

UNIFORMLY S -ARTINIAN RINGS AND MODULES

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ABSTRACT. Let R be a commutative ring with identity and S a multiplicative subset of R . An R -module M is said to be a u - S -Artinian module if there is $s \in S$ such that any descending chain of submodules of M is S -stationary with respect to s . The notion of u - S -Artinian modules is characterized in terms of $(S$ -MIN)-conditions and u - S -cofinite properties. A ring R is said to be u - S -Artinian if R itself is a u - S -Artinian module. It is then shown that any u - S -semisimple ring is u - S -Artinian. It is proved that a ring R is u - S -Artinian if and only if R is u - S -Noetherian, the u - S -Jacobson radical $\text{Jac}_S(R)$ of R is S -nilpotent and $R/\text{Jac}_S(R)$ is a u - $S/\text{Jac}_S(R)$ -semisimple ring. Besides, some examples are given to distinguish Artinian rings, u - S -Artinian rings, and S -Artinian rings.

1. INTRODUCTION

Throughout this article, all rings are commutative rings with identity and all modules are unitary. Let R be a ring. We denote by $\text{Max}(R)$ the set of all maximal ideals of R . A subset S of R is called a multiplicative subset of R if $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Early in 2002, Anderson and Dumitrescu [2] introduced the so-called S -Noetherian ring R , in which for any ideal I of R , there exists a finitely generated ideal K of R such that $sI \subseteq K \subseteq I$ for some $s \in S$. Note that Cohen's Theorem, the Eakin–Nagata Theorem and the Hilbert Basis Theorem for S -Noetherian rings are also given in [2]. The notion of S -Noetherian rings provides a good direction for S -generalizations of other classical rings (see, for example, [3, 6, 7, 10, 12]). However, it is often difficult to study these S -generalizations of classical rings via a module-theoretic approach. The essential difficulty is that the selected element $s \in S$ is often not “uniform” in their definitions. To overcome this difficulty for Noetherian properties, Qi et al. [8] recently introduced the notions of uniformly S -Noetherian rings, which are S -Noetherian rings such that s is independent of I in the definition of S -Noetherian rings. They also introduced the notion of u - S -injective modules and then characterized uniformly S -Noetherian rings in terms of u - S -injective modules. Some other uniform S -versions of rings and modules,

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such as semisimple rings, von Neumann regular rings, projective modules and flat modules, are introduced and studied by the present authors in [18, 19].

In 2020, Sengelen et al. [16] introduced the notions of S -Artinian rings for which any descending chain of ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m \supseteq \cdots$ of R satisfies an S -stationary condition, i.e., there exist $s \in S$ and $k \in \mathbb{Z}^+$ such that $sI_k \subseteq I_n$ for all $n \geq k$. In the definition of S -Artinian rings, it is easy to see that although the element $s \in S$ is independent of n , it is certainly dependent on the given descending chain of ideals. Recently, Özen et al. [15] extended the notion of S -Artinian rings to that of S -Artinian modules by replacing descending chains of ideals by those of submodules. And then, Omid et al. [11] introduced the notion of weakly S -Artinian modules, for which every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_m \supseteq \cdots$ of submodules of M is weakly S -stationary, i.e., there exists $k \in \mathbb{Z}^+$ such that for each $n \geq k$, $s_n N_k \subseteq N_n$ for some $s_n \in S$. In the definition of weakly S -Artinian modules, it is easy to see that the element s_n is dependent both on n and the descending chain of modules. So the notion of weakly S -Artinian modules is certainly a “weak” version of that of S -Artinian modules. In this article, we introduce and study the “uniform” S -version of Artinian rings and modules (we call them u - S -Artinian rings and modules) such that the element s given in the definition of S -Artinian rings and modules is both independent of n and the descending chain of ideals or submodules. Obviously, we have the following implications:

$$\boxed{\text{Artinian rings}} \implies \boxed{u\text{-}S\text{-Artinian rings}} \implies \boxed{S\text{-Artinian rings}}$$

But neither of the implications can be reversed (see Example 2.7 and Example 3.4). Denote by $\text{Jac}_S(R)$ the u - S -Jacobson radical of a ring R (see Definition 4.2). Then it is also worth to mention that a ring R is u - S -Artinian if and only if R is u - S -Noetherian, the u - S -Jacobson radical $\text{Jac}_S(R)$ of R is S -nilpotent and $R/\text{Jac}_S(R)$ is a u - $S/\text{Jac}_S(R)$ -semisimple ring (see Theorem 4.9).

The related notions of uniformly S -torsion theory originally emerged in [18], and we give a quick review below. An R -module T is called u - S -torsion (with respect to s) provided that there exists $s \in S$ such that $sT = 0$. A sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

of R -modules is called a short u - S -exact sequence (with respect to s) if $s\text{Ker}(f) = s\text{Coker}(f) = 0$, $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$ for some $s \in S$. An R -homomorphism $f : M \rightarrow N$ is a u - S -monomorphism (resp., u - S -epimorphism, u - S -isomorphism) (with respect to s) provided $\text{Ker}(f)$ is (resp., $\text{Coker}(f)$ is, both $\text{Ker}(f)$ and $\text{Coker}(f)$ are) u - S -torsion (with respect to s). Recall from [19] that an R -module P is called u - S -projective provided that the induced sequence

$$0 \rightarrow \text{Hom}_R(P, A) \rightarrow \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C) \rightarrow 0$$

is u - S -exact for any u - S -exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. Suppose M and N are R -modules. We say M is u - S -isomorphic to N if there exists a u - S -isomorphism $f : M \rightarrow N$. A family $\mathcal{C}$ of R -modules is said to be closed under u - S -isomorphisms if, whenever M is u - S -isomorphic to N and M is in $\mathcal{C}$, N is also in $\mathcal{C}$. Note that the class of u - S -projective modules is closed under u - S -isomorphisms. One can deduce

from [19, Lemma 2.1] that the existence of u - S -isomorphisms of two R -modules is actually an equivalence relationship.

2. UNIFORMLY S -ARTINIAN MODULES

Let R be a ring, S a multiplicative subset of R and M an R -module. Suppose $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ is a descending chain of submodules of M . The family $\{M_i\}_{i \in \mathbb{Z}^+}$ is said to be S -stationary (with respect to s) if there exist $s \in S$ and $k \in \mathbb{Z}^+$ such that $sM_k \subseteq M_n$ for every $n \geq k$, and M is called an S -Artinian module if each descending chain of submodules $\{M_i\}_{i \in \mathbb{Z}^+}$ of M is S -stationary (see [15, Definition 1]). Note that in the definition of S -Artinian module, the element s is dependent on the given descending chain of submodules. The main purpose of this section is to introduce and study a “uniform” version of S -Artinian modules.

Definition 2.1. Let R be a ring and S a multiplicative subset of R . An R -module M is called a u - S -Artinian (abbreviates uniformly S -Artinian) module (with respect to s) provided that there exists $s \in S$ such that each descending chain $\{M_i\}_{i \in \mathbb{Z}^+}$ of submodules of M is S -stationary with respect to s .

Trivially, if $0 \in S$, then every R -module is u - S -Artinian. If $S_1 \subseteq S_2$ are multiplicative subsets of R and M is u - S_1 -Artinian, then M is obviously u - S_2 -Artinian. Note that Artinian modules are exactly u - $\{1\}$ -Artinian modules. So all Artinian modules are u - S -Artinian modules for any multiplicative set S . Next we give a u - S -Artinian module which is not Artinian.

Example 2.2. Let $R = \mathbb{Z}$ be the ring of integers, p a prime in R and $M = \mathbb{Z}_p[[x]]$ the set of all formal power series over $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$. Set $S = \{p^n \mid n \in \mathbb{N}\}$. Then M is not Artinian since the descending chain

$$\langle x \rangle \supseteq \langle x^2 \rangle \supseteq \cdots \supseteq \langle x^n \rangle \supseteq \cdots$$

is not stationary. However, since M is obviously u - S -torsion, M is a u - S -Artinian module.

Let S be a multiplicative subset of R . We set

$$S^* = \{r \in R \mid rt \in S \text{ for some } t \in R\}$$

and call it the saturation of S . A multiplicative set S is said to be saturated if $S = S^*$. Trivially, we have $S \subseteq S^*$ for all multiplicative subset S of R .

Proposition 2.3. Let R be a ring, S a multiplicative subset of R and M an R -module. Let S^* be the saturation of S . Then M is a u - S -Artinian R -module if and only if M is a u - S^* -Artinian R -module.

Proof. Suppose M is a u - S -Artinian R -module. Then M is trivially a u - S^* -Artinian R -module since $S \subseteq S^*$. Now, suppose M is a u - S^* -Artinian R -module. Then there is $r \in S^*$ such that each descending chain of submodules $\{M_i\}_{i \in \mathbb{Z}^+}$ of M is S^* -stationary with respect to r , i.e., there exists $k \in \mathbb{Z}^+$ such that $rM_k \subseteq M_n$ for each $n \geq k$. Since $r \in S^*$, $rt \in S$ for some $t \in R$. Note that $rtM_k \subseteq rM_k \subseteq M_n$. Hence M is a u - S -Artinian R -module. $\square$

Let R be a ring, M an R -module and S a multiplicative subset of R . For any $s \in S$, there is a multiplicative subset $S_s = \{1, s, s^2, \dots\}$ of S . We denote by M_s the localization of M at S_s . Note that $M_s \cong M \otimes_R R_s$.

Lemma 2.4. *Let R be a ring, S a multiplicative subset of R and M an R -module. If M is a u - S -Artinian R -module, then there exists an element $s \in S$ such that M_s is an Artinian R_s -module.*

Proof. Let s be an element in S such that each descending chain of submodules $\{M_i\}_{i \in \mathbb{Z}^+}$ of M is S -stationary with respect to s . Let $M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ be a descending chain of R_s -submodules of M_s . Considering the natural homomorphism $f : M \rightarrow M_s$, we have a descending chain of submodules of M as follows:

$$f^{-1}(M_1) \supseteq f^{-1}(M_2) \supseteq \dots \supseteq f^{-1}(M_n) \supseteq \dots$$

There is a $k \in \mathbb{Z}^+$ such that $sf^{-1}(M_k) = f^{-1}(sM_k) = f^{-1}(M_k) \subseteq f^{-1}(M_n)$ for each $n \geq k$ since M_k is an R_s -module. Hence $M_k = M_n$ for each $n \geq k$. Consequently, M_s is an Artinian R_s -module. $\square$

Remark 2.5. The converse of Lemma 2.4 also does not hold in general. Let $R = k[[x]]$ be the formal power series ring over a field k . Let $S = \{1, x, x^2, \dots\}$. Then R_S is a field, and so is an Artinian R_S -module. However, R is not a u - S -Artinian R -module as R is not Artinian (see Proposition 3.2).

Recall from [1, 13] that a multiplicative subset S of R is said to satisfy the maximal multiple condition if there exists an $s \in S$ such that $t|s$ for each $t \in S$. Both finite multiplicative subsets and the multiplicative subsets that consist of units satisfy the maximal multiple condition. Certainly, u - S -Artinian modules are S -Artinian modules. And the following result shows that the converse also holds for multiplicative sets satisfying the maximal multiple condition.

Proposition 2.6. *Let R be a ring, S a multiplicative subset of R satisfying the maximal multiple condition, and M an R -module. Then M is a u - S -Artinian module if and only if M is an S -Artinian module.*

Proof. If M is a u - S -Artinian module, then trivially M is S -Artinian. Let $s \in S$ be such that $t|s$ for each $t \in S$. Suppose M is an S -Artinian module and $\{M_i\}_{i \in \mathbb{Z}^+}$ a descending chain of submodules of M . Then there exists $t \in S$ such that $tM_k \subseteq M_n$ for each $n \geq k$. So $sM_k \subseteq tM_k \subseteq M_n$ for each $n \geq k$. Hence M is a u - S -Artinian module. $\square$

However, the following example shows S -Artinian modules are not u - S -Artinian modules in general.

Example 2.7. Let R be a valuation domain whose valuation group is $G = \prod_{\aleph} \mathbb{Z}$, the Hahn product of $\aleph$ copies of $\mathbb{Z}$ with lexicographic order, where $\aleph$ is an uncountable regular cardinal (see [9]). Let $S = R \setminus \{0\}$ be the set of all nonzero elements of R . Then R itself is an S -Artinian R -module but not u - S -Artinian.

Proof. First, we claim that R is an S -Artinian R -module. Indeed, let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R . We may assume that each I_i is not equal to 0. Then, for each I_i , there exists a nonzero element $r_i \in I_i$ such that $v(r_i) \in G$. Moreover, we can assume that all r_i satisfy $v(r_i) \leq v(r_{i+1})$. Since $\aleph$ is an uncountable regular cardinal, $\varinjlim v(r_i) < \aleph$. So there is an element $x \in G$ such that $x \geq v(r_i)$ for each i . Suppose $v(s) = x$. Then $0 \neq s \in \bigcap_{i=1}^{\infty} I_i$. Hence $sI_k \subseteq I_n$ for each $n \geq k$. Consequently, R is an S -Artinian R -module. Now, we claim that R is not a u - S -Artinian R -module. On the contrary, suppose R is u - S -Artinian. Then, by Lemma 2.4, there is an $s \in S$ such that R_s is an Artinian domain, which is exactly a field. Let r be a nonzero element such that $v(r) > nv(s)$ for all non-negative integers n . Then r is not a unit in R_s , which is a contradiction. Consequently, R is not a u - S -Artinian R -module. (Note that it can also be easily deduced by Proposition 3.2.) $\square$

Lemma 2.8. *Let R be a ring and S a multiplicative subset of R . Let M and N be R -modules. If M is u - S -isomorphic to N , then M is u - S -Artinian if and only if N is u - S -Artinian.*

Proof. Let M be u - S -Artinian with respect to $s \in S$ and $f : M \rightarrow N$ a u - S -isomorphism. Suppose $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ is a descending chain of submodules of N . Then $f^{-1}(N_1) \supseteq f^{-1}(N_2) \supseteq \cdots \supseteq f^{-1}(N_n) \supseteq \cdots$ is a descending chain of submodules of M . Then there is $k \in \mathbb{Z}^+$ such that $sf^{-1}(N_{k_s}) \subseteq f^{-1}(N_n)$ for any $n \geq k$. Hence $s(\text{Im}(f) \cap N_k) = sf(f^{-1}(N_k)) \subseteq f(f^{-1}(N_n)) = \text{Im}(f) \cap N_n$ for any $n \geq k$. Note that we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(f) \cap N_n & \longrightarrow & N_n & \longrightarrow & (\text{Im}(f) + N_n)/\text{Im}(f) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Im}(f) \cap N_k & \longrightarrow & N_k & \longrightarrow & (\text{Im}(f) + N_k)/\text{Im}(f) \longrightarrow 0 \end{array}$$

with $(\text{Im}(f) + N_n)/\text{Im}(f)$ and $(\text{Im}(f) + N_k)/\text{Im}(f)$ submodules of $\text{Coker}(f)$, and hence u - S -torsion. It follows by [8, Lemma 2.11] that $s^2N_k \subseteq N_n$. Consequently, N is u - S -Artinian with respect to s^2 .

Suppose N is u - S -Artinian. Then by [19, Lemma 2.1], there is a u - S -isomorphism $g : N \rightarrow M$. So we can show that M is u - S -Artinian similarly. $\square$

Proposition 2.9. *Let R be a ring and S a multiplicative subset of R . Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an u - S -exact sequence. Then B is u - S -Artinian if and only if A and C are u - S -Artinian. Consequently, a finite direct sum $\bigoplus_{i=1}^n M_i$ is u - S -Artinian if and only if each M_i is u - S -Artinian ($i = 1, \dots, n$).*

Proof. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an u - S -exact sequence. Then there is an exact sequence $0 \rightarrow \text{Im}(f) \rightarrow B \rightarrow \text{Coker}(f) \rightarrow 0$, with $\text{Im}(f)$ u - S -isomorphic to A and $\text{Coker}(f)$ u - S -isomorphic to C . So we can assume that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence by Lemma 2.8. If B is u - S -Artinian, then it is easy to verify

that A and C are u - S -Artinian. Now suppose A and C are u - S -Artinian. Let $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$ be a descending chain of submodules of B . Then

$$B_1 \cap A \supseteq B_2 \cap A \supseteq \cdots \supseteq B_n \cap A \supseteq \cdots$$

is a descending chain of submodules of A , and

$$(B_1 + A)/A \supseteq (B_2 + A)/A \supseteq \cdots \supseteq (B_n + A)/A \supseteq \cdots$$

is a descending chain of submodules of $C \cong B/A$. So there are $s_1, s_2 \in S$ and $k \in \mathbb{Z}^+$ such that $s_1 B_k \cap A \subseteq B_n \cap A$ and $s_2(B_k + A) \subseteq B_n + A$ for any $n \geq k$. Then one can verify that $s_1 s_2 B_k \subseteq B_n$ for any $n \geq k$. Hence, B is u - S -Artinian. $\square$

Let $\mathfrak{p}$ be a prime ideal of R . We say an R -module M is u - $\mathfrak{p}$ -Artinian provided that M is u -($R \setminus \mathfrak{p}$)-Artinian. The next result gives a local characterization of Artinian modules.

Proposition 2.10. *Let R be a ring and M an R -module. Then the following statements are equivalent:*

- (1) M is Artinian;
- (2) M is u - $\mathfrak{p}$ -Artinian for any $\mathfrak{p} \in \text{Spec}(R)$;
- (3) M is u - $\mathfrak{m}$ -Artinian for any $\mathfrak{m} \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ be a descending chain of submodules of M . Then, for each $\mathfrak{m} \in \text{Max}(R)$, there exist $s_{\mathfrak{m}} \notin \mathfrak{m}$ and $k_{\mathfrak{m}} \in \mathbb{Z}^+$ such that $s_{\mathfrak{m}} M_{k_{\mathfrak{m}}} \subseteq M_n$ for each $n \geq k_{\mathfrak{m}}$. Note that R can be generated by the set $\{s_{\mathfrak{m}} \mid \mathfrak{m} \in \text{Max}(R)\}$. So there is a finite subset $\{s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_t}\}$ that generates R . Let $k = \max\{k_{\mathfrak{m}_1}, \dots, k_{\mathfrak{m}_t}\}$. Then $M_k = \langle s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_t} \rangle M_k \subseteq \sum_{i=1}^t (s_{\mathfrak{m}_i} M_{k_{\mathfrak{m}_i}}) \subseteq M_n$ for all $n \geq k$. Hence M is Artinian. $\square$

Let R be a ring and S a multiplicative subset of R . Recall from [15] that an R -module M is said to be S -cofinite (called finitely S -cogenerated in [15, Definition 3]) if, for each nonempty family of submodules $\{M_i\}_{i \in \Delta}$ of M , $\bigcap_{i \in \Delta} M_i = 0$ implies that $s(\bigcap_{i \in \Delta'} M_i) = 0$ for some $s \in S$ and a finite subset $\Delta' \subseteq \Delta$. If $S = \{1\}$, then S -cofinite modules are exactly the classical cofinite modules.

Definition 2.11. Let R be a ring and S a multiplicative subset of R . An R -module M is called u - S -cofinite (with respect to s) if there is an $s \in S$ such that for each nonempty family of submodules $\{M_i\}_{i \in \Delta}$ of M , $\bigcap_{i \in \Delta} M_i = 0$ implies that $s(\bigcap_{i \in \Delta'} M_i) = 0$ for a finite subset $\Delta' \subseteq \Delta$.

Obviously, cofinite R -modules are u - S -cofinite, and u - S -cofinite R -modules are S -cofinite.

Proposition 2.12. *Let R be a ring, M an R -module and S a multiplicative subset of R . Then the following statements hold:*

- (1) If $S_1 \subseteq S_2$ are multiplicative subsets of R and M is u - S_1 -cofinite, then M is u - S_2 -cofinite.
- (2) Let S^* be the saturation of S . Then M is u - S -cofinite if and only if M is u - S^* -cofinite.

Proof. (1) is trivial.

(2) If M is u - S -cofinite, then M is also u - S^* -cofinite by (1). Suppose M is u - S^* -cofinite. Then there is an $r \in S^*$ such that for each nonempty family of submodules $\{M_i\}_{i \in \Delta}$ of M , $\bigcap_{i \in \Delta} M_i = 0$ implies that $r(\bigcap_{i \in \Delta'} M_i) = 0$ for a finite subset $\Delta' \subseteq \Delta$. Let $s := rt \in S$ for some $t \in R$. Then $s(\bigcap_{i \in \Delta'} M_i) = 0$. Hence M is u - S -cofinite. $\square$

Let $\mathfrak{p}$ be a prime ideal of R . We say an R -module M is u - $\mathfrak{p}$ -cofinite provided that M is u -($R \setminus \mathfrak{p}$)-cofinite. The next result gives a local characterization of cofinite modules.

Proposition 2.13. *Let R be a ring and M an R -module. Then the following statements are equivalent:*

- (1) M is cofinite;
- (2) M is u - $\mathfrak{p}$ -cofinite for any $\mathfrak{p} \in \text{Spec}(R)$;
- (3) M is u - $\mathfrak{m}$ -cofinite for any $\mathfrak{m} \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $\{M_i\}_{i \in \Delta}$ be a family of submodules of M such that $\bigcap_{i \in \Delta} M_i = 0$. Then, for each $\mathfrak{m} \in \text{Max}(R)$, there exist $s_{\mathfrak{m}} \notin \mathfrak{m}$ and $k_{\mathfrak{m}} \in \mathbb{Z}^+$ such that $s_{\mathfrak{m}}(\bigcap_{i \in \Delta'_{\mathfrak{m}}} M_i) = 0$ for a finite subset $\Delta'_{\mathfrak{m}} \subseteq \Delta$. Note that R can be generated by a finite subset $\{s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_t}\}$. Let $\Delta' = \bigcap_{i=1}^t \Delta'_{\mathfrak{m}_i}$. Then

$$\bigcap_{i \in \Delta'} M_i = \langle s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_t} \rangle \left(\bigcap_{i \in \Delta'} M_i \right) \subseteq \sum_{i=1}^t \left(s_{\mathfrak{m}_i} \bigcap_{i \in \Delta'_{\mathfrak{m}_i}} M_i \right) = 0.$$

Hence, M is cofinite. $\square$

Definition 2.14. Let $\mathcal{N}$ be a nonempty family of submodules of M . Then $N \in \mathcal{N}$ is called an S -minimal element of $\mathcal{N}$ with respect to s if, whenever $N' \subseteq N$ for some $N' \in \mathcal{N}$, we have $sN \subseteq N'$. We say M satisfies the $(S\text{-MIN})$ -condition with respect to s if every nonempty family of submodules of M has an S -minimal element with respect to s .

It follows from [5, Proposition 10.10] that an R -module M is Artinian if and only if every factor module M/N is finitely cogenerated, if and only if M satisfies the (MIN) -condition. Recently, Özen et al. extended this result to s -Artinian modules in [15, Theorem 3]. Now we give a uniform S -version of [5, Proposition 10.10].

Theorem 2.15. *Let R be a ring, S a multiplicative subset of R and M an R -module. Let $s \in S$. Then the following statements are equivalent:*

- (1) M is a u - S -Artinian module with respect to s .
- (2) M satisfies the $(S\text{-MIN})$ -condition with respect to s .
- (3) For any nonempty family $\{N_i\}_{i \in \Gamma}$ of submodules of M , there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $s \bigcap_{i \in \Gamma_0} N_i \subseteq \bigcap_{i \in \Gamma} N_i$.
- (4) Every factor module M/N is u - S -cofinite with respect to s .

Proof. (1) $\Rightarrow$ (2): Suppose that M is a u - S -Artinian module with respect to s . Let $\mathcal{N}$ be a nonempty set of submodules of M . On the contrary, suppose $\mathcal{N}$ has no S -minimal element of $\mathcal{N}$ with respect to s . Take $N_1 \in \mathcal{N}$. Then there exists $N_2 \in \mathcal{N}$ such that $N_1 \supseteq N_2$ and $sN_1 \not\subseteq N_2$. Iterating these steps, we can obtain a descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ such that $sN_k \not\subseteq N_{k+1}$ for any k . This implies M is not a u - S -Artinian module with respect to s , which is a contradiction.

(2) $\Rightarrow$ (3): Let $\{N_i\}_{i \in \Gamma}$ be a nonempty family of submodules of M . Let $\mathcal{A}$ be the set of all intersections of finitely many N_i . Then each $N_i \in \mathcal{A}$, and so $\mathcal{A}$ is nonempty. So there is an S -minimal element, say $A = \bigcap_{i \in \Gamma_0} N_i$, of $\mathcal{A}$. Choose $j \in \Gamma_0 - \Gamma$ and note that $N_j \cap A \subseteq A$ and $N_j \cap A \in \mathcal{A}$. By the S -minimality of A , we have $sA \subseteq N_j \cap A$. Since j is arbitrary, we conclude that $sA \subseteq \bigcap_{i \in \Gamma} N_i$.

(3) $\Rightarrow$ (1): Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of submodules of M . Then there is a positive integer k such that $sN_k = s \bigcap_{i=1}^k N_i \subseteq \bigcap_{i=1}^{\infty} N_i$. So $sN_k \subseteq N_n$ for any $n \geq k$.

(2) $\Rightarrow$ (4): Suppose $\bigcap_{i \in \Gamma} N_i/N = 0$ for some family of submodules $\{N_i/N\}_{i \in \Gamma}$ of M/N . Then $\bigcap_{i \in \Gamma} N_i = N$. Set $\mathcal{A} = \{\bigcap_{i \in \Gamma'} N_i \mid \Gamma' \subseteq \Gamma \text{ is a finite subset}\}$. By (3), $\mathcal{A}$ has an S -minimal element with respect to s , say $M_0 = \bigcap_{i \in \Gamma_0} N_i$ for some finite subset $\Gamma_0 \subseteq \Gamma$. So, for any $k \in \Gamma - \Gamma_0$, we have $sM_0 \subseteq M_0 \cap N_k$. Thus $sM_0 \subseteq M_0 \cap (\bigcap_{k \in \Gamma - \Gamma_0} N_k) \subseteq \bigcap_{k \in \Gamma} N_k = N$. Consequently, M/N is u - S -cofinite with respect to s .

(4) $\Rightarrow$ (1): Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of submodules of M . Set $N = \bigcap_{i=1}^{\infty} N_i$. By assumption, M/N is u - S -cofinite with respect to s . Note that $\bigcap_{i=1}^{\infty} N_i/N = 0$ in M/N . Then there is a positive integer k such that $s \bigcap_{i=1}^k N_i/N = 0$ by (4). So $sN_k \subseteq N_n$ for all $n \geq k$. So M is a u - S -Artinian module with respect to s . $\square$

3. BASIC PROPERTIES OF u - S -ARTINIAN RINGS

Recall from [16, Definition 2.1] that a ring R is called an S -Artinian ring if any descending chain of ideals $\{I_i\}_{i \in \mathbb{Z}^+}$ of R is S -stationary with respect to some $s \in S$. Note that s is determined by the descending chain of ideals in the definition of S -Artinian rings. Now we introduce a “uniform” version of S -Artinian rings.

Definition 3.1. Let R be a ring and S a multiplicative subset of R . Then R is called a u - S -Artinian (abbreviates uniformly S -Artinian) ring (with respect to s) provided that R is a u - S -Artinian R -module (with respect to s), that is, there exists $s \in S$ such that each descending chain $\{I_i\}_{i \in \mathbb{Z}^+}$ of ideals of R is S -stationary with respect to s .

Since u - S -Artinian rings are u - S -Artinian modules over themselves, the results in Section 2 also hold for u - S -Artinian rings. In particular, u - S -Artinian rings are S -Artinian. However, S -Artinian rings are not u - S -Artinian in general. A counterexample was given in Example 2.7. If $0 \in S$, then every ring R is u - S -Artinian. So u - S -Artinian rings are not Artinian in general. A multiplicative set S is said to be regular if every element in S is a nonzero divisor. The following proposition shows that u - S -Artinian rings are exactly Artinian for any regular multiplicative set S .

Proposition 3.2. *Let R be a ring and S a regular multiplicative subset of R . If R is a u - S -Artinian ring, then R is an Artinian ring.*

Proof. Let s be an element in S . Consider the descending chain $Rs \subseteq Rs^2 \subseteq \cdots$ of ideals of R . Then there exists k such that $sRs^k \subseteq Rs^n$ for any $n \geq k$. In particular, we have $s^{k+1} = rs^{k+2}$ for some $r \in R$. Since s is a nonzero divisor, we have $1 = rs$, and thus s is a unit. So R is an Artinian ring. $\square$

In order to give a non-trivial u - S -Artinian ring which is not Artinian, we consider the direct product of u - S -Artinian rings.

Proposition 3.3. *Let $R = R_1 \times R_2$ be a direct product of rings R_1 and R_2 , and $S = S_1 \times S_2$ a direct product of multiplicative subsets of R_1 and R_2 . Then R is a u - S -Artinian ring if and only if R_i is a u - S_i -Artinian ring for each $i = 1, 2$.*

Proof. Suppose R is a u - S -Artinian ring with respect to (s_1, s_2) . Let $\{I_i^1\}_{i \in \mathbb{Z}^+}$ be a descending chain of ideals of R_1 . Then $\{I_i^1 \times 0\}_{i \in \mathbb{Z}^+}$ is a descending chain of ideals of R . Then there exists an integer k such that $(s_1, s_2)(I_k^1 \times 0) \subseteq I_n^1 \times 0$ for all $n \geq k$. Hence $s_1 I_k^1 \subseteq I_n^1$ for all $n \geq k$. So R_1 is a u - S_1 -Artinian ring with respect to s_1 . Similarly, R_2 is a u - S_2 -Artinian ring with respect to s_2 . On the other hand, suppose R_i is a u - S_i -Artinian ring with respect to s_i for each $i = 1, 2$. Let $\{I_i = I_i^1 \times I_i^2\}_{i \in \mathbb{Z}^+}$ be a descending chain of ideals of R . Then there exists an integer k_i such that $s_i I_{k_i}^i \subseteq I_n^i$ for all $n \geq k_i$. Set $k = \max\{k_1, k_2\}$. Then $(s_1, s_2)I_k = s_1 I_k^1 \times s_2 I_k^2 \subseteq I_n^1 \times I_n^2 = I_n$ for all $n \geq k$. So R is a u - S -Artinian ring. $\square$

The promised non-Artinian u - S -Artinian rings are given as follows.

Example 3.4. Let $R = R_1 \times R_2$ be a product of R_1 and R_2 , where R_1 is an Artinian ring but R_2 is not Artinian. Set $S = \{1\} \times \{1, 0\}$. Then R is a u - S -Artinian ring but not Artinian by Proposition 3.3.

Let R be a ring and S a multiplicative subset of R . Recall from [19] that an R -module M is called u - S -semisimple (with respect to s) provided that any u - S -short exact sequence $0 \rightarrow A \xrightarrow{f} M \xrightarrow{g} C \rightarrow 0$ is u - S -split (with respect to s), i.e., there exists an R -homomorphism $h : M \rightarrow A$ such that $h \circ f = s\text{Id}_A$ for some $s \in S$. A ring R is called a u - S -semisimple ring if every free R -module is u - S -semisimple. The rest of this section is devoted to showing that any u - S -semisimple ring is u - S -Artinian. First, we introduce the notion of u - S -simple modules.

Definition 3.5. An R -module M is said to be u - S -simple (with respect to s) provided that M is not u - S -torsion with respect to some $s \in S$, and any proper submodule of M is u - S -torsion with respect to s .

Since any proper submodule of a u - S -simple R -module is u - S -torsion, we have that any u - S -simple R -module is u - S -semisimple by [19, Lemma 2.1]. Moreover, we have the following result.

Proposition 3.6. *Suppose M is a u - S -simple R -module. Then $M^{(\aleph)}$ is a u - S -semisimple R -module for any ordinal $\aleph$.*

Proof. Suppose M is a u - S -simple R -module with respect to s and N is a submodule of $M^{(\aleph)}$ such that $M^{(\aleph)}/N$ is not u - S -torsion with respect to s . Set $\Gamma = \{\alpha \subseteq \aleph \mid s(N \cap M^{(\alpha)}) = 0\}$. For each $i \in \aleph$, we set M^i to be the i -th component of $M^{(\aleph)}$. Then we claim there is $i \in \aleph$ such that $s(N \cap M^i) = 0$, and hence Γ is not empty. Indeed, on the contrary, suppose $N \cap M^i$ is not u - S -torsion for any $i \in \aleph$. Since M is u - S -simple with respect to s , we have $N \cap M^i = M^i \cong M$, and hence $N = M^{(\aleph)}$, which is a contradiction. Let Λ be a chain in Γ . Then $s(N \cap \bigcup_{\alpha \in \Lambda} M^{(\alpha)}) = \bigcup_{\alpha \in \Lambda} s(N \cap M^{(\alpha)}) = 0$. So Λ has an upper bound. By Zorn's Lemma, there is a maximal element, say β , in Γ . Set $L = M^{(\beta)}$. We claim that $M^{(\aleph)}$ is u - S -isomorphic to $N + L$. Otherwise, since $M^j \cong M$ is u - S -simple, there is an $M^j \not\subseteq N + L$ for some $j \in \aleph$. Hence $N \cap M^{(\beta \cup \{j\})}$ is u - S -torsion with respect to s , which contradicts the maximality of β . Since $N \cap L$ is u - S -torsion, N is u - S -isomorphic to $N/(N \cap L)$ and $(N + L)/(N \cap L)$ is u - S -isomorphic to $(N + L)$, which is also u - S -isomorphic to $M^{(\aleph)}$. Considering the split monomorphism $g : N/(N \cap L) \rightarrow (N + L)/(N \cap L)$, we have that the natural embedding map $N \rightarrow M$ is also a u - S -split monomorphism. $\square$

Theorem 3.7. *Let R be a ring and S a multiplicative subset of R . Suppose R is a u - S -semisimple ring. Then R is a u - S -Artinian ring.*

Proof. Suppose R is a u - S -semisimple ring. Let $\aleph$ be a cardinal larger than $2^{\sharp(R)} \cdot \aleph_0$, where $\sharp(R)$ is the cardinal of R . Then the free R -module $R^{(\aleph)}$ is u - S -semisimple with respect to some $s \in S$. And so every subquotient of $R^{(\aleph)}$ is also u - S -semisimple with respect to some $s \in S$. Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R . Note that there are at most $2^{\sharp(R)} \cdot \aleph_0$ such chains. Set $R = I_0$. Consider the exact sequence $\xi_i : 0 \rightarrow I_i \rightarrow I_{i-1} \rightarrow I_{i-1}/I_i \rightarrow 0$ for any positive integer i . Since R is a u - S -semisimple ring, each I_{i-1}/I_i is u - S -projective by [19, Theorem 3.5]. So, by [19, Corollary 2.10], each ξ_i is u - S -split with respect to s . Hence, by [19, Lemma 2.4], there is a u - S -isomorphism $f_i : I_{i-1} \rightarrow I_i \oplus I_{i-1}/I_i$ with respect to s for each i . So there are u - S -isomorphisms

$$R \xrightarrow{f_1} I_1 \bigoplus R/I_1 \xrightarrow{f_2 \oplus \text{Id}} I_2 \bigoplus I_1/I_2 \bigoplus R/I_1 \rightarrow \cdots$$

$$\xrightarrow{f_k \oplus \text{Id}} I_k \bigoplus \left(\bigoplus_{i=0}^k I_i/I_{i-1} \right) \rightarrow \cdots$$

Assume $f(1) \in \bigoplus_{i=1}^k I_{i-1}/I_i \subseteq \bigoplus_{i=1}^\infty I_{i-1}/I_i$, where $f = \lim_{\rightarrow k} ((f_k \oplus \text{Id}) \circ \cdots \circ f_1)$. Then R is u - S -isomorphic to $\bigoplus_{i=0}^k I_i/I_{i-1}$ with respect to s . And so I_k is u - S -torsion with respect to s . Hence $sI_k/I_n = 0$, i.e., $sI_k \subseteq I_n$ for all $n \geq k$. So R is a u - S -Artinian ring. $\square$

Corollary 3.8. *Let R be a ring and S a multiplicative subset of R . Suppose R is a u - S -semisimple ring. Then any S -finite R -module is u - S -Artinian.*

Proof. Since the class of u - S -Artinian modules is closed under u - S -isomorphisms, we just need to show that any finitely generated R -module is u - S -Artinian, which can easily be deduced by Proposition 2.9 and Theorem 3.7. $\square$

4. A CHARACTERIZATION OF u - S -ARTINIAN RINGS

It is well known that a ring R is Artinian if and only if R is a Noetherian ring with its Jacobson radical $\text{Jac}(R)$ nilpotent and $R/\text{Jac}(R)$ a semisimple ring (see [17, Theorem 4.1.10]). In this section, we will give a “uniform” S -version of this result. We begin with the notion of u - S -maximal submodules and the u - S -Jacobson radical of a given R -module.

Definition 4.1. Let R be a ring, S a multiplicative subset of R and $s \in S$. Then a submodule N is said to be u - S -maximal in an R -module M with respect to s provided that

- (1) M/N is not u - S -torsion with respect to s ;
- (2) if $N \subsetneq H \subseteq M$, then M/H is u - S -torsion with respect to s .

Note that an R -module N is a u - S -maximal submodule of M (with respect to s) if and only if M/N is u - S -simple (with respect to s). If M does not have any u - S -maximal submodule with respect to s , then we set $\text{Jac}_s(M) = M$. Otherwise, we denote by $\text{Jac}_s(M)$ the intersection of all u - S -maximal submodules of M with respect to s . The submodule $\text{Jac}_s(M)$ of M is called the u - S -Jacobson radical of M with respect to s .

Definition 4.2. Let R be a ring, S a multiplicative subset of R and M an R -module. Then the u - S -Jacobson radical of M is defined to be $\text{Jac}_S(M) = \bigcap_{s \in S} \text{Jac}_s(M)$ under the above notions.

First, we have the following observation.

Lemma 4.3. Let M be an R -module. Then

$$\begin{aligned} \text{Jac}_s(M/\text{Jac}_s(M)) &= 0, \\ \text{Jac}_S(M/\text{Jac}_S(M)) &= 0. \end{aligned}$$

Proof. We just note that an R -module $N/\text{Jac}_s(M) \subseteq M/\text{Jac}_s(M)$ is u - S -maximal with respect to s if and only if $N + \text{Jac}_s(M) \subseteq M$ is u - S -maximal with respect to s . $\square$

Proposition 4.4. Suppose M is a u - S -Artinian R -module with $\text{Jac}_S(M)$ u - S -torsion. Then there exists $T \subseteq M$ such that $sT = 0$ and $M/T \subseteq \bigoplus_{i=1}^t S_i$, where each S_i is u - S -simple with respect to s for some $s \in S$. Consequently, M is a u - S -Noetherian R -module.

Proof. If M has no u - S -maximal submodule, then we may assume $T = \text{Jac}_S(M) = M$ is u - S -torsion. So the assertion trivially holds. Now suppose M has a u - S -maximal submodule. Then the intersection of all u - S -maximal submodules of M is u - S -torsion. Since M is u - S -Artinian, there exists a finite family of u - S -maximal submodules, say $\{M_1, \dots, M_n\}$, of M such that $T := \bigcap_{i=1}^n M_i$ is u - S -torsion with respect to some $t \in S$ by Theorem 2.15. Note that M/T is a submodule of $\bigoplus_{i=1}^n M/M_i$, where M/M_i is u - S -simple with respect to some $s_i \in S$. Set $s = ts_1 \cdots s_n$. Then each $S_i := M/M_i$ is u - S -simple with respect to s . One can

easily check that M/T , as a submodule of $\bigoplus_{i=1}^n M/M_i$, is u - S -Noetherian with respect to s . Thus M is u - S -Noetherian with respect to s . $\square$

Proposition 4.5. *If R is a u - S -Artinian ring, then $R/\text{Jac}_S(R)$ is a u - $S/\text{Jac}_S(R)$ -semisimple ring.*

Proof. Write $J = \text{Jac}_S(R)$, $\bar{R} = R/J$ and $\bar{S} = S/\text{Jac}_S(R)$. Since R is a u - S -Artinian ring, $\bar{R}$ is a u - S -Artinian R -module by Proposition 2.9. By Lemma 4.3, $\text{Jac}_S(\bar{R}) = 0$. It follows from Proposition 4.4 that there exists an element $s \in S$ and a submodule T of $\bar{R}$ such that $sT = 0$ and $\bar{R}/T \subseteq \bigoplus_{i=1}^t S_i$ with each S_i u - S -simple with respect to s . Now let $F = \bar{R}^{(\aleph)}$ be a free $\bar{R}$ -module with $\aleph$ an arbitrary cardinal. Then there is a short exact sequence $0 \rightarrow T^{(\aleph)} \rightarrow \bar{R}^{(\aleph)} \rightarrow (\bar{R}/T)^{(\aleph)} \rightarrow 0$. By Proposition 4.4, $(\bar{R}/T)^{(\aleph)}$ is a submodule of $(\bigoplus_{i=1}^t S_i)^{(\aleph)} \cong \bigoplus_{i=1}^t (S_i)^{(\aleph)}$ which is u - S -semisimple by Proposition 3.6. Hence $(\bar{R}/T)^{(\aleph)}$ is also u - S -semisimple by [19, Proposition 3.3]. Since $sT^{(\aleph)} = 0$, $(\bar{R})^{(\aleph)}$ is a u - S -semisimple R -module. So $(\bar{R})^{(\aleph)}$ is actually a u - $\bar{S}$ -semisimple $\bar{R}$ -module, that is, $\bar{R}$ is a u - $\bar{S}$ -semisimple ring. $\square$

Lemma 4.6. *Let R be a ring, S a multiplicative subset of R and S^* the saturation of S . Suppose $r \in R$ and $s \in S$. If $r - s \in \text{Jac}_S(R)$, then $r \in S^*$.*

Proof. Assume on the contrary that $r \notin S^*$. Then we claim R/Rr is not u - S -torsion. Indeed, if $t(R/Rr) = 0$ for some $t \in S$, then $t = rr'$ for some $r' \in R$. So $r \in S^*$, which is a contradiction. We also claim that there exists a u - S -maximal ideal I of R such that $r \in I$. Indeed, let Λ be the set of ideals J of R that contain r satisfying that R/J is not u - S -torsion. One can check that the union of any ascending chain in Λ is also in Λ . So there is a maximal element I in Λ by Zorn's Lemma. Hence I is a u - S -maximal ideal of R . Since $\text{Jac}_S(R) \subseteq I$, we have $s \in I$. Then $s(R/I) = 0$, which is a contradiction. So $r \in S^*$. $\square$

Proposition 4.7 (Nakayama Lemma for S -finite modules). *Let R be a ring, $I \subseteq \text{Jac}_S(R)$, S a multiplicative subset of R and M an S -finite R -module. If $sM \subseteq IM$ for some $s \in S$, then M is u - S -torsion.*

Proof. Let F be a finitely generated submodule, say generated by $\{m_1, \dots, m_n\}$, of M satisfying $s'M \subseteq F$ and $sM \subseteq IM$ for some $s', s \in S$. Then $ss'F \subseteq IF$. So we can assume M itself is generated by $\{m_1, \dots, m_n\}$. Then we have $a := s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0$, where $a_i \in I^i$ by [14, Theorem 2.1]. Note that $aM = 0$ and $a - s^n \in I$. By Lemma 4.6, $a \in S^*$, where S^* is the saturation of S . It follows that there is $r \in R$ such that $ar \in S$. Hence $arM = 0$, and so M is u - S -torsion. $\square$

Let I be an ideal of R . We say I is S -nilpotent if $sI^k = 0$ for some integer k and $s \in S$.

Proposition 4.8. *Let R be a ring and S a multiplicative subset of R . Suppose R is a u - S -Artinian ring. Then $\text{Jac}_S(R)$ is S -nilpotent.*

Proof. Suppose R is a u - S -Artinian ring with respect to some $t \in S$. Write $J = \text{Jac}_S(R)$. Consider the descending chain

$$J \supseteq J^2 \supseteq \dots \supseteq J^n \supseteq J^{n+1} \supseteq \dots$$

Then there exists an integer k such that $tJ^k \subseteq J^n$ for some $n \geq k$. We claim that $sJ^k = 0$ for some $s \in S$. On the contrary, set $\Gamma = \{I \subseteq R \mid sJ^k I \neq 0 \text{ for all } s \in S\}$. Since $R \in \Gamma$, Γ is nonempty. So there is an S -minimal element I in Γ by Theorem 2.15. Let $x \in I$ be such that $sJ^k x \neq 0$ for all $s \in S$. Then $0 \neq stJ^k x \subseteq sJ^{k+1}x$. So $sJ^{k+1}x \neq 0$ for any $s \in S$. Hence $Jx \in \Gamma$. Since $Jx \subseteq Rx \subseteq I$, there exists $s_1 \in S$ such that $s_1 Rx \subseteq s_1 I \subseteq Jx \subseteq Rx$ by the S -minimality of I . So there exists $s_2 \in S$ such that $s_2 Rx = 0$ by Proposition 4.7, which contradicts $sJ^k x \neq 0$ for all $s \in S$. $\square$

Recently, Qi et al. [8, Definition 2.1] introduced the notion of u - S -Noetherian rings. A ring R is called a u - S -Noetherian (abbreviates uniformly S -Noetherian) ring provided there exists an element $s \in S$ such that for any ideal I of R , $sI \subseteq K$ for some finitely generated sub-ideal K of I . Finally, we will show the promised result.

Theorem 4.9. *Let R be a ring and S a multiplicative subset of R . Then the following statements are equivalent:*

- (1) R is a u - S -Artinian ring;
- (2) R is a u - S -Noetherian ring, $\text{Jac}_S(R)$ is S -nilpotent and $R/\text{Jac}_S(R)$ is a u - $S/\text{Jac}_S(R)$ -semisimple ring.

Proof. (1) $\Rightarrow$ (2): Suppose R is a u - S -Artinian ring with respect to some $s \in S$. We just need to prove that R is u - S -Noetherian, because the other two statements are shown in Proposition 4.8 and Proposition 4.5 respectively. Write $J = \text{Jac}_S(R)$. By Proposition 4.8, there exists an integer m such that $tJ^m = 0$ for some $t \in S$. We will show that R is u - S -Noetherian by induction on m . Let $m = 1$. It follows by Proposition 4.4 that R is u - S -Noetherian. Now, let $m > 1$. Set $\bar{R} = R/J^{m-1}$. Then $\bar{R}$ is also u - S -Artinian by Proposition 2.9. Note that $\text{Jac}_S(\bar{R}) = J/J^{m-1}$. So $\text{Jac}_S(\bar{R})^{m-1}$ is also u - S -torsion. Hence $\bar{R}$ is u - S -Noetherian by induction. Since $tJ^m = 0$, tJ^{m-1} can also be seen as an ideal of R/J . Since R/J is a u - S/J -semisimple ring, R/J is also u - S/J -Noetherian by [19, Corollary 3.6]. So tJ^{m-1} , and hence J^{m-1} , are both u - S -Noetherian R -modules since J^{m-1} is u - S -isomorphic to tJ^{m-1} . Considering the exact sequence $0 \rightarrow J^{m-1} \rightarrow R \rightarrow R/J^{m-1} \rightarrow 0$, we have that R is also u - S -Noetherian by [8, Lemma 2.12].

(2) $\Rightarrow$ (1): Write $J = \text{Jac}_S(R)$. We may assume R is u - S -Noetherian with respect to s such that $sJ^m = 0$ and R/J is a u - S -semisimple R -module with respect to s . We claim that J is a u - S -Artinian R -module. Since sJ^{m-1} is an S -finite R/J -module and R/J is a u - S -semisimple ring, we have that sJ^{m-1} is u - S -Artinian by Corollary 3.8. Consider the sequence $0 \rightarrow sJ^{m-1} \rightarrow sJ^{m-2} \rightarrow sJ^{m-2}/sJ^{m-1} \rightarrow 0$. Since sJ^{m-2}/sJ^{m-1} is an S -finite R/J -module, we have that sJ^{m-2}/sJ^{m-1} is u - S -Artinian by Corollary 3.8. Since sJ^{m-1} is u - S -Artinian, sJ^{m-2} is also u - S -Artinian

by Proposition 2.9. Iterating these steps, we have that sJ is also u - S -Artinian. So J is also u - S -Artinian since J is u - S -isomorphic to sJ .

Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a descending chain of ideals of R . Consider the following natural commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_i \cap J & \longrightarrow & I_i & \longrightarrow & (I_i + J)/J \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_{i+1} \cap J & \longrightarrow & I_{i+1} & \longrightarrow & (I_{i+1} + J)/J \longrightarrow 0. \end{array}$$

Since J is u - S -Artinian, there is an integer k_1 such that $s(I_{k_1} \cap J) \subseteq I_n \cap J$ for $n \geq k_1$. Since R/J is a u - S/J -semisimple ring, R/J is also a u - S/J -Artinian ring by Theorem 3.7. Hence there is an integer k_2 such that $s((I_{k_2} + J)/J) \subseteq (I_n + J)/J$ for $n \geq k_2$. Taking $k = \max\{k_1, k_2\}$, we can easily deduce that $sI_k \subseteq I_n$ for $n \geq k$. Hence R is a u - S -Artinian ring. $\square$

Let R be a commutative ring and M an R -module. Then the idealization of R by M , denoted by $R(+M)$, is equal to $R \oplus M$ as R -modules with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. It is easy to verify that $R(+M)$ is a commutative ring with identity $(1, 0)$ (see [4] for more details). Note that there is a natural exact sequence of $R(+M)$ -modules:

$$0 \rightarrow 0(+M) \rightarrow R(+M) \xrightarrow{\pi} R \rightarrow 0.$$

Let S be a multiplicative subset of R . Then it is easy to verify that $S(+M) = \{(s, m) \mid s \in S, m \in M\}$ is a multiplicative subset of $R(+M)$. Now, we give a u - S -Artinian property on idealizations.

Proposition 4.10. *Let R be a commutative ring, S a multiplicative subset of R and M an R -module. Then $R(+M)$ is a u - $S(+M)$ -Artinian ring if and only if R is a u - S -Artinian ring and M is an S -finite R -module.*

Proof. Suppose $R(+M)$ is a u - $S(+M)$ -Artinian ring. Then $R \cong R(+M)/0(+M)$ is a u - S -Artinian ring essentially by Proposition 2.9. By Theorem 4.9, $R(+M)$ is a u - $S(+M)$ -Noetherian ring. So $0(+M)$ is an $S(+M)$ -finite ideal of $R(+M)$, which implies that M is an S -finite R -module.

Suppose R is a u - S -Artinian ring and M is an S -finite R -module. Then M is u - S -Artinian R -module by Proposition 2.9. Let $I^\bullet : I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of ideals of $R(+M)$. Then there is a descending chain of ideals of R : $\pi(I^\bullet) : \pi(I_1) \supseteq \pi(I_2) \supseteq \cdots$, where $\pi : R(+M) \twoheadrightarrow R$ is the natural epimorphism. Thus there is an element $s' \in S$ which is independent of $I^\bullet$ satisfying that there exists $k' \in \mathbb{Z}^+$ such that $s'\pi(I_{k'}) \subseteq \pi(I_n)$ for any $n \geq k'$. Similarly, $I^\bullet \cap 0(+M) : I_1 \cap 0(+M) \supseteq I_2 \cap 0(+M) \supseteq \cdots$ is a descending chain of sub-ideals of $0(+M)$ which is equivalent to a descending chain of submodules of M . So there is an element $s'' \in S$ satisfying that there exists $k'' \in \mathbb{Z}^+$ such that $s''(I_{k''} \cap 0(+M)) \subseteq I_n \cap 0(+M)$ for any $n \geq k''$. Let $k = \max(k', k'')$ and $n \geq k$. Consider the following natural

commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I_n \cap 0(+)M & \longrightarrow & I_n & \longrightarrow & \pi(I_n) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & I_k \cap 0(+)M & \longrightarrow & I_k & \longrightarrow & \pi(I_k) \longrightarrow 0.
 \end{array}$$

Set $s = s's''$. Then we have $sI_k \subseteq I_n$ for any $n \geq k$. So $R(+)M$ is a u - $S(+)M$ -Artinian ring. $\square$

Taking $S = \{1\}$ in Proposition 4.10, we can easily deduce the following classical result.

Corollary 4.11. *Let R be a commutative ring and M an R -module. Then $R(+)M$ is an Artinian ring if and only if R is an Artinian ring and M is a finitely generated R -module.*

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