(ii) $\Rightarrow$ (i) **Case 1:** Assume that M_2 is torsion-free. Hence $M_2 \cong Q$ has the exchange property by [28, Theorem 1.21]. Therefore, for any non-trivial decomposition $M = A \oplus B$ of M, it is easily seen that one of the summands is isomorphic to R and the other is isomorphic to Q. In addition, note that $\mathrm{Rad}(Q) = Q$ and R is noetherian. Then $\mathrm{Hom}_R(Q, R) = 0$ and $\mathrm{Im}\, f \ll Q$ for any $f \in \mathrm{Hom}_R(R, Q)$. Using these arguments, we obtain that the R-module M is a DCAU-module.

Case 2: Assume that M_2 is torsion with the stated property. Then M_2 is an abelian module by [32, Theorem 3.10]. Let $M = A \oplus B$ be a non-trivial decomposition of M. Then $M_2 = T(M) = T(A) \oplus T(B)$ is a direct summand of M. Therefore there exist $A' \leq A$ and $B' \leq B$ such that $A = T(A) \oplus A'$ and $B = T(B) \oplus B'$. This implies that $A' \oplus B' \cong R$. Since R is indecomposable, without loss of generality we can assume that $A' \cong R$ and $B' = 0$. This yields $B = T(B) \subseteq M_2$ and hence $\mathrm{Rad}(B) = B$ by [18, Lemma 2.1]. Since M_2 is abelian, $\mathrm{Hom}_R(T(B), T(A)) = 0$. Moreover, it is clear that $\mathrm{Hom}_R(T(B), R) = 0$. Thus $\mathrm{Hom}_R(B, A) = 0$. Now take $f \in \mathrm{Hom}_R(A, B)$. Then $f(T(A)) = 0$ as $T(M)$ is abelian. It follows that $f(A) = f(A')$ is cyclic and so $\mathrm{Im}\, f \ll B$ since $\mathrm{Rad}(B) = B$. Consequently, M is a DCAU-module. □

The next result is analogous to Proposition 4.7.

Corollary 4.8. *Let R be a Dedekind domain and let $M = M_1 \oplus M_2$ be an R-module such that $M_1 \cong R$. Then the following are equivalent:*

(i) *M is a DCEU-R-module.*
(ii) *M_2 is a torsion module such that for each nonzero prime ideal $\mathfrak{p}$ of R, either $T_{\mathfrak{p}}(M_2) \cong R(\mathfrak{p}^\infty)$ or $T_{\mathfrak{p}}(M_2) \cong R/\mathfrak{p}^n$ for some non-negative integer n.*

Proof. This follows from Proposition 3.15 and Corollary 4.4. □

Recall that a submodule N of a module M is called *fully invariant* if $f(N) \subseteq N$ for every $f \in \mathrm{End}_R(M)$. According to [32], a module M is called a *duo module* provided every submodule of M is fully invariant. Let R be a Dedekind domain. Then every R-module M with $\mathrm{Ann}_R(M) \neq 0$ is a direct sum of cyclic submodules by [36, Theorem 6.14]. Next, we characterize DCAU- and DCEU-R-modules which are direct sums of cyclic submodules.

Corollary 4.9. *Let R be a Dedekind domain and let M be an R-module which is a direct sum of cyclic submodules. Then the following are equivalent:*

(i) *M is a DCAU-module.*
(ii) *M is an abelian module.*
(iii) *M is a duo module.*
(iv) *Either $M \cong R$ or there exist distinct nonzero prime ideals $\mathfrak{p}_i$ $(i \in I)$ of R and submodules M_i $(i \in I)$ of M such that $M = \bigoplus_{i \in I} M_i$ and, for each $i \in I$, $M_i \cong R/{\mathfrak{p}_i}^{n_i}$ for some non-negative integer n_i.*

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) These implications are obvious.

(i) $\Rightarrow$ (iv) This follows from Proposition 2.7, Corollary 4.4, and Remark 4.6 (ii).

(iv) $\Rightarrow$ (iii) Use [32, Theorem 3.10]. □

The next result is an immediate consequence of Corollaries 4.4 and 4.8.

Corollary 4.10. *Let R be a Dedekind domain and let M be an R-module which is a direct sum of cyclic submodules. Then the following are equivalent:*

(i) *M is a DCEU-module.*

(ii) *There exist distinct nonzero prime ideals $\mathfrak{p}_i$ $(i \in I)$ of R and submodules F and M_i $(i \in I)$ of M such that $M = F \oplus (\bigoplus_{i\in I} M_i)$, either $F = 0$ or $F \cong R$ and, for each $i \in I$, $M_i \cong R/{\mathfrak{p}_i}^{n_i}$ for some non-negative integer n_i.*

In the next result we characterize the class of R-modules whose submodules are DCAU.

Proposition 4.11. *Let R be a Dedekind domain with quotient field Q, and let M be an R-module. Then the following are equivalent:*

(i) *Every submodule of M is a DCAU-module.*

(ii) (a) *M is a torsion-free module which is isomorphic to an R-submodule of Q, or*

(b) *M is a torsion module such that for every nonzero prime ideal $\mathfrak{p}$ of R, either $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^\infty)$ or $T_{\mathfrak{p}}(M) \cong R/\mathfrak{p}^n$ for some non-negative integer n.*

Proof. (i) $\Rightarrow$ (ii) Assume that M is not torsion. Then M is uniform by Corollary 3.9. Therefore M is torsion-free and it is isomorphic to an R-submodule of Q by [22, Theorem 4.1]. Now assume that M is a torsion module. From Corollary 4.4, it follows that M has the structure described in condition (b) since for every nonzero prime ideal $\mathfrak{p}$ of R and every positive integer n, the R-module $A = R(\mathfrak{p}^\infty) \oplus R/\mathfrak{p}^n$ has a submodule B which is isomorphic to $R/\mathfrak{p}^n \oplus R/\mathfrak{p}^n$ and clearly B is not a DCAU-module.

(ii) $\Rightarrow$ (i) (a) This follows from the fact that every R-submodule of Q is indecomposable.

(b) Let $\mathfrak{p}$ be a nonzero prime ideal of R and let n be a positive integer. Note that if N is an R-module which is isomorphic either to $R(\mathfrak{p}^\infty)$ or to $R/\mathfrak{p}^n$, then for every nonzero proper submodule $L \leq N$, $L \cong R/\mathfrak{p}^s$ for some positive integer s. Now use Corollary 4.4. □

5. Modules over discrete valuation rings

Unless otherwise stated, throughout this section R will denote a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$ and quotient field Q. Our aim in this section is to describe the structure of DCAU- and DCEU-R-modules. We begin by investigating the structure of DCAU-modules. It is well known that an R-module M is divisible if and only if $pM = M$ if and only if M is injective if and only if M is isomorphic to a direct sum of copies of Q and Q/R (see, for example, [18, Lemma 2.1] and [20, Theorem 7]). A module M is said to be *reduced* if M does not have nonzero divisible submodules. Recall that an R-module M is said to be *bounded* if $p^n M = 0$ for some positive integer n. Note that for any R-module M, $T(M) = T_{\mathfrak{m}}(M)$.

Lemma 5.1. *Let $M = M_1 \oplus M_2$ be an R-module such that $M_1 \cong R/\mathfrak{m}^n$ for some positive integer n and M_2 is a nonzero torsion-free submodule of M. Then the following are equivalent:*

(i) *M is a DCAU-module.*
(ii) *$M_2 \cong Q$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that M_2 is not divisible. Then $M_2 \neq pM_2$ and hence $M_2/p^nM_2 \neq 0$. We claim that $p^nM_2 \neq p^{n-1}M_2$. Suppose, to the contrary, that $p(p^{n-1}M_2) = p^{n-1}M_2$. Then $p^{n-1}M_2$ is injective. Therefore $M_2 = p^{n-1}M_2 \oplus N$ for some submodule $N \leq M_2$. This yields $p^{n-1}N = 0$. But M_2 is torsion-free, hence $N = 0$ and so $M_2 = p^{n-1}M_2$, a contradiction. It follows that $p^{n-1}(M_2/p^nM_2) \neq 0$. Moreover, since M_2/p^nM_2 is a bounded R-module, $M_2/p^nM_2 = \bigoplus_{i \in I} C_i$ is a direct sum of cyclic submodules C_i $(i \in I)$ by [23, Theorem 7.1]. Therefore there exists $j \in I$ such that $p^{n-1}C_j \neq 0$. But $p^nC_j = 0$, so $C_j \cong R/\mathfrak{m}^n$. This clearly implies that M_1 is a homomorphic image of M_2. This contradicts the fact that M is a DCAU-module. It follows that M_2 is divisible. Since M_2 is torsion-free, M_2 is isomorphic to a direct sum of copies of Q (see [23, Theorem 6.3]). In addition, note that M_2 is a DCAU-module by Proposition 2.7. Then $M_2 \cong Q$.

(ii) $\Rightarrow$ (i) Since M_1 is torsion and M_2 is torsion-free, clearly $\mathrm{Hom}_R(M_1, M_2) = 0$. Also, we have $\mathrm{Hom}_R(M_2, M_1) = 0$ as M_2 is divisible and M_1 is reduced. Moreover, both M_1 and M_2 are DCAU, since they are indecomposable. Now, using [19, Proposition 4.15], it follows that M is a DCAU-module. □

From the preceding lemma, we derive the structure of reduced DCAU-R-modules which are not torsion-free.

Proposition 5.2 ([21, Theorem 3.7])**.** *Let M be a nonzero reduced module which is not torsion-free. Then M is a DCAU-module if and only if $M \cong R/\mathfrak{m}^n$ for some positive integer n.*

Proof. The sufficiency is obvious. Conversely, assume that M is DCAU. By Theorem 4.3, there exists a submodule $L \leq M$ such that $M = T(M) \oplus L$ and $T(M) \cong R/\mathfrak{m}^n$ for some positive integer n. Note that L is a DCAU-module by Proposition 2.7. Since M is reduced, it follows from Lemma 5.1 that $L = 0$ and so $M \cong R/\mathfrak{m}^n$. □

Next, we deal with reduced torsion-free R-modules. We will need the following lemma.

Lemma 5.3. *Let M be a torsion-free R-module. Then M is reduced if and only if $\bigcap_{n\geq 1} p^nM = 0$.*

Proof. Assume that M is not reduced. Then M has a nonzero submodule N which is isomorphic to Q. Hence $N \subseteq \bigcap_{n\geq 1} p^nM$ and so $\bigcap_{n\geq 1} p^nM \neq 0$. This proves the sufficiency. Conversely, suppose that $\bigcap_{n\geq 1} p^nM \neq 0$ and take $0 \neq x \in \bigcap_{n\geq 1} p^nM$. Then for any positive integer n, there exists an element $x_n \in M$ such that $x = p^nx_n$. Moreover, x_n is unique as M is torsion-free. Now consider the R-homomorphism $f : Q \to M$ defined by $f(r/p^n) = rx_n$ for every $r \in R$ and every positive integer n.

A trivial verification shows that f is well defined and it is an R-monomorphism. This implies that M is not reduced, and the proof is complete. □

Proposition 5.4. *Let M be a torsion-free reduced R-module. Then the following are equivalent:*

(i) *M is a DCAU-module.*
(ii) *M is an abelian module.*

Proof. (i) ⇒ (ii) Let $M = A \oplus B$ be a non-trivial decomposition of M and let $f : A \to B$ be an R-homomorphism. Since M is a DCAU-module, we have $f(A) \ll B$ and so $f(A) \subseteq \mathrm{Rad}(B) = pB$. Let $C = \{b \in B \mid pb \in f(A)\}$. Clearly, C is a submodule of B with $pC = f(A)$. Consider the mapping $\theta : f(A) \to C$ defined by $\theta(pc) = c$ for all $c \in C$. Since M is torsion-free, it follows that θ is well defined and it is an R-isomorphism. Hence $h = \mu\theta f : A \to B$ is an R-homomorphism, where $\mu : C \to B$ denotes the inclusion map. Since M is a DCAU-module, we conclude that $h(A) \ll B$ and so $h(A) \subseteq pB$. But $h(A) = C$, hence $C \subseteq pB$. Therefore $f(A) = pC \subseteq p^2B$. We continue in this fashion to obtain that $f(A) \subseteq \bigcap_{n\geq 1} p^nB$. But $\bigcap_{n\geq 1} p^nB = 0$ by Lemma 5.3. Then $f(A) = 0$ and consequently M is abelian.
(ii) ⇒ (i) This is clear. □

Let M be an R-module over a Dedekind domain R. It is well known that $d(M)$, the sum of all divisible (injective) submodules of M, is an injective R-module. Moreover, there exists a reduced submodule N of M such that $M = d(M) \oplus N$ (see, for example, [20, Theorem 8]). Moreover, it is easily seen that for any decomposition $M = A \oplus B$ of M, we have $d(M) = d(A) \oplus d(B)$.

Lemma 5.5. *Let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $M_1 \cong Q$ and M_2 is an abelian reduced module. Assume that M_1 is not a homomorphic image of M_2. Then M is a DCAU-module.*

Proof. Let $M = A \oplus B$ be a non-trivial decomposition of M. Since $d(M) = d(A) \oplus d(B) = M_1$ is indecomposable, it follows that $d(A) = M_1$ or $d(B) = M_1$ and hence $M_1 \subseteq A$ or $M_1 \subseteq B$. Without loss of generality we can assume that $M_1 \subseteq A$. Therefore there exists a submodule $C \leq A$ such that $A = M_1 \oplus C$. Thus $M = M_1 \oplus C \oplus B$ and so $C \oplus B \cong M_2$ is abelian and reduced. Take an R-homomorphism $f : A \to B$. Then $f(C) = 0$ as $\mathrm{Hom}_R(C, B) = 0$. Moreover, $f(M_1) = 0$ since $f(M_1)$ is divisible and B is reduced. It follows that $f = 0$. Now take an R-homomorphism $g : B \to A$. Note that $g(B) \subseteq M_1$ since $\mathrm{Hom}_R(B, C) = 0$. By assumption, M_1 is not a homomorphic image of B. Then $g(B)$ is a proper submodule of M_1. But M_1 is a hollow module by [18, Lemma 2.4]. Therefore $g(B) \ll M_1$ and hence $g(B) \ll A$. This completes the proof. □

Theorem 5.6. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field Q. Let M be a nonzero R-module. Then the following are equivalent:*

(i) *M is a DCAU-module.*
(ii) *M is isomorphic to one of the following modules:*

(1) $Q^a \oplus R^b$, *$a, b \in \{0, 1\}$;*

(2) $(Q/R)^a \oplus R^b$, *$a, b \in \{0, 1\}$;*

(3) $Q^a \oplus (R/m^n)^b$, *$a, b \in \{0, 1\}$ and n is a positive integer;*

(4) $(Q/R)^a \oplus (R/\mathfrak{m}^n)^b$, *$a, b \in \{0, 1\}$ and n is a positive integer;*

(5) *$0 \neq L$ is abelian such that $d(L) = T(L) = 0$ and L is not isomorphic to R;*

(6) *$Q \oplus L$, where $0 \neq L$ is abelian such that $d(L) = T(L) = 0$, L is not isomorphic to R, and Q is not a homomorphic image of L.*

Proof. (i) ⇒ (ii) Clearly, $M = d(M) \oplus L$ for some reduced submodule L of M. Suppose that M is a DCAU-module. Then, both $d(M)$ and L are DCAU-modules by Proposition 2.7. Since the modules $Q \oplus Q$, $Q \oplus Q/R$ and $Q/R \oplus Q/R$ are not DCAU, we see that either $d(M) = 0$ or $d(M) \cong Q$ or $d(M) \cong Q/R$. On the other hand, if L is not torsion-free, then $L \cong R/\mathfrak{m}^n$ for some positive integer n (see Proposition 5.2). Now assume that L is torsion-free. Then L is abelian by Proposition 5.4. Moreover, note that the R-module $R \oplus R$ is not DCAU. So given a free R-module F, F is DCAU if and only if $F \cong R$. It follows that if $L \neq 0$ such that L is not isomorphic to R, then L is not a free R-module. Therefore, by [23, Theorem 9.4], L contains a free submodule B such that L/B is a nonzero injective module. This implies that Q/R is a homomorphic image of L and so $Q/R \oplus L$ is not a DCAU-module. Using all the arguments provided above, we conclude that M satisfies the condition (ii).

(ii) ⇒ (i) (1), (2), (3) and (4) follow from Example 2.4 (iv). (5) is trivial, and (6) follows from Lemma 5.5. □

A discrete valuation ring R is called a *complete discrete valuation ring* if R is a complete topological ring with respect to the p-adic topology. For example, the ring $\widehat{\mathbb{Z}}_p$ and $F[[x]]$ are complete discrete valuation rings, where $\widehat{\mathbb{Z}}_p = \mathbb{Q}_p^*$ is the ring of p-adic integers. Next, we present an explicit description of the structure of DCAU-modules over complete discrete valuation rings.

Corollary 5.7. *Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field Q. Let M be a nonzero R-module. Then the following are equivalent:*

(i) *M is a DCAU-module.*

(ii) *M is isomorphic to one of the following modules: $Q^a \oplus R^b$, $Q^a \oplus (R/\mathfrak{m}^n)^b$, $(Q/R)^a \oplus R^b$, or $(Q/R)^a \oplus (R/\mathfrak{m}^n)^b$ for some positive integer n with $a, b \in \{0, 1\}$.*

Proof. (i) ⇒ (ii) By Theorem 5.6, we only need to show that R has no nonzero abelian reduced torsion-free R-modules which are not isomorphic to R. To see this, assume, to the contrary, that there exists a nonzero abelian R-module L such that $T(L) = d(L) = 0$ and L is not isomorphic to R. Then, by [23, Corollary 11.6], there exist nonzero submodules A and B of L such that $L = A \oplus B$ and A is cyclic.

It is clear that $A \cong R$, since L is torsion-free. Using again [23, Corollary 11.6], we infer that B has a nonzero cyclic direct summand such that $B \cong R$. Hence L has a direct summand which is isomorphic to $R \oplus R$. This contradicts the fact that L is abelian.

(ii) $\Rightarrow$ (i) This follows from Theorem 5.6. □

The second part of this section will be devoted to the determination of the structure of DCEU-modules over a discrete valuation ring. We begin by showing a dualization of Proposition 5.2.

Proposition 5.8. *Let M be a nonzero reduced R-module which is not torsion-free. Then the following are equivalent:*

(i) *M is a DCEU-module.*
(ii) *$M = M_1 \oplus M_2$ such that $M_1 \cong R/\mathfrak{m}^n$ for some positive integer n and M_2 is an abelian reduced torsion-free submodule of M.*

Proof. (i) $\Rightarrow$ (ii) By [23, Corollary 7.3], M has a direct summand M_1 which is isomorphic to $R/\mathfrak{m}^n$ for some positive integer n. Let M_2 be a submodule of M such that $M = M_1 \oplus M_2$. Assume that M_2 is not torsion-free. Then similarly M_2 has a direct summand which is isomorphic to R/m^k for some positive integer k. There is no loss of generality in assuming that $k \leq n$. This clearly implies that $R/\mathfrak{m}^k$ is isomorphic to a submodule of $R/\mathfrak{m}^n$. On the other hand, since M is a DCEU-module, it follows from Proposition 2.7 that $R/\mathfrak{m}^n \oplus R/\mathfrak{m}^k$ is a DCEU-module. This is a contradiction (see Lemma 3.11 (ii)). Hence $T(M_2) = 0$. Now, using Proposition 2.7 and [19, Proposition 3.20], we deduce that M_2 is abelian.

(ii) $\Rightarrow$ (i) This follows from the fact that $T(M) = M_1$ is DCEU and Proposition 4.5. □

The following lemma will be used to establish our next theorem.

Lemma 5.9. *Let M be a nonzero abelian reduced torsion-free R-module. Then the following hold:*

(1) *$M \oplus (Q/R)$ is a DCEU-module.*
(2) *The following are equivalent:*
   (i) *$M \oplus Q \oplus (Q/R)$ is a DCEU-module.*
   (ii) *$M \oplus Q$ is a DCEU-module.*
   (iii) *$M \oplus Q$ is an abelian module.*
   (iv) *$\mathrm{Hom}_R(M, Q) = 0$.*

Proof. (1) By Proposition 4.5.

(2) (i) $\Rightarrow$ (ii) Use Proposition 2.7.

(ii) $\Rightarrow$ (iii) Since $M \oplus Q$ is torsion-free, this implication follows from [19, Proposition 3.20].

(iii) $\Rightarrow$ (iv) This is clear by [19, Lemma 3.17].

(iv) $\Rightarrow$ (i) Note that $\mathrm{Hom}_R(Q, M) = 0$ since M is reduced. Hence $M \oplus Q$ is abelian by [19, Propositions 3.8 and 3.20]. Now the result follows from Proposition 4.5 and the fact that Q/R is a torsion R-module. □

Theorem 5.10. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field Q. Then the following are equivalent for an R-module M:*

(i) *M is a DCEU-module.*

(ii) *$M = M_1 \oplus M_2 \oplus M_3$ such that $M_1 \cong Q^a \oplus (Q/R)^b$, $M_2 \cong (R/\mathfrak{m}^n)^c$ for some positive integer n, and M_3 is an abelian torsion-free reduced R-module, where $a, b, c \in \{0, 1\}$ with $(b, c) \neq (1, 1)$ and $\mathrm{Hom}_R(M_3, Q) = 0$ if $a = 1$.*

Proof. (i) $\Rightarrow$ (ii) Note that $M = d(M) \oplus N$ for some reduced submodule $N \leq M$. Moreover, $d(M)$ is isomorphic to a direct sum of copies of Q and Q/R. On the other hand, note that the modules Q^2 and $(Q/R)^2$ are not DCEU. Also, for every positive integer n, the module $(R/\mathfrak{m}^n) \oplus (Q/R)$ is not a DCEU-module (see Lemma 3.11 (ii)). Now this implication follows from Propositions 2.7 and 5.8, Lemma 5.9 and [19, Proposition 3.20].

(ii) $\Rightarrow$ (i) Suppose that M satisfies the stated condition. Using Proposition 2.7, we are reduced to proving the following cases:

Case 1: Assume $M_1 \cong Q \oplus Q/R$, $c = 0$, and $M_3 \neq 0$ with $\mathrm{Hom}_R(M_3, Q) = 0$. Then M is a DCEU-module by Lemma 5.9(2).

Case 2: Assume $M_1 \cong Q$, $M_2 \cong R/\mathfrak{m}^n$, and $M_3 \neq 0$ with $\mathrm{Hom}_R(M_3, Q) = 0$. Note that $M_1 \oplus M_3$ is an abelian module by Lemma 5.9(2). Now, applying Proposition 4.5, we conclude that M is a DCEU-module.

Case 3: Assume $M_1 = 0$, $M_2 \cong R/\mathfrak{m}^n$, and $M_3 \neq 0$. From Proposition 5.8, it follows that M is DCEU.

Case 4: Assume $M_1 \cong Q/R$, $c = 0$, and $M_3 \neq 0$. This case follows from Lemma 5.9 (1). ☐

Corollary 5.11. *Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field Q. Then the following are equivalent for a nonzero R-module M:*

(i) *M is a DCEU-module.*

(ii) *M is isomorphic to one of the following modules: $Q^a \oplus (Q/R)^b$, $Q^a \oplus (R/\mathfrak{m}^n)^b$, $(Q/R)^a \oplus R^b$, or $(R/\mathfrak{m}^n)^a \oplus R^b$ for some positive integer n with $a, b \in \{0, 1\}$.*

Proof. By arguments similar to those in the proof of Corollary 5.7 ((i) $\Rightarrow$ (ii)), it can easily be seen that a reduced torsion-free R-module L is DCEU if and only if $L \cong R$. Now the result follows from Theorem 5.10. ☐

The next remark should be compared with Remark 4.6.

Remark 5.12. Let M be a DCAU-(DCEU-)R-module. From Theorems 5.6 and 5.10, it follows that the torsion submodule $T(M)$ is a direct summand of M (see also Theorem 4.3).

The next example illustrates that the condition “R is complete” in the hypotheses of Corollaries 5.7 and 5.11 is not superfluous.

Example 5.13. Let R be a discrete valuation ring which is not complete. Then R has an indecomposable R-module M of rank ≥ 2 by [23, Theorem 19.8]. Clearly, M is not isomorphic to any one of the R-modules R, Q, Q/R, and $R/\mathfrak{m}^n$, $n \geq 1$

(see [23, Proposition 4.6]). On the other hand, M is both a DCAU- and a DCEU-module since M is indecomposable. We conclude from Theorems 5.6 and 5.10 that M is a torsion-free reduced R-module.

6. Rings whose DCAU-(DCEU-)modules are DCEU (DCAU)

A ring R is said to be *right DCAU* (*right DCEU*) if the right R-module R_R is DCAU (DCEU). We next characterize when a ring R is right DCAU (or right DCEU). The following proposition can be obtained by using [19, Lemma 2.2 and Corollary 4.5], but we will give a direct proof of this result.

Proposition 6.1. *The following are equivalent for a ring R with Jacobson radical* $\mathrm{J}(R)$*:*

(i) *R is right DCAU.*
(ii) *For every idempotent $e \in R$, $(1-e)Re = Re \cap (1-e)R \subseteq \mathrm{J}(R)$.*

Proof. It is clear that $(1-e)Re = Re \cap (1-e)R$.

(i) $\Rightarrow$ (ii) Let e be an idempotent of R. Take $y \in Re \cap (1-e)R$. Then $y = ae = (1-e)b$ for some elements $a, b \in R$. Consider the R-homomorphism $f : eR \to (1-e)R$ defined by $f(x) = ax$ for any $x \in eR$. Note that $f(er) = aer = (1-e)br$ for all $r \in R$. So f is well defined. Since R_R is a DCAU-R-module, it follows that $f(eR) \ll (1-e)R$. This implies that $y = ae = f(e) \in \mathrm{J}(R)$.

(ii) $\Rightarrow$ (i) Let $R_R = A \oplus B$ be a decomposition of R_R. Then there exists an idempotent $e \in R$ such that $A = eR$ and $B = (1-e)R$. Let $f : A \to B$ be an R-homomorphism. Hence there exists $b \in B = (1-e)R$ such that $f(er) = br$ for all $r \in R$. Thus $f(A) = bR$. Moreover, note that $b = f(e) = f(ee) = be$. Therefore $b \in Re \cap (1-e)R$. But $Re \cap (1-e)R \subseteq \mathrm{J}(R)$ by hypothesis. Hence $b \in \mathrm{J}(R) \cap B$. This yields $b \in \mathrm{Rad}(B)$ and so $\mathrm{Im}\, f = bR \ll B$. This completes the proof. □

If x is an element of a ring R, the right annihilator of x in R is the set $\mathrm{ann}_r(x) = \{r \in R \mid xr = 0\}$.

Proposition 6.2. *The following are equivalent for a ring R with right singular ideal* $\mathrm{Z}(R_R)$*:*

(i) *R is right DCEU.*
(ii) *For every idempotent $e \in R$, $(1-e)Re \subseteq \mathrm{Z}(R_R)$.*

Proof. (i) $\Rightarrow$ (ii) Let e be an idempotent of R. Take $a \in (1-e)Re$ and consider the R-homomorphism $f : eR \to (1-e)R$ defined by $f(x) = ax$ for all $x \in eR$. Clearly, f is well defined. Note that $\mathrm{Ker}\, f = \{ex \mid x \in R \text{ and } x \in \mathrm{ann}_r(a)\}$. Since R_R is a DCEU-module, it follows that $\mathrm{Ker}\, f$ is essential in eR. Therefore $eR/\mathrm{Ker}\, f$ is a singular module. So aR is a singular module. In particular, $a \in \mathrm{Z}(R_R)$. This shows that $(1-e)Re \subseteq \mathrm{Z}(R_R)$.

(ii) $\Rightarrow$ (i) Let $R_R = A \oplus B$ be a decomposition of R_R. Then there exists an idempotent $e \in R$ such that $A = eR$ and $B = (1-e)R$. Let $f : A \to B$ be an R-homomorphism. Hence there exists $b \in (1-e)R$ such that $f(er) = br$ for all $r \in R$. As $f(e(1-e)) = 0$, we have $b(1-e) = 0$ and so $b = be \in Re$. This yields $b \in Re \cap (1-e)R = (1-e)Re$. By hypothesis, it follows that $b \in \mathrm{Z}(R_R)$.

Hence $I = \mathrm{ann}_r(b)$ is essential in R_R. Moreover, it is clear that for any $x \in I$, we have $b(ex) = (be)x = bx = 0$ and $b((1-e)x) = be((1-e)x) = b(e(1-e))x = 0$. Consequently, $eI \subseteq I$ and $(1-e)I \subseteq I$. Therefore, $eI \oplus (1-e)I = I \leq^{\mathrm{ess}} R_R = eR \oplus (1-e)R$. This implies that $eI \leq^{\mathrm{ess}} eR$ by [2, Proposition 5.20 (2)]. But $\mathrm{Ker}\, f = eI$, so $\mathrm{Ker}\, f \leq^{\mathrm{ess}} eR$. This finishes the proof. □

Recall that a ring R is called *right continuous* if it satisfies the following two conditions:

(C_1) Every right ideal is essential in a direct summand of R_R.
(C_2) Whenever A and B are right ideals of R such that $A \cong B$ and B is a direct summand of R_R, A is a direct summand of R_R.

It is well known that every right self-injective ring is right continuous (see [28, Proposition 2.1]).

Corollary 6.3. *Let R be a ring such that $\mathrm{J}(R) = \mathrm{Z}(R_R)$ (e.g., R is right continuous). Then R is right DCAU if and only if R is right DCEU.*

Proof. This follows from Propositions 6.1 and 6.2 and [4, Corollary 2.1.30]. □

The next remark can also be obtained by using [19, Propositions 3.20 and 4.18].

Remark 6.4. From Propositions 6.1 and 6.2, it follows that if R is a right DCAU (resp., right DCEU) ring such that $\mathrm{J}(R) = 0$ (resp., $\mathrm{Z}(R_R) = 0$), then R is an abelian ring (see, for example, [4, Proposition 1.2.2]).

Recall that a ring R is said to be *strongly regular* if for each $x \in R$ there exists $y \in R$ such that $x^2 y = x$; equivalently, R is an abelian von Neumann regular ring (see [15, p. 28]).

Proposition 6.5. *Let M be an R-module with $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

(i) *M is a DCAU-module such that S is a von Neumann regular ring.*
(ii) *M is a DCEU-module such that S is a von Neumann regular ring.*
(iii) *S is a strongly regular ring.*

Proof. (i) ⇒ (iii) and (ii) ⇒ (iii) Since S is a von Neumann regular ring, it follows from [35, Theorem 4] that $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$ are direct summands of M for every $f \in S$. Assume for example that M is a DCAU-module. Let $M = A \oplus B$ be a decomposition of M and let $f \in \mathrm{Hom}_R(A, B)$. Then $\mathrm{Im}\, f \ll B$. Now consider the R-endomorphism g of M defined by $g(a) = f(a)$ and $g(b) = 0$ for every $(a, b) \in A \times B$. Clearly, $\mathrm{Im}\, g = \mathrm{Im}\, f$ is a direct summand of B and hence $f = 0$. Therefore M is abelian. That is, S is an abelian ring (see [6, Theorem 4.4]). It follows that S is a strongly regular ring. The proof of the DCEU case is similar.

(iii) ⇒ (i) and (iii) ⇒ (ii) are clear. □

Example 6.6. From the previous proposition, it follows that a von Neumann regular ring R is right DCAU if and only if R is right DCEU if and only if R is abelian.

From [19, Proposition 4.10] and [26, Theorem 5.6], it follows that every DCAU-module (resp., DCEU-module) is SDSF (resp., SSF). So for any class $\mathcal{C}$ of R-modules which is closed under finite direct sums, there is no ring R for which every R-module in $\mathcal{C}$ is DCAU (or DCEU). On the other hand, we think that it is a good idea to investigate the class of rings R for which every SSF R-module is DCEU. Unfortunately, we have not been able to provide a characterization of this class of rings. But in the next proposition we give an example of such rings.

Lemma 6.7. *Let M be an SSF module such that* $\mathrm{Soc}(M) \leq^{\mathrm{ess}} M$ *and* $\mathrm{Rad}(M) = 0$. *Then M is a DCEU-module.*

Proof. Note that every simple submodule of M is a direct summand of M since $\mathrm{Rad}(M) = 0$. By using similar arguments as in Remark 2.10, we infer that $\mathrm{Soc}(M)$ is an SSF module. Now take a non-trivial decomposition $M = A \oplus B$ of M and let $f : A \to B$ be an R-homomorphism. Therefore, for every simple submodule S_a of A and every simple submodule S_b of B, S_a and S_b are not isomorphic. Thus $f(\mathrm{Soc}(A)) = 0$ and so $\mathrm{Soc}(A) \subseteq \mathrm{Ker}\, f$. But $\mathrm{Soc}(A) \leq^{\mathrm{ess}} A$ by [2, Propositions 5.20 (2) and 9.19]. Then $\mathrm{Ker}\, f \leq^{\mathrm{ess}} A$. It follows that M is a DCEU-module. □

Recall that a ring R is called *right semiartinian* if every nonzero cyclic R-module has nonzero socle. Equivalently, every nonzero right R-module has nonzero socle. The next result is an immediate consequence of the preceding lemma.

Proposition 6.8. *Let R be a right semiartinian right V-ring. Then every SSF R-module is a DCEU-module.*

Next, we wish to investigate some properties of the rings R for which every right (left) DCAU-R-module (resp., right (left) DCEU-R-module) is DCEU (resp., DCAU). Such rings will be called *right (left) AE-rings* (resp., *right (left) EA-rings*). The ring R is called an AE-ring (resp., an EA-ring) in case R is both a right and a left AE-ring (resp., a right and a left EA-ring). We begin with a lemma that provides a source of examples of these kinds of rings. First recall that a ring R is called *right Bass* if every nonzero right R-module has a maximal submodule. It is well known that a ring R is right Bass if and only if $\mathrm{Rad}(M) \ll M$ for every nonzero right R-module M (see, for example, [9, 2.21 or Exercises 2.22 (2)]).

Lemma 6.9. *The following statements hold:*

(i) *Let R be a right semiartinian ring such that for every DCAU-R-module M which is not semisimple we have* $\mathrm{Soc}(M) = \mathrm{Rad}(M)$. *Then R is a right AE-ring.*

(ii) *Let R be a right Bass ring such that for every DCEU-R-module M which is not semisimple we have* $\mathrm{Z}(M) \subseteq \mathrm{Rad}(M)$. *Then R is a right EA-ring.*

Proof. Clearly, a semisimple module is DCAU if and only if it is DCEU (see Example 2.3).

(i) Take a DCAU-R-module M which is not semisimple and let $M = A \oplus B$ be a decomposition of M. By assumption, $\mathrm{Soc}(A) \oplus \mathrm{Soc}(B) = \mathrm{Soc}(M) = \mathrm{Rad}(M) =$

$\mathrm{Rad}(A)\oplus\mathrm{Rad}(B)$ (see [2, Proposition 9.19]). Thus $\mathrm{Soc}(A) = \mathrm{Rad}(A)$ and $\mathrm{Soc}(B) = \mathrm{Rad}(B)$. Let $f : A \to B$ be an R-homomorphism. Then $\mathrm{Im}\, f \ll B$ as M is DCAU. Hence $\mathrm{Im}\, f \subseteq \mathrm{Rad}(B) = \mathrm{Soc}(B)$. But $A/\mathrm{Ker}\, f \cong \mathrm{Im}\, f$, so $A/\mathrm{Ker}\, f$ is semisimple. Therefore $\mathrm{Rad}(A/\mathrm{Ker}\, f) = 0$ and hence $\mathrm{Rad}(A) \subseteq \mathrm{Ker}\, f$ by [2, Proposition 9.14]. It follows that $\mathrm{Soc}(A) \subseteq \mathrm{Ker}\, f$. Since R is right semiartinian, we have that $\mathrm{Soc}(A) \leq^{\mathrm{ess}} A$ (see [37, p. 183, Proposition 2.5]). Thus, $\mathrm{Ker}\, f \leq^{\mathrm{ess}} A$. This proves that M is a DCEU-module.

(ii) Let M be a DCEU-R-module which is not semisimple. Take a decomposition $M = A \oplus B$ of M and let $f : A \to B$ be an R-homomorphism. Then $\mathrm{Ker}\, f \leq^{\mathrm{ess}} A$ and so $A/\mathrm{Ker}\, f$ is a singular module. Since $A/\mathrm{Ker}\, f \cong \mathrm{Im}\, f$, it follows that $\mathrm{Im}\, f \subseteq \mathrm{Z}(B)$. It is clear that $\mathrm{Z}(M) = \mathrm{Z}(A) \oplus \mathrm{Z}(B)$. So, by hypothesis, we have that $\mathrm{Z}(A) \oplus \mathrm{Z}(B) \subseteq \mathrm{Rad}(A) \oplus \mathrm{Rad}(B)$. Hence $\mathrm{Z}(B) \subseteq \mathrm{Rad}(B)$. Therefore $\mathrm{Im}\, f \subseteq \mathrm{Rad}(B)$. But $\mathrm{Rad}(B) \ll B$ as R is right Bass. Hence $\mathrm{Im}\, f \ll B$. It follows that M is a DCAU-module. □

We next present an explicit example of a ring satisfying the conditions of the preceding lemma.

Example 6.10. Let R be a commutative local perfect ring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$. If R is a field then clearly R is both an AE-ring and an EA-ring. Assume that R is not a field. Then $\mathfrak{m} \neq 0$ and $\mathrm{Soc}(R) = \mathfrak{m}$ by [2, Proposition 15.17]. Let M be an R-module. It is well known that $\mathrm{Rad}(M) = \mathfrak{m}M$. Moreover, since $\mathfrak{m}^2 = 0$, $\mathfrak{m}\,\mathrm{Rad}(M) = 0$ and hence $\mathrm{Rad}(M) \subseteq \mathrm{Soc}(M)$. Moreover, we have $\mathrm{Soc}(R)\mathrm{Z}(M) = 0$ (see [24, Lemma 7.2 (2)]). Therefore $\mathfrak{m}\mathrm{Z}(M) = 0$, which implies that $\mathrm{Z}(M) \subseteq \mathrm{Soc}(M)$. In addition, the ring R is (right) semiartinian and (right) Bass by [2, Theorem 28.4].

(i) Let M be a DCAU-R-module which is not semisimple. Let S be a simple submodule of M. Assume that S is not contained in $\mathrm{Rad}(M)$. Then S is not small in M. So there exists a proper submodule L of M such that $S + L = M$. Since S is simple, $S \cap L = 0$ and so $S \oplus L = M$. As R is perfect, L has a maximal submodule K. Moreover, since R is local, there exists an R-isomorphism $\theta : L/K \to S$. Let $\pi : L \to L/K$ be the natural epimorphism and set $f = \theta\pi : L \to S$. Then $\mathrm{Im}\, f \ll S$ since M is DCAU. This implies that $S = 0$, a contradiction. It follows that $\mathrm{Soc}(M) \subseteq \mathrm{Rad}(M)$. Therefore, $\mathrm{Soc}(M) = \mathrm{Rad}(M)$. Now, using Lemma 6.9 (i), we infer that R is an AE-ring.

(ii) Let M be a DCEU-R-module which is not semisimple. Suppose that M has a simple submodule S that is a direct summand of M. Then $M = S \oplus N$ for some nonzero submodule $N \leq M$. Since R is local and perfect, N has a simple submodule S_1 such that $S \cong S_1$ by [2, Theorem 28.4]. So there exists a monomorphism $\mu : S \to N$. This contradicts the fact that M is DCEU. It follows that every simple submodule of M is small in M. Thus $\mathrm{Soc}(M) \subseteq \mathrm{Rad}(M)$ and hence $\mathrm{Soc}(M) = \mathrm{Rad}(M)$. Thus, $\mathrm{Z}(M) \subseteq \mathrm{Rad}(M)$. Now, applying Lemma 6.9 (ii), we obtain that R is an EA-ring.

(iii) From (i) and (ii), it follows that for every maximal ideal $\mathfrak{m}$ of a commutative perfect ring R, the ring $R/\mathfrak{m}^2$ is both an AE-ring and an EA-ring.

(iv) Let $\mathfrak{m}$ be a maximal ideal of a Dedekind domain R. Then $R/\mathfrak{m}^2$ is an artinian (and so a perfect) ring by [36, Theorem 6.14]. Now (i) and (ii) imply that $R/\mathfrak{m}^2$ is both an AE- and an EA-ring.

Lemma 6.11. *Let $R = R_1 \oplus R_2$ be a ring decomposition of a ring R. Then R is a right AE-ring (right EA-ring) if and only if so are R_1 and R_2.*

Proof. Let M be an R-module. It is clear that $M = M_1 \oplus M_2$ with $M_i = MR_i$ for $i = 1, 2$. Moreover, the submodules of each M_i are the same whether M_i is regarded as an R_i-module or as an R-module. It is not hard to see that $\mathrm{Hom}_R(M_i, M_j) = 0$ for every $i \neq j \in \{1, 2\}$.

To prove the sufficiency for EA-rings, let M be a DCEU-R-module. Then each M_i is DCEU as an R-module and so also as an R_i-module ($i = 1, 2$) by Proposition 2.7. Since each R_i is an EA-ring, it follows that each M_i is DCAU as an R_i-module and so also as an R-module. Now, using [19, Proposition 4.15], we see that M is a DCAU-R-module. Therefore R is an EA-ring. Similarly, we can prove the sufficiency for AE-rings by using Proposition 2.7 and [19, Proposition 3.8]. The necessity follows from the fact that, for each $i \in \{1, 2\}$, every R_i-module N_i is an R-module and the lattices of R_i-submodules and R-submodules of N_i coincide. □

Recall from [1] that an R-module M is called *coretractable* if $\mathrm{Hom}_R(M/K, M) \neq 0$ for any proper submodule K of M.

Lemma 6.12. *Let R be a commutative semiartinian von Neumann regular ring and assume that R is an EA-ring. Then R is semisimple.*

Proof. To prove that R is semisimple, it suffices to show that R is a perfect ring since $\mathrm{J}(R) = 0$. Using [1, Theorem 3.14], the proof is completed by showing that every cyclic R-module is coretractable. To show this, let I be an ideal of R and set $M = R/I$. Let N be a proper submodule of M. Then $N = X/I$ for some proper ideal X of R with $I \subseteq X$. Let L be a maximal ideal of R such that $X \subseteq L$ and set $K = L/I$. Clearly, $M \oplus M/K$ is not a DCAU-module. Therefore $M \oplus M/K$ is not a DCEU-module since R is an EA-ring. Note that every R-module has an essential socle as R is semiartinian. Then $E(\mathrm{Soc}(M) \oplus M/K) = E(M \oplus M/K)$. So Corollary 3.18 implies that $\mathrm{Soc}(M) \oplus M/K$ is not SSF. On the other hand, it is easily seen that M is an abelian R-module and hence M is a DCEU-R-module. Thus $\mathrm{Soc}(M)$ is SSF by Remark 2.10. It follows that $\mathrm{Hom}_R(M/K, \mathrm{Soc}(M)) \neq 0$. This yields $\mathrm{Hom}_R(M/K, M) \neq 0$. Since $\frac{M/N}{K/N} \cong M/K$, we infer that $\mathrm{Hom}_R(M/N, M) \neq 0$. This shows that M is coretractable, as required. □

Remark 6.13. It is easily seen that every factor ring of an EA-ring (resp., AE-ring) is again an EA-ring (resp., AE-ring).

Theorem 6.14. *Let R be a commutative ring. Then*

(i) *If R is an EA-ring, then R is a perfect ring.*
(ii) *If R is a perfect ring with $\mathrm{J}(R)^2 = 0$, then R is both an AE-ring and an EA-ring.*

Proof. (i) Suppose that R is an EA-ring. First let us show that R is semiartinian. Assume, to the contrary, that $\mathrm{Soc}(R/I) = 0$ for some proper ideal I of R. Let $\mathfrak{m}$ be a maximal ideal of R such that $I \subseteq \mathfrak{m}$ and set $M = R/I \oplus R/\mathfrak{m}$. It is clear that R/I is an abelian R-module. Hence Corollary 3.13 implies that M is a DCEU R-module. Since R is an EA-ring, it follows that M is a DCAU-module. But $R/\mathfrak{m}$ is a homomorphic image of the R-module R/I, so $R/\mathfrak{m} \ll R/\mathfrak{m}$, a contradiction. Consequently, R is a semiartinian ring. Moreover, it is easily seen that $R/\mathrm{J}(R)$ is an EA-ring. In addition, note that $R/\mathrm{J}(R)$ is semiartinian and von Neumann regular by [13, Corollary 3.33E]. It follows from Lemma 6.12 that $R/\mathrm{J}(R)$ is a semisimple ring. Hence R is perfect by [37, p. 189, Proposition 5.1].

(ii) Assume that R is a perfect ring such that $\mathrm{J}(R)^2 = 0$. It is well known that $R = R_1 \times \cdots \times R_n$ is a finite direct product of commutative perfect local rings R_i $(1 \leq i \leq n)$ (see, for example, [25, Theorem 23.24]). Note that $\mathrm{J}(R_i)^2 = 0$ for every $1 \leq i \leq n$. Now, applying Example 6.10 and Lemma 6.11, we conclude that R is both an AE- and an EA-ring. □

Example 6.15. Let D be a discrete valuation ring with maximal ideal $\mathfrak{m}$. Let n be a positive integer and consider the ring $R = D/\mathfrak{m}^n$. Note that R is an artinian ring (see [36, Theorem 6.14]) and $\mathrm{J}(R)^2 \neq 0$ if $n \geq 3$. On the other hand, by using Theorems 5.6 and 5.10, it is easily seen that a nonzero R-module M is DCAU if and only if M is DCEU if and only if $M \cong R/\mathfrak{m}^s$ for some positive integer s with $1 \leq s \leq n$. In particular, R is both an AE- and an EA-ring.

Corollary 6.16. *The following are equivalent for a commutative ring R:*

(i) *R is an EA-ring.*
(ii) *R is a semilocal ring such that $R_{\mathfrak{m}}$ is an EA-ring for every maximal ideal $\mathfrak{m}$ of R.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 6.11 and Theorem 6.14, R is semilocal and $R = R_1 \times \cdots \times R_n$ is a finite direct product of commutative local EA-rings R_i $(1 \leq i \leq n)$. It is not difficult to see that any localization of R at a maximal ideal of R is isomorphic to R_i for some $i \in \{1, 2, \ldots, n\}$.

(ii) $\Rightarrow$ (i) By Theorem 6.14, each $R_{\mathfrak{m}}$ is a perfect ring. From [11, Theorem A], it follows that R is a Bass ring. But R is semilocal, so R is a perfect ring (see [11, Theorem A] and [2, Theorem 28.4]). By using the argument in ((i) $\Rightarrow$ (ii)), we infer that R is a direct product of local EA-rings. Hence R is an EA-ring by Lemma 6.11. □

Next, we provide another subclass of the class of right AE-rings. Then we present an AE-ring which is not an EA-ring.

Example 6.17. (i) Let R be a right V-ring (e.g., R is a commutative von Neumann regular ring) and let M be a DCAU-R-module. Then M is a non-cosingular module by [38, Proposition 2.5]. Hence M is an abelian module by Proposition 3.1. Therefore M is a DCEU-module. It follows that R is a right AE-ring.

(ii) Let R be a commutative von Neumann regular ring which is not semisimple.

(a) From (i) and Theorem 6.14, it follows that R is an AE-ring that is not an EA-ring. For an explicit example, we can take $R = \prod_{n=1}^{\infty} F_n$, where $F_n = \mathbb{Z}/2\mathbb{Z}$ for every $n \geq 1$.

(b) It is well known that for every maximal ideal $\mathfrak{m}$ of R, the localization $R_{\mathfrak{m}}$ is a field. On the other hand, R is not semilocal since R is not semisimple. This shows that the condition "R is semilocal" is not superfluous in Corollary 6.16.

The next corollary follows easily from Theorem 6.14.

Corollary 6.18. *If R is a commutative domain, then R is an EA-ring if and only if R is a field.*

Recall that a module M is called a *Hamsher module* provided each nonzero submodule $N \leq M$ has a maximal submodule (see [12, p. 201]).

Proposition 6.19. *Every right AE-ring is a right Bass ring.*

Proof. Let R be a right AE-ring. We want to prove that R is a right Bass ring. By [12, Theorem 1], it is sufficient to prove that the injective hull $E(S)$ of each simple R-module S is a Hamsher module. To show this, assume, to the contrary, that R has a simple R-module S such that $E(S)$ contains a nonzero submodule N with $\mathrm{Rad}(N) = N$. Clearly, $N \neq S$ and $S \subseteq N$. Consider the R-module $M = N \oplus S$. Note that $\mathrm{Hom}_R(N, S) = 0$. Moreover, it is clear that every nonzero submodule of $E(S)$ is a uniform module. So N is an abelian R-module since N is indecomposable. Now, using Corollary 3.4, we conclude that M is a DCAU-module. Since R is a right AE-ring, it follows that M is a DCEU-module. From Corollary 3.13, we have $\mathrm{Hom}_R(S, N) = 0$, a contradiction. The result follows. □

Corollary 6.20. *Let R be a commutative domain. Then R is an AE-ring if and only if R is a field.*

Proof. The sufficiency is obvious. Conversely, suppose that R is an AE-ring. Then R is a Bass ring by Proposition 6.19. Since R is a commutative domain, it follows from [16, Lemma 2] that R is a field. □

Proposition 6.21. *Let R be a right hereditary ring. Then R is a right AE-ring if and only if R is a right V-ring.*

Proof. The sufficiency is clear by Example 6.17 (i). Conversely, suppose, to the contrary, that R has a simple right R-module S that is not injective. Then $S \oplus E(S)$ is a DCAU-module by Example 2.4 (ii). Since R is a right AE-ring, it follows that $S \oplus E(S)$ is a DCEU-module, a contradiction. Therefore R is a right V-ring. □

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