NEW REPRESENTATIONS OF WHITTAKER'S CONFLUENT HYPERGEOMETRIC FUNCTION

por J. BROMBERG
N. York

By substituting an Euler integral for an equivalent Beta function in the summand of the conventional series for function $M_{k,m}(x)$, a generalization of the Poisson type integral representation of cylindrical and associated functions is obtained, which further, by expressing it as a series of known integrals of this type, yields an expansion of Whittaker's function in terms of Bessel and Struve functions of increasing orders.

1. *Integral Representations.* Whittaker's solution $M_{k,\pm m}(x)$ of the differential equation

$$y'' + [-1/4 + k/x + (1/4 - m^2)/x^2]\, y = 0, \tag{1}$$

whose classic expression (¹) may be written as

$$M_{k,m}(x) = [\Gamma(2m+1)/\Gamma(-k+m+1/2)] \,.\, x^{m+1/2} \,.$$
$$e^{-1/2x} \,.\, \sum_{n=0,1}^{\infty} \{[\Gamma(-k+m+1/2+n)/\Gamma(2m+1+n)] \,.\, x^n/n!\}, \tag{2}$$

appears, after multiplying, in (2), th esummand and dividing the constant factor by $\Gamma(k+m+1/2)$, as

$$M_{k,m}(x) = C_{k,m}\, x^{m+1/2}\, e^{-1/2x}$$
$$\sum_{n=0,1}^{\infty} [B(-k+m+1/2+n,\ k+m+1/2) \,.\, x^n/n!], \tag{2a}$$

(¹) [1], § 16.1 (p. 337, bottom). The bracketed figures in these footnotes refer to the order numbers in the list of references at the end of the text.

where, as in the rest of the present calculation,

$$C_{k,m} \equiv 1/B(-k+m+1/2,\ k+m+1/2) \qquad (3)$$

and the B-functions have the usual meaning

$$B(p,q) \equiv [\Gamma(p)\,.\,\Gamma(q)]/\Gamma(p+q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt\ (^2). \qquad (4)$$

We substitute in (2a) the series $\sum_{n=0,1}^{\infty}[(-1/2\,x)^s/s!]$ for $e^{-1/2x}$, transform the double series thus obtained by Cauchy's rule, with $N \equiv n+s$, and introduce the unit factor $N!/N!$ into the summand:

$$M_{k,m}(x) = C_{k,m}\,x^{m+1/2}\,.\sum_{n=0,1}^{\infty}[(x^N/N!)\,.\sum_{s=0,1}^{s=N}(\{N!/[(N-s)!\,s!]\}\,.$$
$$(-1/2)^s\,.\,B(-k+m+1/2+N-s,\ k+m+1/2))].$$

According to (4) and interchanging summation and integration we have, with the usual notation for the binomial coefficient appearing in the braces,

$$M_{k,m}(x) = C_{k,m}\,x^{m+1/2}\sum_{N=0,1}^{\infty}((x^N/N!)$$
$$\int_0^1 \{\sum_{s=0,1}^{N}[\binom{N}{s}(-1/2/t)^s]\}\,.\,t^{-k+m-1/2+N}(1-t)^{k+m-1/2}\,dt).$$

By the binomial formula the factor in braces is equivalent to $[1-1/(2t)]^N$, and so, with

$$\xi \equiv 1/2\,x,$$

$$M_{k,m}(x) = C_{k,m}\,x^{m+1/2}\sum_{N=0,1}^{\infty}\{[(-1/2\,x)^N/N!]$$

(2) [1], §§ 12.4, 12.41 (pp. 253-5).

$$\int_0^1 (1-2t)^N . t^{-k+m-1/2} (1-t)^{k+m-1/2} dt\} =$$

$$= 2^{\pm 2k} C_{k,m} x^{m+1/2} \sum_{N=0,1}^{\infty} \{[(-\xi)^N/N!]$$

$$\int_0^1 (1-2t)^N [t(1-t)]^{\pm k+m-1/2} [1 \mp (1-2t)]^{\mp 2k} dt\} .$$

In our range of integration it is $|1-2t| \leqq 1$ and the binomial formula may be applied to the $\mp 2k$*—th* power before dt (except, possibly, at a terminus, but we shall ignore, temporarily, this singularity):

$$M_{k,m}(x) = 2^{\pm 2k} C_{k,m} x^{m+1/2} \sum_{N=0,1}^{\infty} ([(-\xi)^N/N!]$$

$$\sum_{p=0,1}^{\infty} \{(\mp 1)^p \binom{\mp 2k}{p} \int_0^1 [t(1-t)]^{\pm k+m-1/2} (1-2t)^{N+p} dt\}) \qquad (5)$$

Denoting the value of the integral in (5) by A, we have, with $t = \sin^2 \psi$,

$$A = \int_0^{1/2\pi} (\sin \psi . \cos \psi)^{2(\pm k+m-1/2)} \cos^{N+p} (2\psi) \sin 2\psi . d\psi =$$

$$= 2^{2(\mp k-m)} \int_0^{1/2\pi} \sin^{2(\pm k+m)} (2\psi) . \cos^{N+p} (2\psi) . d(2\psi) =$$

$$= 2^{2(\mp k-m)} \int_0^{\pi} \sin^{2(\pm k+m)} \varphi . \cos^{N+p} \varphi \, d\varphi .$$

In the range of our last integration $\sin \varphi$ is symmetrical and $\cos \varphi$ is antisymmetrical with respect to the ordinate through the middle of the range. Therefore:

for $(N+1)$ odd it is $\int_{\pi/2}^{\pi} = -\int_{0}^{\pi/2}$ and $A=0$;

for $(N+p)$ even it is $\int_{\pi/2}^{\pi} = +\int_{0}^{\pi/2}$ and $A=$

$$= 2^{2(\mp k - m)+1} \cdot \int_{0}^{\pi/2} \cos^{2[\pm k + m + 1/2(N+p)]} \varphi \cdot \tan^{2(\pm k + m)} \varphi \cdot d\varphi =$$

$$= 2^{2(\mp k - m)} \Gamma(\pm k + m + 1/2) \cdot \Gamma[1/2 (N+p) + 1/2]$$
$$/\Gamma[\pm k + m + 1 + 1/2 (N+p)] \text{ (3)}.$$

Applying to the second Gamma-function Legendre's duplication formula (4)

$$\Gamma(z+1/2) = \pi^{1/2} \cdot 2^{-(2z-1)} \cdot \Gamma(2z) / \Gamma(z), \qquad (6)$$

$$A = 2^{2(\mp k - m)} \pi^{1/2} \{\Gamma(\pm k + m + 1/2)$$
$$/\Gamma[\pm k + m + 1 + 1/2 (N+p)]\} .$$
$$\{2^{-(N+p)} (N+p)! / [1/2 (N+p)]!\} . \qquad (7)$$

We have now to substitute into (5) the value of the integral A from (7) for the terms with $(N+p)$ even and to drop there the terms with $(N+p)$ odd; as, consequently, $(-1)^N (\mp 1)^p = (-1)^{N+p} (\pm 1)^p = (\pm 1)^p$ and as also $[(N+p)!/N!]\, \xi^N = \partial^p (\xi^{N+p}) / \partial \xi^p$, we can write:

$$M_{k,m}(x) = 2^{-2m} \pi^{1/2} C_{k,m} \Gamma^{(\pm k+m+1/2)} \cdot x^{m+1/2} \left[\left(\sum_{N=0,2}^{\infty} \sum_{p=0,2}^{\infty} + \sum_{N=1,3}^{\infty} \sum_{p=1,3}^{\infty} \right) \right.$$

$$\left. \left((\pm 1)^p \binom{\mp 2k}{p} \frac{\partial^p}{\partial \xi^p} \left\{ \frac{(1/2\, \xi)^{N+p}}{[1/2(N+p)]!\, \Gamma[\pm k + m + 1 + 1/2(N+p)]} \right\} \right) \right] \qquad (8)$$

After another application of Cauchy's rule, with $N+p \equiv M$,

---

(3) [2] tab. 42, Nº 6 (p. 71) and tab. 17, Nº 19 (p 44), with a condition of validity to our $\pm k + m < 0$, which is insufficient, as will be seen; [11], p. 164, (11).

(4) [1], § 12.15 (p. 240).

the summation becomes

$$\left(\sum_{M=0,2}^{\infty}\sum_{p=0,2}^{M}+\sum_{M=2,4}^{\infty}\sum_{p=1,3}^{M-1}\right)(\pm 1)^p\binom{\mp 2k}{p}$$

$$\partial^p\{(1/2\,\xi)^M/[(1/2\,M)!\,\Gamma(\pm k+m+1+1/M)]\}/\partial\xi^p)=\sum_{M=0,2}^{\infty}\sum_{p=0,1}^{M}(\ldots),$$

with the understanding, originally, that $0\leqq p\leqq M$; but, as, for $p>M$, it is $\partial^p(\xi^M)/\partial\xi^p=0$, we may replace $\sum_{p=0,1}^{M}$ by $\sum_{p=0,1}^{\infty}$. With $2M$ used for M in both the summation symbol and the summand, our sum becomes now

$$\sum_{M=0,1}^{\infty}\sum_{p=0,1}^{\infty}((\pm 1)^p\binom{\mp 2k}{p}$$
$$\partial^p\{(1/2\,\xi)^{2M}/[M!\,\Gamma(\pm k+m+1+M)]\}/\partial\xi^p),$$

and, because of the well-known definition of the modified Bessel function

$$I_\nu(z)=\pi^{-1/2}\,[(\Gamma(\nu+1/2)]^{-1}\,(1/2\,z)^\nu$$
$$\int_0^\pi\left[\begin{smallmatrix}\cosh\\ \exp\end{smallmatrix}(\pm z\cos\vartheta)\right]\sin^{2\nu}\vartheta\,d\vartheta\ (^5)\ (^6)=$$
$$=\sum_{n=0,1}^{\infty}\{(1/2\,z)^{\nu+2n}/[n!\,\Gamma(\nu+1+n)]\}\ (^7) \tag{9}$$

we may now rewrite (8) as

---

(5) [3], § 3.71, (9) (p. 79).

(6) Choice of sign independent of choice between exp. and cosh; if cosh is used, it is $\int_0^\pi=2\int_0^{1/2\pi}$, $\int_{-1/2\pi}^{1/2\pi}=2\int_0^{1/2\pi}$ and $\int_{-\infty}^{\infty}=2\int_0^{\infty}$.

(7) [3], § 3,7, (2) (p. 77).

(8) Thus most of the beginning of the present article down to this point could have been dropped; we keep it because of a more general heuristic value possibly inherent to the used method of replacing a Beta function by its integral representation.

$$M_{k,m}(x) = 2^{2-m}\,\pi^{1/2}\,C_{k,m}\,\Gamma\,(\pm k + m + 1/2)\,x^{m+1/2}\cdot$$

$$\sum_{p=0,1}^{\infty}\{(\pm 1)^p \binom{\mp 2k}{p}\,\partial^p\,[I_{\pm k+m}(\xi)/(1/2\,\xi)^{\pm k+m}]/\partial\xi^p\}\,. \quad (10)$$

It may also be written symbolically:

$$M_{k,m}(x) = 2\,\pi^{1/2}\,C_{k,m}\,\Gamma\,(\pm k + m + 1/2)\,(1/2\,\xi)^{m+1/2}$$
$$(1 \pm \partial/\partial\xi)^{\mp 2k}\,[I_{\pm k+m}(\xi)/(1/2\,\xi)^{\pm k+m}]. \quad (11)$$

We substitute into (10) or (11), for the Bessel function, its Poisson type integral representation (9), perform the differentiation $\partial/\partial\xi$ behind the integration symbol, interchange summation and integration and cancel constants, then use the binomial formula:

$$M_{k,m}(x) = 2\,C_{k,m}(1/2\,\xi)^{m+1/2}$$
$$\int_0^{\pi}\{\sum_{p=0,1}^{\infty}[\binom{\mp 2k}{p}\cos^p\vartheta]\}\,[\exp(\pm\,\xi\cos\vartheta)]\,\sin^{2(\pm k+m)}\vartheta\,d\vartheta =$$
$$= 2\,C_{k,m}(1/2\,\xi)^{m+1/2}$$
$$\int_0^{\pi}(1+\cos\vartheta)^{\mp 2k}\,.\,\exp(\pm\,\xi\cos\vartheta)\,.\,\sin^{2(\pm k+m)}\vartheta\,d\vartheta =$$
$$= 2\,C_{k,m}(1/2\,\xi)^{m+1/2}$$
$$\int_0^{\pi}[\exp(\pm\,\xi\cos\vartheta)]\,\tan^{\pm 2k}(1/2\,\vartheta)\,.\,\sin^{2m}\vartheta\,d\vartheta = \quad (12)$$
$$= 2\,C_{k,m}(1/2\,\xi)^{m+1/2}$$
$$\int_0^{\pi}\left({\cosh \atop \exp}\{\pm[\xi\cos\vartheta + 2k\log\tan(1/2\,\vartheta)]\}\right)\sin^{2m}\vartheta\,d\vartheta\ (^6). \quad (13)$$

Instead of investigating, at each application, the legality of using the binomial formula and of interchanging integration with summation or differentiation, it is simpler to check our main result (12), and to obtain simultaneously the condition of its validity, by substituting it into the original equation (1). A still shorter way [8] is suggested by the similarity of formula (12), once it has been derived, to a formula given by Nielsen:

$$B[1/2(p+\nu)+1, 1/2(p-\nu)+1] = 2^{-p}$$

$$\int_0^{1/2\pi} \tan^{\nu}\omega \, . \, \sin^{p+1}(2\omega) \, . \, d\omega = 2\int_0^{1/2\pi} \tan^{\nu}\omega \, . \, (1/2\sin 2\omega)^{p+1} d\omega \; (^9), \qquad (14)$$

valid, according to the same authority [10], for

$$\mathcal{R}(p \pm \nu + 1) > -1. \qquad (15)$$

In order to apply (14) (which is, incidentally, a simple corollary from our (4)) to the B-function in (2a), we assume

$$1/2(p \pm \nu) + 1 = n + m - k + 1/2 \text{ and } 1/2(p \mp \nu) + 1 = m + k + 1/2;$$

$$\text{whence } p = n + 2m - 1 \text{ and } \nu = \pm(n - 2k); \quad n = 0, 1, 2 \ldots$$

Condition (15) of validity now splits into: $2n + \mathcal{R}(2m - 2k) > -1$ and $\mathcal{R}(2m + 2k) > -1$; these two conditions reduce, for the critical value $n = 0$, to

$$\mathcal{R}(\pm k + m) > 1/2, \qquad (16)$$

or, for the case that both k and m are real,

$$-|k| + m > -1/2.$$

(For $M_{k,-m}(x)$, the second in the fundamental systen of solutions of (1), the condition of validity becomes even more stringent: $\mathcal{R}(\pm k - m) > -1/2$, or $-|k| - m > -1/2$).

As the series in (2a) converges uniformly, we may interchange in it summation and integration [11] and rewritte it as

---

(9) [4], p. 379.
(10) [4], p. 373, eq. Γ_{16}.
(11) [1], § 4.7 (p. 78-9).

$$M_{k,m}(x) = 2\, C_{k,m}\, x^{m+1/2}\, e^{-1/2x}$$

$$\int_0^{1/2\pi} \{\sum_{n=0,1}^{\infty} [(1/2 \sin 2\omega)^{n+2m} \tan^{\pm(n-2k)} \omega \,.\, x^n/n!]\}\, d\omega =$$

$$= 2\, C_{k,m}\, x^{m+1/2}\, e^{-1/2x}$$

$$\int_0^{1/2\pi} (\sum_{n=0,1}^{\infty} \{[(\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\, \omega)^2 \,.\, x]^n/n!\})\, (1/2 \sin 2\omega)^{2m} \,.\, \tan^{\mp 2k} \omega \,.\, d\omega =$$

$$= C_{k,m}\, x^{m+1/2}\, e^{-1/2x} \int_0^{1/2\pi} \{\exp[1/2\, x(1 \mp \cos 2\omega)]\}$$

$$(1/2 \sin 2\omega)^{2m}\ \tan^{\mp 2k} \omega\, d(2\omega) = 2^{-2m}\, C_{k,m}\, x^{m+1/2}$$

$$\int_0^{\pi} [\exp(\mp 1/2\, x \cos \vartheta)] \tan^{\mp 2k} (1/2\, \vartheta) \,.\, \sin^{2m} \vartheta\, d\vartheta,$$

which is identical with (12), from which also (10) and (11) could now be easily derived by reversing the procedure used above.

It is natural to substitute in (13) $1/2\,\pi + \vartheta'$ for ϑ and thus to obtain

$$M_{k,m}(x) = 2\, C_{k,m} (1/2\, \xi)^{m+1/2}$$

$$\int_{-1/2\pi}^{1/2\pi} (\{\begin{smallmatrix}\exp\\ \cosh\end{smallmatrix} \{\mp[\xi \sin \vartheta' - 2k \log \tan(1/4\,\pi + 1/2\, \vartheta')]\})\, \cos^{2m} \vartheta'\, d\vartheta' \text{ (6)},$$

in order to conveniently introduce the anti-gudermannian

$$\varphi \equiv gd^{-1}\, \vartheta' = \log \tan(1/4\,\pi + 1/2\ \vartheta') \text{ (12)}; \qquad (17)$$

with $gd^{-1}(-1/2\,\pi) = -\infty$; $gd^{-1}(1/2\,\pi) = \infty$; $\sin \vartheta' = \tanh \varphi$ (13); $\cos \vartheta' = 1/\cosh \varphi$ (13); $d\vartheta' = d\varphi/\cosh \varphi$ (14), it is

---

(12) [5], Addenda, § 3, 2nd line (p. 58).
(13) [5], Addenda, § 13, 3rd line (p. 58).
(14) [5], Addenda, § 9, last line (p. 56).

$$M_{k,m}(x) = 2\,C_{k,m}(1/2\,\xi)^{m+1/2}$$

$$\int_{-}^{\infty} \left\{{}^{\exp}_{\cosh}\,[\pm(\xi \tanh\varphi - 2k\,\varphi)]\right\} \cosh^{-2m-1}\varphi\, d\varphi\ (^{6})\ (^{15}). \qquad (18)$$

For $k = 0$ equation (1) degenerates into one of the standard forms of the Bessel equation, solved by a Bessel function (of an imaginary argument, or modified) ([16]). From (11), and using later also (6), we obtain the relation

$$M_{0,m}(x) = 2\,\pi^{1/2}\,C_{0,m}\,\Gamma\,(m+1/2)\,.\,(1/2\,\xi)^{m+1/2}\,.\,I_m(\xi)/(1/2\,\xi)^m = \\ = 2^{2m}\,\Gamma\,(m+1)\,.\,x^{1/2}\,.\,I_m(1/2\,x), \qquad (19)$$

which is essentially identical with the «second Kummer formula» [17]) and also otherwise well-know ([18]). But from (18) we would have now

$$M_{0,m}(x) = 2\,C_{0,m}(1/2\,\xi)^{m+1/2} \int_{-}^{\infty} [{}^{\exp}_{\cosh}(\pm\,\xi \tanh\varphi)] \cosh^{-2m-1}\varphi\, d\varphi\ (^{6}),$$

and by comparing this with (19), we obtain

$$I_\nu(z) = [\pi^{1/2}\,\Gamma\,(\nu+1/2)]^{-1}\,(1/2\,z)^\nu$$

$$\int_{-}^{\infty} [{}^{\exp}_{\cosh}\ (\pm\,z \tanh\varphi)] \cosh^{-2\nu-1}\varphi\, d\varphi\ (^{6}); \qquad (20)$$

this could have been derived also directly from (9) by introducing, again, for ϑ, first ϑ' and then the anti-gudermannian φ, according to (17). Hyperbolic, instead of trigonometric, functions could be used, with occasional advantage, in the Poisson type integral representations of numerous other Bessel and associated functions ([19]). We single out for this treatment, in view

([15]) A contact (in the case of $m = 0$) appears possible with Sharpe's integral: [3], § 4.43, (2) (p. 105).

([16]) [5], VIII, § 7, 3rd case (p. 146).

([17]) [1], § 16.11, (II) (p. 338).

([18]) [1], § 17.212 (p. 360-1); [6], eq. (1,12) (p. 129); [10], C 2.273 (p. 474); follows also from [7], § 18.43, title (p. 276).

([19]) An integral representation close enough to this type seems to have of a later utilization, the modified Struve function

$$L_\nu(z) = 2\,[\pi^{1/2}\,\Gamma\,(\nu+1/2)]^{-1}\,(1/2\,z)^\nu \int_0^{1/2\pi} \sinh(z\cos\vartheta)\,\sin^{2\nu}\vartheta\,d\vartheta =$$

$$= \sum_{p=0,1}^{\infty} \{(1/2\,z)^{\nu+2p+1}/[\Gamma\,(p+3/2)\,.\,\Gamma\,(\nu+p+3/2)]\}\;(^{20}) \qquad (21)$$

and write

$$L_\nu(z) = 2\,[\pi^{1/2}\,\Gamma\,(\nu+1/2)]^{-1}\,(1/2\,z)^\nu$$
$$\int_0^\infty \sinh(z\tan\varphi)\,.\,\cosh^{-2\nu-1}\varphi\,.\,d\varphi. \qquad (22)$$

Also the condition (16) of validity reduces, for $k=0$, to the well-known condition $\mathcal{R}(\nu) > -1/2$ for (9) and (21), and consequently for (20) and (22).

2. *Connection of the Whittaker function with Bessel and Struve functions.* By the addition theorem for the braced factor in the integrand of (18),

$$M_{k,m}(x) = 4\,C_{k,m}(1/2\,\xi)^{m+1/2} \int_0^\infty [\cosh(\xi\tanh\varphi)\cosh 2k\,\varphi\,-$$
$$-\sinh(\xi\tanh\varphi)\sinh 2k\,\varphi]\,\cosh^{-2m-1}\varphi\,.\,d\varphi. \qquad (23)$$

In order to transform this expression with the aid of (20) and (22), it remains to expand $\cosh 2k\,\varphi$ and $\sinh 2k\,\varphi$ into converging series in powers of $\cosh\varphi$. Expansions of this type (for a non-qualified $2k$) are not, probably, available in the literature, but can be derived when it is noticed that

---

been given for the first time in [8], p. 142, (3), where it suffices to substitute $x\cosh\varphi$ for α in order to obtain $J_0(x) = 2\,\pi^{-1}\int_0^\infty \sin\,(x\cosh\varphi)\,d\varphi$, as apparently noticed by the author of [9], p. 294, who, however, derives this formula also by a different method. For a few more general formulae of Mehler's type see [3], § 6.13 (p. 170) and § 6.2 (p. 180).

(20) [3], § 10.4, (11) (p. 329).

$$v(u) \equiv \begin{matrix}\exp \\ \cosh \\ \sinh\end{matrix} (\pm \mu \cosh^{-1} u),$$

where $u = \cosh \varphi$ and φ is real, satisfies, in each case, the differential equation

$$(u^2 - 1)(d^2v/du^2) + u(dv/du) - \mu^2 v = 0 \text{ (21)}.$$

The two solutions of the usual type in descending (because $|u| \geqq 1$) series of powers of u, with the indefinite constant factors omitted, appear as

$$\cosh^{\pm\mu}\varphi \mp \mu \sum_{p=1,2}^{\infty} [4^{-p} f(\mp \mu, p) \cosh^{\pm\mu - 2p} \varphi] =$$

$$= \mp \mu \sum_{p=0,1}^{\infty} [4^{-p} f(\mp \mu, p) \cosh^{\pm\mu - 2p} \varphi], \qquad (24)$$

where

$$f(j, p) \equiv \Gamma(j + 2p) / [\Gamma(p+1) \,.\, \Gamma(j + 1 + p)], \qquad (25)$$

with the understanding that the equality in (24) is correct also for $\mu = 0$, as in this case the first (for $p = 0$) term on the right side becomes $(\mp \mu)\, \Gamma(\mp \mu)/[\Gamma(\mp \mu + 1)] = 1$, like on the left side, and all other terms vanish on either side. The series in (24) are convergent by the elementary ratio test for $\cosh \varphi > 1$ and by the Raabe test for $\cosh \varphi = 1$. The numerical factors necessary for combining linearly the two series (24) into $\begin{matrix}\cosh \\ \sinh\end{matrix} (\mu\varphi)$ are found by causing φ to increase beyond any positive limit, with μ assumed first positive, then negative, and comparing the limit values of the series with $\lim_{\varphi \to \infty} \begin{matrix}\cosh \\ \sinh\end{matrix} (\mu\varphi)$ in each case. The resulting formulae are:

$$\begin{matrix}\cosh \\ \sinh\end{matrix} (\mu\varphi) = 2^{\mu - 1} (-\mu) \sum_{p=0,1}^{\infty} [4^{-p} f(-\mu, p) \,.\, \cosh^{\mu - 2p} \varphi] \pm 2^{-\mu - 1} (+\mu)$$

$$\sum_{p=0,1}^{\infty} [4^{-p} f(\mu, p) \,.\, \cosh^{-\mu - 2p} \varphi]. \qquad (26)$$

It follows from their derivation that they hold only for a real

---

(21) [10], C 2.235 (p. 453).

and positive φ; the restriction, however, is merely formal, as μ is unrestricted.

Interchanging summation and integration in the uniformly converging series (26) ([11]) and applying to it (23) for $\mu = \pm 2k$ and then using (20) and (22),

$$M_{k,m}(x) = 4k\, C_{k,m} (1/2\, \xi)^{m+1/2} \int_0^\infty \sum_{p=0,1}^{\infty} \{2^{-2k-2p} f(2k, p)$$
$$[\cosh(\xi \tanh \varphi) + \sinh(\xi \tanh \varphi)] \cosh^{-2k-2m-2p-1} \varphi -$$
$$- 2^{2k-2p} f(-2k, p)$$
$$[\cosh(\xi \tanh \varphi) - \sinh(\xi \tanh \varphi)] \cosh^{2k-2m-2p-1} \varphi\}\, d\varphi =$$
$$= k\pi^{1/2}\, C_{k,m} \sum_{p=0,1}^{\infty} \{f(2k, p)\, x^{-(k-1/2+p)}\, \Gamma(k + m + 1/2 + p)$$
$$[I_{k+m+p}(1/2\, x) + L_{k+m+p}(1/2\, x)] -$$
$$- f(-2k, p)\, x^{-(-k-1/2+p)}\, \Gamma(-k + m + 1/2 + p)$$
$$[I_{-k+m+p}(1/2\, x) - L_{-k+m+p}(1/2\, x)]\} . \qquad (27)$$

It is possible to derive this result from (12), instead of from (18), by using Poisson integrals of the original type and certain formulas partly appearing in handbooks ([22]), where, however, no hints as to their derivation or further authorities are given. By retaining the $\begin{smallmatrix}\text{Bessel}\\\text{Struve}\end{smallmatrix}$ functions alone in the brackets of (27) we obtain the part of $M_{k,m}(x)$ which is $\begin{smallmatrix}\text{even}\\\text{odd}\end{smallmatrix}$ in k. Further, the factors in (27) clearly split into two groups of functions of which one does and the other does not contain m. We finally represent the Whittaker function in the formally simple shape

$$M_{k,m}(x) = k\, C_{k,m} \sum_{p=0,1}^{\infty} [K(k - 1/2 + p, p, x) \,.\, M^{(1)}{}_{k+m+p}(1/2\, x) -$$
$$- K(-k - 1/2 + p, p, x) \,.\, M^{(2)}{}_{-k+m+p}(1/2\, x)], \quad (28)$$

where

$$\begin{matrix} M_\nu^{(1)}(z) \\ M_\nu^{(2)} \end{matrix} = \pi^{1/2}\, \Gamma(\nu + 1/2)\, [I_\nu(z) \pm L_\nu(z)]\, ([23]) =$$

([22]) [11], p. 265, (9).

([23]) The essential part of $M_\nu{}^{(2)}(z)$ is known in the literature: [3], § 13.51, (7) and after (p. 425). In the case of $\nu = 0$ a relation to Theisinger's integral appears probable: [3], § 10.46 (p. 338).

$$= 2(1/2\,z)^{\nu} \int_0^{1/2\pi} \exp(\pm z \cos \vartheta) \sin^{2\nu} \vartheta\, d\vartheta =$$

$$= 2(1/2\,z)^{\nu} \int_0^{\infty} \exp(\pm z \tanh \varphi) \cosh^{-2\nu-1} \varphi\, d\varphi = \pi^{1/2}\,\Gamma\,(\nu + 1/2)$$

$$\sum_{n=0,1}^{\infty} \{(\pm 1)^n (1/2\,z)^{\nu+n} / [\Gamma(1/2\,n+1)\,\Gamma(\nu + 1/2\,n + 1)]\}\ ; \quad (29)$$

$$K(\nu, p, z) \equiv f(2\nu + 1 - 2p, p)\, z^{-\nu} = \{\Gamma(2\nu+1)/[\Gamma(p+1)$$
$$\Gamma(2\nu + 2 - p)]\}\, z^{-\nu} = p^{-1} \binom{2\nu}{p-1} z^{-\nu},$$

for $p = 0, 1, 1 \ldots$, with, for $p = 0$, the convention $K(\nu, 0, z) = (2\nu+1)^{-1} z^{-\nu}$, and, for $p = 1$, the usual assumption $\binom{2\nu}{0} = 1$. In our case it is easy to see that $K(\pm k - 1/2 + p, p, x) = [\Gamma(\pm 2k + 2p)/\Gamma(\pm 2k + 1 + p)]/[p!\, x^{\pm k - 1/2 + p}] \neq \infty$ for any $0 < x < \infty$, at $p \neq 0$; also for $p = 0$ it is $K(\pm k - 1/2, 0, x) = (\pm 2k)^{-1} x^{\mp k + 1/2} \neq \infty$, as it is assumed $k \neq 0$. It may also be written $K(\nu, p, z) \equiv s(2\nu + 2 - p, p) \,.\, (pz^{\nu})^{-1}$, with reference to the recently introduced special function $s(a, b) \equiv \Gamma(a + b - 1)/[\Gamma(a)\, F(b)] = [(a + b - 1) \,.\, B(a, b)]^{-1}$ (24).

As, from (9) or (20), $I_{\nu}(-z) = (-1)^{\nu} I_{\nu}(z)$ and, from (21) or (22), $L_{\nu}(-z) = -(-1)^{\nu}\, L_{\nu}(z)$, it follows, as also from the last form in (29), that $M_{\nu}^{(1)}(-z) = (-1)^{\nu} M_{\nu}^{(2)}(z)$ and $M_{\nu}^{(2)}(-z) = (-1)^{\nu}\, M_{\nu}^{(1)}(z)$; as, besides, obviously, $K(\nu, p, -z) = (-1)^{\nu}\, K(\nu, p, z)$, the well known property $M_{k,m}(-z) = (-1)^{m+1/2}\, M_{-k,m}(z)$ («Kummer's first formula») (25) is derived in a particularly easy way from our (28).

For an estimate of the convergence of our result, we first apply the asymptotic (for $x \to \infty$) form of the Stirling formula (26): $\Gamma(x+1) \sim (2\pi)^{1/2} e^{-x} x^{x+1/2}$, to each of the three Gamma-functions in (25) and thus obtain the asymptotic value

(24) [11], p. 296, a), (1); *ib.*, (3), an expansion of $s(z,p)$, for p positive and integer, into a series with Stirling numbers. A more detailed theory probably included in the author's book (inaccesible to the writer) on ''Special functions''.

(25) [1], § 16.11, (I) (p. 338); [10], C 2.273 (p. 474).

(26) [1], § 12.33 (p. 253).

of $2^{-j-2p} f(j, p)$:

$$\Gamma(j+2p) \sim (2\pi)^{1/2} e^{-(j+2p-1)} (j+2p-1)^{j+2p-1/2} = (2\pi)^{1/2} e^{-(j+2p-1)}$$
$$[1+(j-1)/(2p)]^{2p} [1+(j-1)/(2p)]^{j-1/2} (2p)^{j+2p-1/2} \sim$$
$$\sim (2\pi)^{1/2} . 1^{j-1/2} . e^{-2p} (2p)^{j+2p-1/2}; \Gamma(p+1) \sim (2\pi)^{1/2} e^{-p} p^{p+1/2};$$
$$\Gamma(j+1+p) \sim (2\pi)^{1/2} e^{-(j+p)} (j+p)^{j+p+1/2} =$$
$$= (2\pi)^{1/2} e^{-(j+p)} [1+(j/p)]^p [1+(j/p)]^{j+1/2} p^{j+p+1/2} \sim$$
$$(2\pi)^{1/2} . 1^{j+1/2} e^{-p} p^{j+p+1/2};$$
$$2^{-j-2p} f(j, p) \sim 1^{-1} . 2^{-1} \pi^{-1/2} e^0 p^{-3/2} = 1/2 \pi^{-1/2} p^{-3/2} \qquad (30)$$

for $j \models \pm 2k$ or any other value.

Now, from the first expression in (27), rewritten in terms of the original Poisson (trigonometric) integrals, we remove such a number N of initial terms that

$$\mathcal{R} (\pm k + m) + N > 1/2, \qquad (31)$$

especially in the case when condition (16) is not satisfied; the finite number of removed singular terms may then be immagined to have been replaced originally by their converging power series from (29), so that result of the present reasoning will not be affected. If, now, N increases indefinitely, it follows from (30) that if the sum of the remainder of the series approaches asymptotically, say, $S(N)$, it is

$$S(N) = 1/2 \pi^{-1/2} \sum_{r=0,1}^{\infty} \{(N+r)^{-3/2} [\int_0^{1/2\pi} e^{1/2x} \cos\vartheta \sin^{2(k+m+N+r)} \vartheta \, d\vartheta -$$
$$\int_0^{1/2\pi} e^{-1/2x} \cos\vartheta \sin^{2(-k+m+N+r)} \vartheta \, d\vartheta]\} .$$

The moduli of the integrands are finite in the entire range, because of (32), and so

$$|S(N)| < 1/2 \pi^{1/2} \sum_{r=0,1}^{\infty} \{(N+r)^{-3/2}$$

$$[\int_0^{1/2\pi} e^{2/1x} \cos\vartheta \sin^{2\mathcal{R}(k+m+N+r)} \vartheta \, d\vartheta +$$

$$\int_0^{1/2\pi} e^{-1/2x\cos\vartheta}\sin^{2\mathcal{R}(-k+m+N+r)}\vartheta\, d\vartheta]\} <$$

$$< 1/2\,\pi^{-1/2}\{\sum_{r=0,1}^{\infty}[(N+r)^{-3/2}]\}$$

$$\int_0^{1/2\pi}[\exp(1/2\,x\cos\vartheta)+\exp(-1/2\,x\cos\vartheta)]\sin\vartheta\, d\vartheta = \pi^{-1/2}$$

$$\{\sum_{r=0,1}^{\infty}[(N+r)^{-3/2}]\}\int_{1/2\pi}^{0}\cosh(\pm 1/2\,x\cos\vartheta)\, d\cos\vartheta.$$

The value of our last integral is $(1/2\,x)^{-1}\sinh(1/2\,x) = \sum_{n=0,1}^{\infty}\{x^{2n}/[2^{2n}(2n+1)!]\}$ and that of the numerical factor preceding it is finite. Thus the serial part of the expression in (27) converges better than the series in the original form (2) of $M_{k,m}(x)$, which, by applying the Stirling formula to the coefficient $\Gamma(-k+m+1/2+n)/\Gamma(2m+1+n)$ therein, proves to approach asymptotically the series $\sum_n\{x^n/[n!\,n^{k+m+1/2}]\}$.

Function $K(\nu; p, z)$ contains, besides the simple monomial $(pz^{\nu})^{-1}$ (27), in which p is a positive integer, merely $\binom{2\nu}{p-1}$ or $s(2\nu+2-p, p)$, for whose adequate tabulation the turn may como some day (28). The tabulation of $M_\nu^{(1)}(z)$ and $M_\nu^{(2)}(z)$, should require little computational work beyond that involved in tabulating $I_\nu(z)$ and $L_\nu(z)$ (29); ν should run through integer and appropriate fractional values; one and the same set of tables should then suffice for both $\nu = k+m+p$ and $\nu = -k+m+p$. In the case of $-|k|+m<0$ (of course, k and m are assumed

(27) On existing tables of functions of this type see [7], section 2 (pp. 23-33).

(28) See, meanwhile. [12], p. 363-4; cp. [7], § 3.57 (p. 39).

(29) On tabulation of $I_\nu(z)$: [7], §18-1-2 (p. 271-3; 280), and [13] p. 224-88; on that of $L_\nu(z)$: [7], § 20.57 (p. 296, 308).

real), function $K_\nu(z)$ [30] would become involved in calculating $I_{-|\nu|}(z)$; as for Struve functions with negative parameters, they seem to have aroused no interest in the literature so far; we hope to turn to this problem in another connection. After the necessary tabulations are completed, a simple multiplication of a few factors would be needed for evaluating a term in the summand of (28), and we saw that the terms converge fast enough. The number of tables and operations involved is as yet perhaps less formidable than the difficulties in tabulating function $M_{k,m}(x)$ envisaged in a recent treatise on mathematical physics [31].

REFERENCES

(American reprints of European books indicated in parentheses)

1. E. T. WHITTAKER and G. N. WATSON, *A course of modern analysis*, 4th ed., Cambridge Univ. Press, 1927 (Macmillan, 1944).
2. D. BIERENS DE HAAN, *Nouv. tables d'intégrales déf.*, Leyden, 1867 (Stechert 1939).
3. G. N. WATSON, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, 1941 (Macmillan 1946).
4. N. NIELSEN, *Handb. d. Theorie d. Cylinderfunktionen*, Teubner, Lpz,
5. E. JANKE u. F. EMDE, *Tables of functions* etc., Teubner, Lpz., 1933 (Dover 1943).
6. A. ERDÉLYI, *Funktionalrelationen mit konfl. hypergeom. Funkt.* I, Math. Zschr., 42, Springer, Berlin, pp. 125-143.
7. A. FLETCHER, J. C. P. MILLER and L. ROSENHEAD, *An Index of mathematical tables*, Scientific Computing Service Ltd., London, 1946.
8. F. G. MEHLER, *Notiz ü d. Dirichlet'schen Integralausdrücke* etc., Math. Annalen, *5* p. 141-144 (1872).
9. H. LAMB, *Hydrodynamics*, 6th ed., Cambr. Univ. Press, 1932 (Dover 1945).
10. E. KAMKE, *Differentialgleichungen. Lösungsmethoden u. Lösungen*, I, 2nd ed., Akad. Verlagsges., Lpz., 1943 (Edwards, 1943).
11. I. M. RYZIK, *Tables of integrals, sums, series and products* (in Russian), OGIZ, Mos.-Lgr., 1943.
12. E. MADELUNG, *Die math. Hilfsmittel d. Physikers*, 3rd ed., Springer, Berlin, 1936 (Dover, 1943).
13. Mathem. tables and other aids to computation, I, Nº 7, July 1944.
14. H. and B. JEFFREYS, *Methods of mathematical physics*, Cambr. Un. Press, 1946.

---

(30) [3], § 3.7 (p. 78).
(31) [14], § 23.02 p. 575.