REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 69, No. 2, 2026, Pages 485–497
Published online: August 21, 2026
https://doi.org/10.33044/revuma.5405

BOTT–DUFFIN (e, f) INVERSES IN e-SYMMETRIC RINGS

SHIBO SHI, YUHENG LIU, AND LONG WANG

Abstract. We investigate several properties of Bott–Duffin (e, f) inverses in e-symmetric rings. In particular, we present new characterizations of Bott–Duffin (e, f) inverses in terms of units in such rings. We also study the EP property of an element a in an ab-symmetric ring, where b is a generalized inverse of a. Finally, we consider the Bott–Duffin decomposition associated with idempotents.

1. Introduction

Throughout this paper we assume that R is a ring with the unity 1. Drazin [3] introduced a class of outer generalized inverses relative to a pair of idempotents e and f, which intermediates between the Bott–Duffin inverse [1] and the (b, c)-inverse [3] in a ring. This particular type of generalized inverse is denominated as the Bott–Duffin (e, f) inverse. The Bott–Duffin (e, f) inverse of a is defined as the unique element $y \in R$, when it exists, that satisfies the equations $y = ey = yf$, $yae = e$ and $fay = f$. Whenever it is more convenient, we shall use the widely accepted shorthand "B-D" for "Bott–Duffin".

In [3, Theorem 2.2], it is proven that a is B-D (e, f) invertible if and only if $e \in Rfae$ and $f \in faeR$. In particular, a has a B-D inverse if and only if $e \in Reae \cap eaeR$. Note that $e \in Reae \cap eaeR$ holds if and only if $eae + 1 - e$ is invertible in R. Consequently, it is natural to determine whether the existence of B-D (e, f) inverses can be characterized by means of the invertibility of specific elements. To our knowledge, the investigation of this topic can be found in [5, 2]. In [5], it has been demonstrated that a is B-D (e, f) invertible if and only if there exist $s, t \in R$ such that both $u = sfae+1-e$ and $v = faet+1-f$ are invertible in R. In [2, Theorem 2.2], under the premise that e and f are self-adjoint idempotents, the criteria for the existence of B-D (e, f) inverses are presented in terms of units. Herein, we recall that $*$ is an involution in R if it is a map $* : R \to R$ such that for all $a, b \in R : (a^*)^* = a, (a + b)^* = a^* + b^*, (ab)^* = b^*a^*$. In this case, we need to

2020 *Mathematics Subject Classification.* Primary 15A09; Secondary 16W10, 16U90, 16E50.
Key words and phrases. symmetric ring, e-symmetric ring, Bott–Duffin (e, f)-inverse, unit.
The research is supported by the National Natural Science Foundation of China (11901510, 11761017), the Natural Science Foundation of Jiangsu Province (BK20200944, BK20220589), and the Anhui Provincial Department of Education Natural Science Research Project (2025AHGXZK30855).

assume that R is a ring with an involution (that is to say, R is a $*$-ring). Moreover, an idempotent e is said to be self-adjoint if $e = e^*$. It has been established that the element a is B-D (e, f) invertible if and only if both $u = (fae)^* fae + 1 - e$ and $v = fae(fae)^* + 1 - f$ are invertible [2].

In the reference [10], the authors introduced a class of weakly symmetric rings, namely e-symmetric rings, where e is an idempotent in R. Recall that R is e-symmetric if the condition $abc = 0$ necessarily implies $acbe = 0$. An idempotent e of a ring R is called left (right) semicentral if $re = ere$ ($er = ere$) for each $r \in R$. It is proven that R is e-symmetric if and only if e is left semicentral and eRe is symmetric. The studies of symmetric rings and their generalized forms can also be traced in a series of works, including [7, 4, 8, 10, 11, 9, 15, 12, 16]. Inspired by the above discussion, in this article, with a focus on the characterizations of B-D (e, f) inverses, we endeavor to further explore how the existence of B-D (e, f) inverses can be characterized by the invertibility of specific elements within the context of e-symmetric rings.

In what follows, some standard terminology and notations in rings are given. We use $E(R)$ to denote the set of all idempotents, and R^{-1} to denote the set of all invertible elements in R. An element $a \in R$ is said to be (von Neumann) regular if there exists an element $x \in R$ such that $axa = a$. Such an x is referred to as a regular (or inner) inverse of a and is denoted by a^-. The group inverse of a is the unique element $a^\sharp$, whenever it exists, satisfying $aa^\sharp a = a$, $a^\sharp a a^\sharp = a^\sharp$ and $aa^\sharp = a^\sharp a$. We use $R^\sharp$ to denote the set of all group invertible elements. The set consisting of all self-adjoint idempotent elements of R (i.e., $e = e^2 = e^*$) will be denoted by $P(R)$. An element $a \in R$ is said to be Moore–Penrose invertible with respect to the involution $*$ if the four equations

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa$$

have a unique common solution. When such a solution exists, it is denoted by $a^\dagger$, and the set of all Moore–Penrose invertible elements in R is denoted by $R^\dagger$. If an element $x \in R$ satisfies both Eqs. (1) and (3), then x is called a $\{1, 3\}$-inverse of a, and is denoted by $a^{(1,3)}$. The set of all $\{1, 3\}$-invertible elements of R is denoted by $R\{1, 3\}$. Analogously, if $x \in R$ satisfies both Eqs. (1) and (4), then x is referred to as a $\{1, 4\}$-inverse of a, and is denoted by $a^{(1,4)}$. The set of all $\{1, 4\}$-invertible elements in R is denoted by $R\{1, 4\}$.

2. Bott–Duffin (e, f) inverses in e-symmetric rings

The inspiration for this section stems from the following results in [10]. In this reference, the authors consider some properties of B-D (e, f) inverses in e-symmetric rings.

Lemma 2.1 ([10, Proposition 2.6]). *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a has a B-D (e, f) inverse, then R is f-symmetric and $eR = fR$.*

In point of fact, a straightforward verification reveals that the converse also holds true.

Lemma 2.2. *Let $e, f \in E(R)$ such that $fR \subseteq eR$. If R is e-symmetric, then R is f-symmetric.*

Proof. For arbitrary elements $a, b, c \in R$, whenever $abc = 0$ holds, it can be deduced that $acbe = 0$. As $fR \subseteq eR$, it follows that $f = ef$. As a result, we have that $acbf = acbef = 0$. This shows that R is also f-symmetric. □

Proposition 2.3. *Let $a \in R$ and $e, f \in E(R)$. If a is B-D (e, f) invertible, then R is e-symmetric if and only if R is f-symmetric.*

Proof. Necessity: As demonstrated in [10, Proposition 2.6], the necessary condition holds. Sufficiency: Suppose that the ring R is f-symmetric. Then, it can be deduced that the idempotent f is left semicentral. Let y denote the B-D (e, f) inverse of a; we have the following sequence of equalities:

$$e = yae = yfae = fyfae = fyae = fe.$$

This implies that $eR \subseteq fR$, and then by Lemma 2.2 we obtain that R is also e-symmetric. □

In the previous proposition, the condition that a is B-D (e, f) invertible must be presupposed. We next explore the equivalent conditions for the existence of B-D (e, f) inverses.

Lemma 2.4. *Let $a \in R$ and $e \in E(R)$. Then the following conditions are equivalent:*

(i) $e \in eaeR \cap Reae$.
(ii) $eae + 1 - e$ *is invertible.*
(iii) $ae + 1 - e$ *is invertible.*
(iv) $ea + 1 - e$ *is invertible.*

Lemma 2.5 (Jacobson's Lemma)**.** *Let $a, b \in R$. Then $1 - ab \in R^{-1}$ if and only if $1 - ba \in R^{-1}$.*

Theorem 2.6. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, the following are equivalent:*

(i) *a is B-D (e, f) invertible.*
(ii) $u = fae + 1 - f$, $v = fae + 1 - e \in R^{-1}$.

In this case, the B-D (e, f) inverse is $y = eu^{-1} = v^{-1}f$.

Proof. (i) ⇒ (ii). In this context, our primary objective is to demonstrate that $f \in Rf(fae)f \cap f(fae)fR$ and $e \in e(fae)eR \cap Re(fae)e$. Once this is established, by Lemma 2.4, we can infer the invertibility of u and v in R. Let y denote the B-D (e, f) inverse of a. First, we show that $f \in f(fae)fR$. Given that $y = ey = yf$ and f is left semicentral (see Proposition 2.3), we have the following equalities:

$$f = fay = fa(ey) = fae(yf) = faefyf = faefy \in f(fae)fR.$$

This clearly indicates that $f \in f(fae)fR$. Furthermore, on account of $eR = fR$, by Lemma 2.1, we conclude $f = ef$ and $e = fe$. Then, we can show that $f \in Rf(fae)f$

as follows:

$$f = fay = fa(yf) = fay(ef) = fay(yae)f = fay(yf)aef \in Rfaef.$$

Next, we turn our attention to proving that $e \in e(fae)eR \cap Re(fae)e$. For the left-hand side of the intersection, we note that

$$e = yae = yfae = y(ef)ae \in Re(fae)e.$$

This implies that $e \in Re(fae)e$. On the other hand, we have

$$e = yae = (fy)ae = (ef)yae = e(fay)yae = efa(ey)yae \in e(fae)eR.$$

This shows that $e \in e(fae)eR$. Combining these, we conclude that $e \in e(fae)eR \cap Re(fae)e$.

(ii) $\Rightarrow$ (i). Note that $fu = ve = fae$. Set $y = eu^{-1} = v^{-1}f$. Then, it immediately follows that $ey = yf = y$. Furthermore, through straightforward calculations, we have $yae = v^{-1}fae = e$ and $fay = faeu^{-1} = f$. Consequently, based on the definition of the B-D (e, f) inverse, a is B-D (e, f) invertible and its B-D (e, f) inverse is y. □

Remark 2.7. In Theorem 2.6, it can be observed that $y = yf = fy$ and $y = ey$. Specifically, the equality $y = ey = yf$ aligns precisely with the definition of the B-D (e, f) inverse. Regarding the equality $y = fy$, since f is left semicentral, we have $y = yf = fyf = fy$. Under the conditions of the theorem, it is thus a worthwhile inquiry to consider whether $y = ye$ holds. Actually, we know that if $y = ye$, then $f = fay = faye = fe = e$. Conversely, when $e = f$, it necessarily follows that $y = yf = ye$. This interconnection can be more formally presented as follows.

Proposition 2.8. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a is Bott–Duffin (e, f) invertible with the B-D (e, f) inverse y, then y is also the B-D (f, e) inverse of a if and only if $e = f$.*

In actuality, when we solely focus on the B-D (e, f) invertibility, it can be proved that if R is e-symmetric and a is B-D (e, f) invertible, then a is also B-D (f, e) invertible. However, according to Proposition 2.8, these two inverses are generally not identical, except when the condition $e = f$ is satisfied.

Lemma 2.9. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a is B-D (f, e) invertible, then R is also f-symmetric and $eR = fR$.*

Proof. To begin with, it suffices for us to demonstrate that $fR \subseteq eR$. Once this is established, it is evident that R is f-symmetric. Indeed, let z be established as the B-D (f, e) inverse of a. Then, by the equalities $f = zaf = (ze)af = ezeaf$, we can conclude that $ef = f$. Thus, R is f-symmetric, from which it further follows that f is left semicentral. Moreover, when considering the equation $e = eaz = eafz = feafz$, it follows that $fe = e$. Based on this established fact, it is evident that eR and fR are equal sets, i.e., $eR = fR$. □

Similar to the proof of Theorem 2.6, we can obtain the following result.

Corollary 2.10. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, the following are equivalent:*

(i) *a is B-D (f, e) invertible.*
(ii) *$u = fae + 1 - f$, $v = fae + 1 - e \in R^{-1}$.*

Proof. (i) $\Rightarrow$ (ii). Analogously to the proof of Theorem 2.6, we also can prove that $f \in Rf(fae)f \cap f(fae)fR$ and $e \in e(fae)eR \cap Re(fae)e$. Indeed, assuming that a is B-D (f, e) invertible, and denoting by z its B-D (f, e) inverse, by Lemma 2.9 we derive the following equality:

$$\begin{aligned} f &= zaf = za(ef) = fzaef = f(ezae)f = f(eaz)zaef \\ &= faezzaef = faezzfaef = faefzzfaef. \end{aligned}$$

This clearly shows that $f \in f(fae)fR \cap Rf(fae)f$. Similarly, we can also prove that $e \in e(fae)eR \cap Re(fae)e$.

(ii) $\Rightarrow$ (i). According to Theorem 2.6, the B-D (e, f) inverse of a is given by $y = eu^{-1} = v^{-1}f$. Let us define $z = ye$. Then it can be readily verified that $fz = fye = ye = z$ and $ze = z$. Additionally, the following equalities hold: $eaz = eaye = eayfe = efayfe = efe = fe = e$, and $zaf = yeaf = yefaf = yfaf = yaf = yaef = ef = f$. This is sufficient to show that a is B-D (f, e) invertible and its B-D (f, e) inverse is z. $\square$

Corollary 2.11. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, then a is B-D (e, f) invertible if and only if a is B-D (f, e) invertible. In this case, $z = ye$ and $y = zf$, where y is the B-D (e, f) inverse of a and z is the B-D (f, e) inverse of a.*

Remark 2.12. By Corollary 2.11, we can see the related properties of B-D (e, f) inverse also hold if we replace the e-symmetric condition with the f-symmetric condition.

Remark 2.13. Let $a \in N(R)$ and $e, f \in E(R)$. Suppose that R is e-symmetric and a is B-D (e, f) invertible. It follows that $e = f = 0$. Indeed, since R is e-symmetric and a is B-D (e, f) invertible, we have $fe = e$ and e is left semicentral. Thus, $(fae)^2 = faefeae = faeae = fa^2e$. It can be shown by induction that $(fae)^k = fa^ke$ for any positive integer k. As $a \in N(R)$, we have $a^m = 0$ for some positive integer m. Hence, $u^m = (fae)^m + 1 - e = 1 - e \in R^{-1}$, and consequently, $e = 0$. Similarly, we can conclude that $f = 0$ as well.

Question 2.14. Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and a is B-D (e, f) invertible with the B-D (e, f) inverse y, then we have $ay = ayf = fayf = f$. One may wonder whether $ya = e$.

Before talking about the question, we first provide the following examples to show that the full matrix ring $M_2(\mathbb{C})$ over the complex field fails to be e-symmetric for any nonzero idempotents. Moreover, the upper triangular matrix ring $T_2(\mathbb{C})$ over the complex field is e-symmetric for certain idempotents.

Example 2.15. Set $R = M_2(\mathbb{C})$ and $e \in E(R)$. If e is left semicentral, then $xe = exe$ for any $x \in R$. Herein, to avoid the trivial case, $e \neq 0$ and $e \notin R^{-1}$, that is, $e \neq I_2$. Indeed, let $e \in R^{-1}$. As $xe = exe$ for all $x \in R$, right-multiplication by e^{-1} gives $x = ex$ for each $x \in R$. Taking $x = I_2$, we obtain $e = I_2$. Moreover, it is well known that $R = M_2(\mathbb{C})$ is not symmetric in the general sense. Based on the above-mentioned non-trivial conditions and the non-symmetry of $R = M_2(\mathbb{C})$, one can conclude that the idempotent e has $\operatorname{rank}(e) = 1$ and has the following forms:

$$e = \begin{pmatrix} \lambda\alpha & \lambda\beta \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix}$$

or

$$e = \begin{pmatrix} \gamma & \gamma u \\ \delta & \delta u \end{pmatrix} = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}.$$

Case 1. Assume that $e = \left(\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & \beta \end{smallmatrix}\right)$. Let $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then, by a computation, we have

$$xe = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix},$$

$$exe = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} \lambda\alpha \\ \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix}.$$

Consequently, we deduce that $\lambda\alpha = 1$ and $\alpha = 0$, which is an obvious contradiction.

Case 2. Suppose $e = \left(\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & u \end{smallmatrix}\right)$. Since $e^2 = e$, we infer that $\left(\begin{smallmatrix} 1 & u \end{smallmatrix}\right)\left(\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}\right) = 1$, that is, $\gamma + u\delta = 1$.

Select $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. By a computation, we obtain

$$xe = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix},$$

$$exe = \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix} \begin{pmatrix} \delta \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix} = \begin{pmatrix} \delta\gamma \\ \delta^2 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}.$$

This implies that $\delta = 0$. Substituting $\delta = 0$ into the equation $\gamma + u\delta = 1$, we get $\gamma = 1$.

Now, choose $x = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. We find that

$$xe = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}$$

and

$$exe = \begin{pmatrix} u \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u \end{pmatrix}.$$

Evidently, $exe \neq xe$.

In conclusion, within the full matrix ring $R = M_2(\mathbb{C})$, if an idempotent $e \in E(R)$ is left semicentral, then e must be either the zero matrix O_2 or the identity matrix I_2. Therefore, we can conclude that the ring $R = M_2(\mathbb{C})$ is not e-symmetric for any nonzero idempotent e.

Example 2.16. Let $R = T_2(\mathbb{C})$ be the subring of 2×2 complex upper triangular matrices, and consider a nonzero idempotent $e \in R$. Easy computations show that e must take one of the following forms:

$$e = I_2 \quad \text{or} \quad e = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad e = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{C}$. If R is e-symmetric, then e is left semicentral. This means that for every $x \in T_2(\mathbb{C})$, the equality $xe = exe$ holds true. It is readily verified that under this condition, the idempotent e is restricted to the forms $e = I_2$ or $e = \left(\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right)$. Indeed, if we take $e = \left(\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right)$ and choose $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $xe \neq exe$. Furthermore, we can establish that R is e-symmetric whenever $e = \left(\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right)$. Indeed, for any elements $x, y, z \in R$ satisfying $xyz = 0$, we have $xzy = \left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. It follows that $xzye = \left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right) = O_2$, which confirms the e-symmetric property. Moreover, it is well known that $T_2(\mathbb{C})$ is not symmetric. Indeed, set $A = I_2$, $B = e_{12}$ and $C = e_{11}$. Then we have $ABC = 0$ while $ACB \neq 0$. Based on the above-mentioned arguments, the following conclusion can be drawn: Let $R = T_2(\mathbb{C})$ and $e \in R$ be a nonzero idempotent. Then, R is e-symmetric if and only if $e = \left(\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right)$ for some $b \in \mathbb{C}$.

The following discussion presents a counterexample to demonstrate that, in general, the assertion posed in Question 2.14 does not hold.

Example 2.17. It suffices to take $R = T_2(\mathbb{C})$ and $a = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $e = f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Clearly, R is e-symmetric and a is B-D (e, f) invertible with $y = e$ being the B-D (e, f) inverse. But one can check that $ay = f$ and $ya = a \neq e$.

Remark 2.18. Let $a \in R$ and $e, f \in E(R)$. Assume that R is e-symmetric and a is B-D (e, f) invertible. Then we can show that $v = fae + 1 - e \in R^{-1}$. By applying Lemma 2.5 and $ef = f$, we can deduce that $s = fa + 1 - e \in R^{-1}$. In light of these results, we propose the following conjectures: The elements $\tilde{s} = af + 1 - e$, $t = ea + 1 - f$ and $\tilde{t} = ae + 1 - f$ are all invertible in R. Furthermore, assume that $t = ea + 1 - f \in R^{-1}$. This assumption entails that $fea + 1 - f \in R^{-1}$ and further that $eaf + 1 - f \in R^{-1}$. Based on the above, it is reasonable to investigate the following result.

Proposition 2.19. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and $eR \subseteq fR$, then the following are equivalent:*

(i) $x_1 = eaf + 1 - f \in R^{-1}$.
(ii) $x_2 = ea + 1 - f \in R^{-1}$.
(iii) $x_3 = ae + 1 - f \in R^{-1}$.
(iv) $x_4 = fae + 1 - f \in R^{-1}$.
(v) $x_5 = eae + 1 - f \in R^{-1}$.

Proof. Since R is e-symmetric, we have $fR \subseteq eR$ under all circumstances. Indeed, given that $x_i \in R^{-1}$ and e is left semicentral, we conclude that for some $r_i, \tilde{r}_i \in R$, $fx_i = fr_ie\tilde{r}_i = efr_ie\tilde{r}_i$, which implies that $f = efr_ie\tilde{r}_ix_i^{-1}$ and $fR \subseteq eR$. When combined with $eR \subseteq fR$, it follows that $ef = f$, $fe = e$ and R is also f-symmetric. This indicates that both e and f are left semicentral. Consequently, we can derive the equalities $ae = afe = fafe = fae = feae = eae$, which gives that $x_3 = x_4 = x_5$. Next, it suffices to prove (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). First, we start with the following equivalence relation based on Lemma 2.5:

$$eaf + 1 - f \in R^{-1} \Leftrightarrow fea + 1 - f = ea + 1 - f \in R^{-1}.$$

This leads to (i) $\Leftrightarrow$ (ii). Moreover, by $eaf = eaef$, we obtain

$$eaf + 1 - f = eaef + 1 - f \in R^{-1} \Leftrightarrow feae + 1 - f \in R^{-1}.$$

Note that $fe = e$ and e is left semicentral. Then we can conclude that $feae = eae = ae$, and then we arrive at the equivalence

$$eaf + 1 - f \in R^{-1} \Leftrightarrow ae + 1 - f \in R^{-1}.$$

This means (ii) $\Leftrightarrow$ (iii). The proof is completed. $\square$

Analogously, we can derive the next result.

Proposition 2.20. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric and $fR \subseteq eR$, then*

$$\begin{aligned} y_1 = eaf + 1 - e \in R^{-1} &\Leftrightarrow y_2 = fa + 1 - e \in R^{-1} \Leftrightarrow y_3 = af + 1 - e \in R^{-1} \\ &\Leftrightarrow y_4 = fae + 1 - e \in R^{-1} \Leftrightarrow y_5 = faf + 1 - e \in R^{-1}. \end{aligned}$$

Proof. Since R is e-symmetric and $fR \subseteq eR$, we can obtain that R is also f-symmetric, and then f is left semicentral. Moreover, by checking all conditions, we can see $e \in fR$. Indeed, for any invertible element y_i, $ey_i \in fR$ and then $e \in fR$, which implies that $e = fe$, $f = ef$ and both e, f are left central. For the subsequent equivalences among y_1 through y_5, refer to the proof of Proposition 2.19. $\square$

Corollary 2.21. *Let $a \in R$ and $e, f \in E(R)$. If R is e-symmetric, the following are equivalent:*

(i) *a is Bott–Duffin (e, f) invertible.*
(ii) $u_1 = eaf + 1 - f$, $v_1 = eaf + 1 - e \in R^{-1}$.
(iii) $u_2 = ae + 1 - f$, $v_2 = af + 1 - e \in R^{-1}$.
(iv) $u_3 = ea + 1 - f$, $v_3 = fa + 1 - e \in R^{-1}$.
(v) $u_4 = ae + 1 - f$, $v_4 = fa + 1 - e \in R^{-1}$.
(vi) $u_5 = ea + 1 - f$, $v_5 = af + 1 - e \in R^{-1}$.

Proof. As R is e-symmetric, e is left semicentral. So by conditions (ii)–(vi), it can be inferred that $fu_i \in eR$ and $ev_i \in fR$, and then $eR = fR$. Hence, the proof can be completed as in the proofs of Theorem 2.6, Proposition 2.19, and Proposition 2.20. $\square$

Remark 2.22. (1) Let $a \in R$ and $e, f \in E(R)$. If $fR \subseteq eR$, we can prove that

$$u_1 = fae + 1 - f \in R^{-1} \Leftrightarrow af + 1 - f \in R^{-1}.$$

Indeed, since $fR \subseteq eR$, we obtain $ef = f$, and consequently, $af + 1 - f = aef + 1 - f \in R^{-1} \Leftrightarrow fae + 1 - f \in R^{-1}$.

(2) Let $a \in R$ and $e, f \in E(R)$. Assume that R is e-symmetric and $eR \subseteq fR$. Then we can show that

$$v_1 = fae + 1 - e \in R^{-1} \Leftrightarrow ea + 1 - e \in R^{-1}.$$

Indeed, from $eR \subseteq fR$ we can deduce that $e = fe$. Then, $ea+1-e = efea+1-e \in R^{-1} \Leftrightarrow feae + 1 - e = fae + 1 - e \in R^{-1}$. Herein, $feae = fae$, because e is left semicentral.

Remark 2.23. Based on the above analysis, we find that the conditions (i) e is left semicentral and (ii) $eR = fR$ are the key to describing the existence of B-D (e, f) inverses. Note that if the condition $u = fae + 1 - f \in R^{-1}$ holds in an e-symmetric ring, we obtain $fu = fae = efae$, and this gives that $fR \subseteq eR$ and R is also f-symmetric. Hence, we may conjecture that if R is e-symmetric, a is Bott–Duffin (e, f) invertible if and only if $u = fae + 1 - f \in R^{-1}$.

Example 2.24. (1) Let D be a division ring and $R = \mathbb{T}_2(D)$ be an upper triangular matrix ring over D. Choose $E = \operatorname{diag}(1, 0)$ and $F = O_2$. Firstly, we know that D is symmetric and R is E-symmetric. By a computation, one can see that $FAE + I_2 - F = I_2$, where $A = E$. However, $FAE + I_2 - E = \operatorname{diag}(0, 1)$ is not invertible in R. This means that A is not Bott–Duffin (E, F) invertible.

In other words, for an arbitrary ring R, given that $e, f \in E(R)$ and R is e-symmetric, the invertibility of the element $u = fae + 1 - f$ does not necessarily imply that a is Bott–Duffin (e, f) invertible.

(2) Consider the ring $R = \mathbb{Z}_6$ and let $a = e = 1$ and $f = 3$. Clearly, R is e-symmetric. Upon computation, it is found that $fae+1-f = 1$ and $fae+1-e = 3$ is not invertible in R.

In fact, it is well established that if a is Bott–Duffin (e, f) invertible, and either of the idempotents e or f is equal to 0, then the remaining idempotent is necessarily 0. An element $e \in E(R)$ is called left minimal idempotent if Re is a minimal left ideal. Write $ME_l^*(R)$ to denote the set of all nonzero left minimal idempotents of R, and $E^*(R)$ to denote the set of all nonzero idempotents of R. If e and $f \in E^*(R)$ and R is e-symmetric, the condition $u = fae + 1 - f \in R^{-1}$ does not imply that a is Bott–Duffin (e, f) invertible, as shown in Example 2.24 (2).

Theorem 2.25. *Let $a \in R$ and $e, f \in ME_l^*(R)$. If R is e-symmetric, the following are equivalent:*

(i) *a is Bott–Duffin (e, f) invertible;*
(ii) $u = fae + 1 - f \in R^{-1}$.

Proof. (i) $\Rightarrow$ (ii). See Theorem 2.6.

(ii) $\Rightarrow$ (i). Firstly, we claim R is also f-symmetric. Indeed, since R is e-symmetric, e is left semicentral. Hence, $fu = fae = efae$ and $f = faeu^{-1} = efaeu^{-1}$. Set $y = eu^{-1}f$. Evidently, $y = ey = yf$. Moreover, a straightforward calculation reveals that $fay = faeu^{-1}f = f$. Next, we aim to demonstrate that $(1 - f)e = 0$. If $(1 - f)e \neq 0$, because $e \in ME_l^*(R)$, we conclude that $R(1 - f)e = Re$, which implies $e = t(1 - f)e$ for some $t \in R$. Multiplying it on the right by f gives $ef = t(1 - f)ef$, and then, by $ef = f$, we have $f = t(1 - f)f = 0$, which is a contradiction. Hence, it follows from $(1 - f)e = 0$ that $ue = fae$ and $e = u^{-1}fae$. This shows that $yae = eu^{-1}fae = e$, and consequently, a is Bott–Duffin (e, f) invertible. ∎

3. Bott–Duffin inverses in e-symmetric $*$-rings

In this section, let R be a $*$-ring. An idempotent $e \in R$ is called a self-adjoint idempotent if $e^* = e$. We use $P(R)$ to denote the set of all self-adjoint idempotents in R. Recall that an element $a \in R$ is EP if $a^\dagger$ and $a^\sharp$ exist and satisfy $a^\dagger = a^\sharp$. It is clear that a is EP if and only if $a^\sharp$ exists and $aa^\sharp \in P(R)$. We now introduce some notations that will be used throughout this section. The symbol $C(R)$ denotes the center of R. That is, $C(R) = \{x : rx = xr \text{ for all } r \in R\}$. In particular, $C(a) = \{x \in R : ax = xa\}$.

Proposition 3.1. *Let $e \in P(R)$. If R is e-symmetric, then $e \in C(R)$.*

Proof. Since R is e-symmetric, we know that e is left semicentral, which means $re = ere$ for any $r \in R$. Taking the adjoint of both sides, we obtain $(re)^* = (ere)^*$, and consequently, $er^* = er^*e = r^*e$. This leads to $re = er$ and $e \in C(R)$. □

Corollary 3.2. *If $a \in R^\dagger$ and R is $aa^\dagger$-symmetric, then a is EP.*

Proof. By the hypothesis, we can prove that R is also $a^\dagger a$-symmetric. Indeed, it can be readily observed that $a^\dagger a = a^\dagger aa^\dagger a = aa^\dagger a^\dagger a \in aa^\dagger R$. By Lemma 2.2, this observation, combined with the lemma, allows us to conclude that R is $a^\dagger a$-symmetric. In accordance with Proposition 3.1, we can conclude that $aa^\dagger, a^\dagger a \in C(R)$, and then $a = a^2a^\dagger = a^\dagger a^2$. Consequently, this implies that $a \in R^\sharp$. Furthermore, since $a^\sharp a = a^\sharp a^2 a^\dagger = aa^\dagger$, we can conclude that a is EP. □

Moreover, similar to Corollary 3.2, we can obtain the following results.

Proposition 3.3. *Suppose that a is regular and a^- is any regular inverse of a. If R is aa^--symmetric, then $a \in R^\sharp$.*

Proof. Similar to the proof of Corollary 3.2, we can obtain that R is also a^-a-symmetric. Hence, a^-a is also left semicentral, and then $a = aa^-(aa^-)a = a(aa^-)a^-(aa^-)a \in a^2R$. Moreover,

$$a = aa^-a(a^-a) = aa^-(a^-a)a(a^-a) = aa^-(a^-a)a \in Ra^2.$$

This leads to $a \in R^\sharp$. □

Corollary 3.4. *If $a \in R\{1,3\}$ and R is $aa^{(1,3)}$-symmetric, then a is EP.*

Proof. By Proposition 3.3, we have $a \in R^\sharp$. Moreover, since R is $aa^{(1,3)}$-symmetric, by Proposition 3.1, we know that $aa^{(1,3)} \in C(R)$, and this leads to $a = aa^{(1,3)}a = a^2a^{(1,3)}$. Multiplying on the left by $a^\sharp$ gives that $a^\sharp a = aa^{(1,3)}$. This shows that a is EP. □

Corollary 3.5. *If $a \in R\{1,4\}$ and R is $aa^{(1,4)}$-symmetric, then a is EP.*

Proof. By Proposition 3.3, we have $a \in R^\sharp$. Moreover, as R is $aa^{(1,4)}$-symmetric, $aa^{(1,4)}$ is left semicentral, and this leads to $a^{(1,4)}a = a^{(1,4)}aa^{(1,4)}a = aa^{(1,4)}a^{(1,4)}a \in aa^{(1,4)}R$. So, we obtain that R is $a^{(1,4)}a$-symmetric, and then $a^{(1,4)}a \in C(R)$ by Proposition 3.1. Based on the fact that $a^{(1,4)}a \in C(R)$, we further conclude that $a = aa^{(1,4)}a = a^{(1,4)}a^2$. Multiplying on the right by $a^\sharp$ gives that $aa^\sharp = a^{(1,4)}a$. This shows that a is EP. □

Similar to the proofs of Corollaries 3.4 and 3.5, we can obtain the following results.

Corollary 3.6. *If $a \in R\{1,3\}$ (or $a \in R\{1,4\}$) and R is $a^{(1,3)}a$- (or $a^{(1,4)}a$-) symmetric, then a is EP.*

We say that a is normal if $a = aa^*a$. In this case, $aa^*, a^*a \in E(R)$. The following result shows that when R is aa^*- or a^*a-symmetric, a is not just group invertible.

Proposition 3.7. *If a is normal and R is aa^*- (or a^*a-) symmetric, then a is EP.*

Proof. Firstly, we claim that if R is aa^*-symmetric, then R is also a^*a-symmetric. Indeed, as $a = aa^*a$, we have $a^*a = a^*aa^*a = aa^*a^*a \in aa^*R$. By Lemma 2.2, we conclude that R is a^*a-symmetric. In accordance with Proposition 3.1, it follows that $aa^*, a^*a \in C(R)$. Hence, we can obtain that $a = (aa^*)a = a(aa^*) \in a^2R$, and $a = a(a^*a) = (a^*a)a \in Ra^2$. This implies $a \in R^\sharp$. Since $a = a^2a^*$, when we premultiply both sides of the equation by $a^\sharp$, we have $aa^\sharp = aa^*$, and consequently, $a^\sharp = a^\dagger = a^*$. □

Definition 3.8 (Bott–Duffin decomposition [6])**.** Let R be a ring and $\alpha \in R$. If $\alpha = g + u$, where $g \in E(R) \cap C(u)$ and u is B-D invertible relative to g, then $\alpha = g + u$ is called a B-D decomposition of α. The idempotent g is called the spectral idempotent of this decomposition.

Furthermore, it is proven in [6] that if y is the B-D inverse of u relative to g, then we can obtain $yu = uy = g$. In what follows, we investigate the B-D decompositions of idempotents in rings.

Theorem 3.9. *Let $e = g + u$ be a B-D decomposition. Then $g = -ug$ and $u^3 = u$.*

Proof. Clearly, $eg = ge$. Since $ge = ege$, we have $g + gu = g + gu + ug + ugu$, which implies that $ug = -ugu$. Let y be the Bott–Duffin inverse of u relative to g. Then $guy = g$, and therefore $g = -ug$. Furthermore, $u + g = e = e^2 = (u + g)(u + g) = u^2 + gu + gu + g = u^2 - g$. Thus, $u^2 - u = g + g$ and $u^3 - u^2 = -g - g$. Therefore, $u^2 - u = -(u^3 - u^2)$, and consequently, $u = u^3$. □

In particular, if $u \in R^{-1}$, then $u^2 = 1$ and $e = e^2 = (u + g)(u + g) = u^2 + gu + gu + g = 1 - g$. Moreover, in this case, u is B-D invertible relative to g with the inverse $y = u^{-1}g$, and $e = (1 - e) + (2e - 1)$.

Definition 3.10 (Strongly clean decomposition [14])**.** Let R be a ring and $\alpha \in R$. If $\alpha = g + u$, where $g \in E(R) \cap C(u)$ and $u \in R^{-1}$, then $\alpha = g + u$ is called a strongly clean decomposition of α.

In fact, whenever $u \in R^{-1}$, the B-D decomposition is the strongly clean decomposition. So, we can obtain the result as follows: For an idempotent, the strongly clean decomposition is unique, that is, $e = (1 - e) + (2e - 1)$. Furthermore, it is well known that e is left semicentral if and only if $1 - e$ is right semicentral. Then the following result can be provided.

Corollary 3.11 ([11, Theorem 4.3])**.** *Let $e \in E(R)$ and $e = u + g$ be a strongly clean decomposition. Then e is left semicentral if and only if g is right semicentral.*

Assume $e = u + g$ is a B-D decomposition. One can see that $u^2 \in E(R)$, and then the following result can be obtained.

Corollary 3.12. *Let $e = g + u$ be a Bott–Duffin decomposition. Then, R is u^2-symmetric iff R is e- and g-symmetric.*

Proof. By Theorem 3.9, we know that $g = -ug$; then $eg = (u + g)g = 0 = ge$, and consequently, $eu = e(e - g) = e = ue$. Hence, $e = u^2 e$ and $g = u^2 g$. This implies that R is e- and g-symmetric if R is u^2-symmetric. Conversely, by checking the proof of Theorem 3.9, we have $u^2 = e + g$, which implies that R is u^2-symmetric if R is e- and g-symmetric. □

Remark 3.13. Let R be a $*$-ring and $e \in E(R)$. Clearly, e^* is an idempotent. Assume that $e = (1 - e^*) + u$ is a clean decomposition (that is, a representation $\alpha = g + u$, where $g \in E(R)$ and $u \in R^{-1}$ [13]). If $eu = ue$, then we have $e^* u = ue^*$. Indeed, $e^* = 1 - e + u$ implies that $e^* u = ue^*$. Hence, in this case, the clean decomposition is a strongly clean decomposition. Moreover, since it is proved that the strongly clean decomposition of idempotents is unique, we can conclude that $1 - e^* = 1 - e$, and thus $e = e^*$. In fact, we can also prove that if $ue = eue$, then $eu = ue$ and $e^* = e$.

Proposition 3.14. *Let $e \in E(R)$ and $e = (1 - e^*) + u$ be a clean decomposition. If $ue = eue$, then $eu = ue$ and $e^* = e$.*

Proof. As $e = (1-e^*)+u$, we have $u = e-1+e^*$, and then $u^2 = 1-e-e^*+ee^*+e^*e$. Set $v = u^2$. It is easy to find that $ev = ve = ee^*e$ and $v^* = v$. Hence, we can see that $ev^{-1} = v^{-1}e$ and $e^*v^{-1} = v^{-1}e^*$. Set $f = e^*ev^{-1}$. Then one can see that $f = v^{-1}e^*e$ and $f^* = f$. As $ue = eue$, we have $e^*e = (1 - e + u)e = ue = eue$, and consequently, $ee^*e = e^*e$. It follows that $f = e^*ev^{-1} = ee^*ev^{-1} = e$. Therefore, $e^* = e$, and hence $u = 2e - 1$ and $eu = ue$. □

Corollary 3.15. *Let $e \in E(R)$ and R be an e-symmetric ring. If $e = (1 - e^*) + u$ is a clean decomposition, then $e^* = e \in C(R)$.*

Proof. Indeed, we know that e is left semicentral if and only if e^* is right semicentral. □

Acknowledgement

The authors sincerely thank the anonymous referee and the editor for their valuable comments and suggestions, which greatly improved the presentation of this paper.

References

[1] R. Bott and R. J. Duffin, On the algebra of networks, *Trans. Amer. Math. Soc.* **74** (1953), 99–109. DOI MR Zbl

[2] N. Castro-González, J. Chen, and L. Wang, Further results on generalized inverses in rings with involution, *Electron. J. Linear Algebra* **30** (2015), 118–134. DOI MR Zbl

[3] M. P. Drazin, A class of outer generalized inverses, *Linear Algebra Appl.* **436** no. 7 (2012), 1909–1923. DOI MR Zbl

[4] G. Kafkas, B. Ungor, S. Halicioglu, and A. Harmanci, Generalized symmetric rings, *Algebra Discrete Math.* **12** no. 2 (2011), 72–84. MR Zbl

[5] Y. Ke and J. Chen, The Bott–Duffin (e, f)-inverses and their applications, *Linear Algebra Appl.* **489** (2016), 61–74. DOI MR Zbl

[6] T. Y. Lam and P. P. Nielsen, Jacobson pairs and Bott–Duffin decompositions in rings, in *Rings, modules and codes*, Contemp. Math. 727, American Mathematical Society, Providence, RI, 2019, pp. 249–267. DOI MR Zbl

[7] J. Lambek, On the representation of modules by sheaves of factor modules, *Canad. Math. Bull.* **14** (1971), 359–368. DOI MR Zbl

[8] G. Marks, Reversible and symmetric rings, *J. Pure Appl. Algebra* **174** no. 3 (2002), 311–318. DOI MR Zbl

[9] F. Meng and J. Wei, e-symmetric rings, *Commun. Contemp. Math.* **20** no. 3 (2018), Article No. 1750039. DOI MR Zbl

[10] F. Meng and J. Wei, Some properties of e-symmetric rings, *Turkish J. Math.* **42** no. 5 (2018), 2389–2399. DOI MR Zbl

[11] F. Meng and J. Wei, (g, e)-symmetric rings, *Algebra Colloq.* **31** no. 2 (2024), 263–270. DOI MR Zbl

[12] F. Meng, J. Wei, and R. Chen, Weak e-symmetric rings, *Comm. Algebra* **51** no. 7 (2023), 3042–3050. DOI MR Zbl

[13] W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977), 269–278. DOI MR Zbl

[14] W. K. Nicholson, Strongly clean rings and Fitting's lemma, *Comm. Algebra* **27** no. 8 (1999), 3583–3592. DOI MR Zbl

[15] L. Ouyang and H. Chen, On weak symmetric rings, *Comm. Algebra* **38** no. 2 (2010), 697–713. DOI MR Zbl

[16] J. Wei, Generalized weakly symmetric rings, *J. Pure Appl. Algebra* **218** no. 9 (2014), 1594–1603. DOI MR Zbl

Shibo Shi ✉
School of Mechanical and Automotive Engineering, South China University of Technology, Guangzhou, China
ShiboMath@163.com

Yuheng Liu
School of Mathematical Sciences, Yangzhou University, Yangzhou, China

Long Wang
School of Mathematical Sciences, Yangzhou University, Yangzhou, China
lwangmath@yzu.edu.cn

Received: September 12, 2025
Accepted: May 12, 2026
Early view: July 29, 2026