ON RESTRICTED PARTITIONS WITH A BASIS OF UNIQUENESS

by

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1. - Let a set, S, of distinct positive integers be called a partitionable set, or shortly a π-set, if it possesses a basis, K, of *unique* restricted partitions, in the following sense: S consists of all integers representable as sums of distinct elements of a sequence,

$$K: \quad k_0 < k_1 < k_2 < \ldots, \qquad (k_0 \geqq 1), \tag{1}$$

of positive integers which has the property that a relation of the form

$$k'_1 + k'_2 + \ldots + k'_a = k''_1 + k''_2 + \ldots + k''_b,$$

where $k'_1 < k'_2 < \ldots$ and $k''_1 < k''_2 < \ldots$ are elements of K, holds only when

$$k'_1 = k''_1, \quad k'_2 = k''_2, \quad \ldots, \quad k'_a = k''_b; \quad a = b.$$

Every sequence K having the latter property determines a π-set S, having K as a basis. Conversely, if a π-set S is given, its basis K is uniquely determined by S. The latter fact can readily be verified by induction, viz., by first characterizing k_0, and then, if $k_0, \ldots, k_j$ in (1) are known, k_{j+1}, in terms of S alone. A formal rewording of this uniqueness proof can be based on the generating relation

$$1 + \sum_{n=1}^{\infty} s_n z^n = \prod_{j=0}^{\infty} (1 + z^{k_j}), \qquad |z| < 1, \tag{2}$$

where s_n denotes the characteristic function of S, i.e.,

(3) $s_n = 1$ or $s_n = 0$ according as n is or is not in S.

It is clear that a given set of integers k_j is a basis, K, if and only if the corresponding product (2) leads to a power series (2) in which no coefficient turns out to have a value distinct from 0 and 1, and that the set S generated by K is then defined by (3). This description of the situation makes clear enough the substantially implicit nature of the requirements involved.

Strictly speaking, (2) holds only if the sequence (1) is infinite, since otherwise the upper limit of the product in (2) must (whereas the upper limit of the sum in (2) may, but need not) be replaced by a finite limit. It is however clear that K is a finite set if and only if S is, and the following considerations deal with an asymptotic question concerning infinite mates, K, S. rather than with the question of enumeration presented by the case of finite mates K, S.

2. - If S is any set of distinct positive integers, let $N(n)$ denote the number of those elements of S which do not exceed n; so that

(4) $$N(n)/n = (s_1 + \ldots + s_n)/n,$$

if s_n is defined by (3). If (4) tends, as $n \to \infty$, to a limit, then S is said to be measurable («in relative measure»), and the limit, which will be denoted by $|S|$, is called the measure of S. It is clear from (4) and (3) that, if

(5) $$|S| = \lim_{n \to \infty} (s_1 + \ldots + s_n)/n$$

exists, then

(6) $$0 \leqq |S| \leqq 1.$$

Since $s_n \geqq 0$, it follows from a Tauberian theorem of Hardy and Littlewood (cf. reference [1] at the end of this paper), that (5) is equivalent to

(7) $$|S| = \lim_{r\to 1}(1-r)\sum_{n=1}^{\infty} s_n r^n, \qquad (0<r<1).$$

By «equivalence» is meant that the existence of either of the limits (5), (7) implies the existence and the identity of both limits.

If this criterion is applied to the case of a π-set S, it follows that S is measurable, and has the measure $|S|$, if and only if the basis, (1), of S is such as to lead to the existence of the limit

(8) $$\lim_{r\to 1} f(r) = |S|,$$

where

(9) $$f(r) = \prod_{j=0}^{\infty}(1+r^{k_j})/(1+r^{2^j}).$$

This is clear from (2) and (7), if recourse is had to Euler's identity

(10) $$\sum_{n=0}^{\infty} z^n = \prod_{j=0}^{\infty}(1+z^{2^j}), \qquad |z|<1$$

(which, in view of the criterion (2), means that the set of all positive integers is a π-set, with

(11) $$1, 2, 4, 8, \ldots$$

as basis).

3. - It is clear that, if

(12) $$k_0^* < k_1^* < k_2^* < \ldots$$

is a subsequence of a sequence (1) and if (1) is a basis, then (12) is a basis and generates a π-set, S^*, contained in the π-set, S, generated by (1). Furthermore, if S is measurable, then S^* is measurable and has the measure

(13) $$|S^*| = |S|/2^h \quad \text{or} \quad |S^*| = 0$$

according as just a finite number or an infinity of elements of

(1) are missing in (12), the exponent h in (13) being the number of those k-values which do not occur in (12). In fact, the truth of (13) is readily verified from the criterion (8), from the definition (9) and from the identity which results if a finite number of factors are omitted on the right of (10).

Let a K be called a 0-basis if the characteristic function, s_n, of the π-set, S, generated by K satisfies

$$(14) \qquad s_1 + \ldots + s_n = o(n) \qquad \text{(i. e., } |S| = 0).$$

For instance, (1) is a 0-basis if $k_n = 3^n$. If it were true that every basis which is not a 0-basis is a subsequence, (12), of Euler's basis, (11), it would follow from (13) that every π-set is measurable, those π-sets which are not of measure 0 having the measure $1/2^h$, where h is a certain positive integer (in fact, the basis (11) generates all positive integers, which form a set of measure 1). But the hypothesis of this conclusion is false. For, if ϑ is any value contained in the interval $0 < \vartheta < 1$ (hence, a value not necessarily of the form $1/2^h$), it is not difficult to construct a basis $K = K_\vartheta$ corresponding to which the π-set, $S = S_\vartheta$, is measurable and has the preassigned ϑ as its measure, $|S_\vartheta|$.

It will remain undecided *whether every or not every π-set is measurable.* What will be proved is a sufficient criterion, along with an explicit condition which can be obtained as a corollary of this criterion, for the measurability of a π-set.

4. - The criterion in question is «Abelian» in nature; it states that; *if* (1) *is a basis for which the limit*

$$(15) \qquad \lim_{n\to\infty} 2^n/k_n$$

exists, then the π-set generated by (1) *is measurable, and its measure is the limit* (15).

In view of the criterion (8), this assertion is equivalent to the statement that, if the existence of (15) is assumed, then the function (9) must tend, as $r \to 1$, to a limit, and that the latter has the same value as (15). Hence, more than the italicized assertion will be proved if it is shown that, whether the limits (8), (15) do or do not exist, the function (9) must satisfy the inequalities

(16) $\liminf_{n\to\infty} 2^n/k_n \leqq \liminf_{r\to 1} f(r), \qquad \limsup_{r\to 1} f(r) \leqq \limsup_{n\to\infty} 2^n/k_n.$

Put

(17) $$f_m(r) = \prod_{n=m}^{\infty} (1+r^{k_n})/(1+r^{2^n}), \qquad (r<1);$$

so that $f_0 = f$, by (9). Since, when n is fixed,

(18) $$(1+r^{k_n})/(1+r^{2^n}) \to 1 \text{ as } r \to 1,$$

it is clear that

(19) $$\limsup_{r\to 1} f(r) = \limsup_{r\to 1} f_m(r)$$

holds for every fixed m. On the other hand, if β denotes the upper limit on the right of the second of the inequalities (16), then either $\beta = \infty$, in which case the second of the assertions (16) is trivial, or else there belongs to every $\varepsilon > 0$ an m having the property that

$$k_n > 2^n/(\beta+\varepsilon) \text{ whenever } n \geqq m = m_\varepsilon.$$

Since $0<r<1$, this means that, if $q = q_\varepsilon$ is defined by

(21) $$q^{\beta+\varepsilon} = r, \text{ so that } 0<q<1,$$

then

$$r^{k_n} < q^{2^n} \text{ whenever } n > m = m_\varepsilon.$$

In view of (17), the last inequality implies that

$$\limsup_{r\to 1} f_m(r) \leqq \limsup_{r\to 1} \prod_{n=m}^{\infty} (1+q^{2^n})/(1+r^{2^n}),$$

where $q = q(r) \to 1$ as $r \to 1$, by (21). Since, corresponding to (18),

$$(1+q^{2^n})/(1+r^{2^n}) \to 1 \text{ as } r \to 1$$

holds for every fixed n, it follows that

$$\limsup_{r\to 1} f_m(r) \leqq \limsup_{r\to 1} \prod_{n=1}^{\infty} (1+q^{2^n})/(1+r^{2^n}).$$

Hence, if (19) is applied on the left, and (10) twice on the right,

$$\limsup_{r\to 1} f(r) \leqq \limsup_{r\to 1} \sum_{n=1}^{\infty} q^n / \sum_{n=1}^{\infty} r^n.$$

Since, as $r \to 1$,

$$\sum_{n=1}^{\infty} q^n / \sum_{n=1}^{\infty} r^n \sim (1-r)/(1-q) \to \beta + \varepsilon,$$

by (21), it follows that

$$\limsup_{r\to\infty} f(r) \leqq \beta + \varepsilon, \text{ and so } \limsup_{r\to\infty} f(r) \leqq \beta.$$

This proves the second of the inequalities (16), and the first is proved in the same way.

It is clear that the existence and the non-vanishing of the limit (15) imply that

(22) $$k_{n+1}/k_n \to 2 \quad \text{as} \quad n \to \infty.$$

It follows therefore from the italicized assertion, just proved, that, if (1) is a basis generating a π-set, S, which is measurable by virtue of the existence of (15), then the basis must satisfy (22) unless $|S|=0$.

5. - The (k', k'')-requirement, specified after (1), is obviously fulfilled by any sequence of positive integers $k_0, k_1, \ldots$ satisfying

(23) $$k_{n+1} > k_0 + \ldots + k_n \text{ for } n = 0, 1, \ldots$$

Since $k_0 \geqq 1$, the inequalities

(24) $$k_n \geqq 2^n, \text{ where } n = 0, 1, 2, \ldots,$$

are necessary for (23). In the inequalities (24), the sign of equality must fail to hold for one n at least, unless (1) is the sequence (11), the basis of all positive integers. Thus (11) appears as the (unique) extremal case of all bases satisfying (23).

A corollary of the italicized criterion is that, *if* $k_0, k_1, \ldots$ *is a sequence of positive integers satisfying* (23), *then it is a basis generating a measurable* π*-set*. In order to conclude this, it is sufficient to ascertain that the limit (15) must exist whenever (23) is satisfied. But this can be assured by, even though it is not explicitly contained in, the arguments applied by R. Salem and D. C. Spencer (cf. reference [2] below).

Since (23) implies the existence of the limit (15), it follows that, if a basis satisfies (23), then (22) must hold unless the π-set is of measure 0. Nevertheless, it is easy to see that, corresponding to every ϑ in the interval $0 < \vartheta < 1$, there exists a basis satisfying (23) and generating a (measurable) π-set the measure of which is ϑ.

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REFERENCES

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[2] R. Salem and D. C. Spencer, *On the influence of gaps on density of integers*, Duke Journal of Mathematics, vol. 9 (1942), pp. 855-872.