# ON THE CONCEPT OF FILTER IN RING THEORY

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0 - *General definitions.* — Let $\mathcal{R}$ be a set in wich we define a ring structure $R = (\mathcal{R}, +, .)$. We may consider also $\mathcal{R}$ as an $R$-leftmodule R in which addition coïncides with ring one and the operation of $R$ (as ring) on R (as module) is the ring multiplication from the left.

If we represent by $\overline{a}$ an element of the module and by $a$ the same element in the ring ($\overline{a}$ is a vector, $a$ is a scalar) the definition given may by written $a \,.\, \overline{b} = \overline{a \,.\, b}$.

We can also build an $R$-right-module R′ with the elements of $R$ by making $a\overline{b} = \overline{ba}$.

Since the theories of $R$-right- and $R$-left-modules are identical, we shall study only the last, and say, in the following, $R$-module for $R$-left-module.

Let $\sigma$ be an $R$-homomorphism of R on B. Necessarily, B is al $R$-module, since

$$\sigma(\overline{a} + \overline{b}) = \sigma(\overline{a}) + \sigma(\overline{b})$$

$$\sigma(a\overline{b}) = a\,\sigma(\overline{b})$$

must hold.

Let us write $\overline{\overline{a}} = \sigma(\overline{a})$, and, since B is a module, it has a «zero-vector» $O'$. If we search the elements of R which are mapped by $\sigma$ on 0′ we have: 1) $O' = \overline{\overline{0}}$, 2) the set is an $R$-submodule of R, that is, a set $\lambda \subseteq$ R, which is an additive group such that $R\lambda \subseteq \lambda$; but the elements of $\lambda$ form in $R$ a left-ideal $I^L$ by the properties stated before.

Conversely, if a set $I^L$ of elements of $R$ is a left ideal, we can find an $R$-homomorphic image B of R in which the set mapped onto the «zero-vector» of B is precisely the submodule formed by the elements of $I^L$.

1 - *The concept of filter.* — Let B be an $R$-homomorphic image of R.

We shall say $\overline{\overline{u}} \in B$ is an $R$-unit vector (or unit vector) of B if and only if $\overline{\overline{ru}} = \overline{\overline{r}}$, for every $r \in R$.

Definition. — *The set $F^L$ of elements of $\mathcal{R}$ which is mapped onto a unit vector $\overline{\overline{u}}$ of* B *by an R-homomorphism of* R *onto* B *shall be called a left filter of R.*

It is known that, in commutative rings, homomorphic mappings may be made of rings onto rings, since right-, left- and two-sided ideals coincide.

In this case the concept of unit-vector may be replaced by the ring unity of the homomorphic image and the definition holds.

2 - *Characterization of filters.* — This paragraph is devoted to prove the following theorem:

Theorem 1. — *The necessary and sufficient conditions for a non-void subset $F^L$ ($F^R$) of $R$ be a left (right) filter of $R$ are:*

*$F_1$) $f_1 - f_2 + u \in F$ for every $f_1, f_2 \in F$, being $u$ a fixed element of $F$.*

*$F_2^R$) If $f \in F$, then $f + r - rf \in F$ for every $r \in R$.*

For right filters, $F_2^L$) must be replaced by

*$F_2^L$) If $f \in F^R$, then $f + r - fr \in F^R$ for every $r \in R$.*

Lemma 1. — *If $F^L$ is a left filter in R, conditions* $F_1$) *and $F_2$) hold.*

Proof. — $F^L$ is a left- filter in $R$ if and only if there exists an $R$-homomorphic image B of R, with a unit-vector $\overline{\overline{u}}$, and $\overline{\overline{f}} = \overline{\overline{u}}$ implies $f \in F^L$ and conversely. Then, $f_1$, $f_2$, $u \in F^L$ implies $\overline{\overline{f_1}} = \overline{\overline{f_2}} = \overline{\overline{u}}$ and $\overline{\overline{f_1 - f_2 + u}} = \overline{\overline{f_1}} - \overline{\overline{f_2}} + \overline{\overline{u}} = \overline{\overline{u}}$, hence $f_1 - f_2 + u \in F^L$, and $F_1$) holds.

If $f \in F^L$ then $\overline{\overline{f}} = \overline{\overline{u}}$ and, for every $r \in R$, $\overline{\overline{f + r - rf}} = \overline{\overline{f}} + \overline{\overline{r}} - \overline{\overline{r}}\,\overline{\overline{f}} = \overline{\overline{f}} + \overline{\overline{r}} - \overline{\overline{r}} = \overline{\overline{F}}$, and $F_2$) holds.

Similar conditions may be proved for right filters.

Lemma 2. — *If in a non-void set F holds* $F_1$) *then $f_1 - f_2 + f_3 \in F$ for every $f_1, f_2, f_3 \in F$ and conversely.*

Proof. $f_1 - f_2 + f_3 = f_3 - (f_2 - f_1 + u) + u$ and

$f_2 - f_1 + u = f \in F$ by $F_1$), then
$f_1 - f_2 + f_3 = f_3 - f + u \in F$.

The converse is trivial.

Lemma 3. — *If $F^L$ is a non-void set with properties* $F_1$) *and* $F_2^L$), *the set* $I^L = \{f_1 - f_2\}$, $f_1, f_2 \in F$ *is a left ideal in R.*

Proof. — $I_1$) Let be $i_1 = f_1 - f_1'$ and $i_2 = f_2 - f_2'$, hence $i_1 - i_2 = f_1 - f_1' + f_2' - f_2$ and, by $F_1$), $f_1 - f_1' + f_2' = f \in F^L$, hence $i_1 - i_2 = f - f_2 \in I^L$.

$I_2$) Let be $i = f_1 - f_2 \in I^L$ and $r \in R$, then
$ri = r(f_1 - f_2) = (r + f_2 - rf_2) - (r + f_1 - rf_1) + f_1 - f_2$
and $r + f_2 - rf_2 \in F^L$, $r + f_1 - rf_1 \in F^L$, by $F_2$), and
$(r + f_2 - rf_2) - (r + f_1 - rf_1) + f_1 = f \in F^L$ by Lemma 2, then

$$ri = f - f_2 \in I^L \qquad \text{q.e.d.}$$

If is given a non-void set $F^L$ with properties $F_1$) and $F_2^L$), the ideal $I^L = \{f_1 - f_2\}$ will be called the *corresponding ideal* of $F^L$.

Lemma 4. — *If $F^L$ is a non-void set with properties* $F_1$) and $F_2^L$), *and $I^L$ its corresponding left ideal* (*R-subspace of* R), *then $F^L$ is the inverse image of only one unit-vector by the R-homomorphism* $R \to R - I^L$.

Proof. — Since $f - f_1 + f_2 = f_3 \in F^L$ or every $f$ of $F^L$ ($f_1$, $f_2 \in F^L$), hence $f = f_3 - f_2 + f_1$ and, fixing $f_1 = u \in F^L$, every $f \in F^L$ may be written $f = i + u$, $i \in I^L$, hence $F^L$ is contained in only one lateral class of $I^L$ and since every $i + u \in F^L$, it will be mapped onto only one element $\overline{\overline{u}}$ of $R - I^L$.

We shall prove $\overline{\overline{u}}$ is a unit vector. Since $r + u - ru \in F^L$, $r + u - ru = f$, hence $\overline{\overline{r}} + \overline{\overline{u}} - \overline{\overline{r}}\,\overline{\overline{u}} = \overline{\overline{u}}$, then $\overline{\overline{r}}\,\overline{\overline{u}} = \overline{\overline{r}}$, as we wish to prove.

Proof of the theorem. — The conditions are necessary by Lemma 1, and sufficient by Lemma 4.

3 - *The ring unities.* — Ring unities play an important rôle in filter theory, as is established by the two following theorems:

Theorem 2. — *If R has a left unity $e^L$ (that is, $re^L = r$ for every $r \in R$), then every R-homomorphic image of* R *has at least one unit-vector.*

Proof. — Let $I^L$ be a left ideal in $R$ and $R - I^L$ the homomorphic image of R given by $I^L$ as kernel. Hence the set $\{e^L+i\}$, $i \in I^L$ is a left filter $F^L$, since $F_1$) and $F_2{}^L$) are verified. Proof: $F_1$) trivial, $F_2{}^L$) $(e^L+i)+r-r(e^L+i)=e^L+$ $+i-ri=e^L+i \in \{e^L+i\}$.

Theorem 3. — *If R has a right unity $e^R$, $e^R \in F^L$ for every $F^L \subseteq R$, then there is not more than one unit-vector in each R-homomorphic image of* R.

Proof. — Let $f \in F^L$, by $F_2$), $f+e^R-e^Rf \in F^L$ and, since $e^R f=f$, $e^R \in F^L$.

Corollary 1. — *If R has a unity (two-sided) to every left (right) ideal corresponds one and only one left (right) filter.*

The converse is true without postulate the existence of unity (Lemma 3).

Now we can prove certain properties between ideal- and filter-lattices.

Lemma 5. — *If $F_1{}^L$ and $F_2{}^L$ are left filters and $F_1{}^L \supseteq F_2{}^L$, then the relation $I_1{}^L \supseteq I_2{}^L$, between its corresponding ideals, holds.*

Proof. — Let $i \in I_2{}^L$, hence $i=f_1-f_2$ $(f_1, f_2 \in F_2{}^L)$ and since $F_2{}^L \subseteq F_1{}^L f_1, f_2 \in F_1{}^L$, then $i \in I_1{}^L$.

Lemma 6. — *Let $I_1{}^L$ and $I_2{}^L$ be left ideals in R and $I_1{}^L \supseteq I_2{}^L$, if some $F_2{}^L$ exists, then an $F_1{}^L$ exists such that $F_1{}^L \supseteq F_2{}^L$,* where $F_i{}^L$ are left-filters which corresponding ideals are the $I_i{}^L$.

Proof. — Let be $f \in F_2{}^L$, hence $\{f+i\}$, $i \in I_1{}^L$ is a left filter: $F_1$) $(f+i_1)-(f+i_2)+(f+i_3)=f+(i_1-i_2+i_3)$, and $i_1-i_2+$ $+i_3=i \in I_1$, hence $(f+i_1)-(f+i_2)+(f+i_3)=f+i \in \{f+i\}$; $F_2{}^L$) Since $f \in F_2{}^L$, $f+r-rf \in F_2{}^L$, hence $i_1=r-rf \in I_2{}^L$ for every $r \in R$, then $(f+i)+r-r(f+i)=f+i+r-rf-ri=f+i_2$ $\in \{f+i\}$.

Let us call $F_1{}^L=\{f+i\}$, $i \in I_1{}^L$. If $f_1 \in F_2{}^L$, $f_1-f=i \in I_2{}^L$ hence $f_1-f=i \in I_1{}^L$, $f_1 \in F_1{}^L$, then $F_1{}^L \supseteq F_2{}^L$.

Theorem 4. — *If R has unity (two sided), ideal- and filter-lattices are isomorphic.*

Proof. — By Lemma 3, to every left- (right-) filter corresponds one and only left- (right-) ideal; if $R$ has unity the converse also holds (Corollary 1).

If we order ideals and filters by inclusion, Lemmas 5 and 6 prove the theorem.

4 - *Two-sided filters.* — Let B be an $R$-homomorphic image of R. If the ideal $I$ defined by this homomorphism is two sided we can give in the set $\mathcal{B}$ of the elements of B a ring structure $B$, homomorphic to $R$, as follows: The sum in $B$ coïncides with the sum in B, the product $\bar{\bar{a}} \cdot \bar{\bar{b}}$ in $B$ is given by $\bar{\bar{a}} \cdot \bar{\bar{b}} = \overline{\overline{ab}}$. It is well known that the operations so defined make in $\mathcal{B}$ a ring $B$. If $B$ has a unity $1'$ (two-sided), the set $F$ of elements of $R$ mapped onto $1'$ will be called a *two-sided filter* in $R$.

Since the unity $1'$ of $B$ has the properties $\bar{\bar{x}} \cdot 1' = 1' \cdot \bar{\bar{x}} = \bar{\bar{x}}$ for every $\bar{\bar{x}} \in B$, if we consider the $R$-monule B, if $f \in F$, $\bar{\bar{f}} = 1'$ and $x\bar{\bar{f}} = \overline{\overline{xf}} = \bar{\bar{x}} \cdot 1' = \bar{\bar{x}}$, hence $F$ is a left filter. Since $I$ is two-sided, we can consider the $R$-right-homomorphic image R′ — I, and, if $f \in F$ $\bar{\bar{f}} = 1'$, hence $x\bar{\bar{f}} = \overline{\overline{fx}} = \bar{\bar{f}} \cdot \bar{\bar{x}} = 1' \cdot \bar{\bar{x}} = \bar{\bar{x}}$ *and* $F$ is a right filter. Then, conditions $F_1$), $F_2{}^L$) and $F_2{}^R$) must hold.

Let $F$ be a non-void set with properties $F_1$), $F_2{}^L$) and $F_2{}^R$). Then, it is simultaneously left- and right-filter, and its corresponding ideal is two-sided (Lemma 3). We can find the ring $R/I$, homomorphic to $R$, in which $F$ is mapped onto a two-sided unity (Lemma 4), then, $F$ is a two sided filter.

Hence, necessary and sufficient conditions for a given non-void set $F \subseteq R$ be a two-sided filter are $F_1$), $F_2{}^L$) and $F_2{}^R$).

In commutative rings, since every filter is two-sided, all the theory may be stated using ring-homomorphisms.

5 - *Further properties.* — We shall prove now some additional properties of filters (left-, right-, or two sided-).

1) *A filter is a multiplicative system.* Proof: Let $x, y \in F$, hence $x + y - xy = z \in F$ by $F_2{}^L$) or $F_2{}^R$), then $xy = x - z + y \in F$ by $F_1$).

2) *If $0 \in F$, then $F = R$.* Proof: Let $F$ be a left-filter (similar proof may be employed for right-filters) and $x \in R$, then $x + 0 - x \cdot 0 = x \in F$, hence $F \supseteq R$, that is $F = R$.

To exclude the case $F = R$ we shall call *proper filter* a filter $F$ for which $0$ is not in $F$.

3) *If $F$ is a proper filter and $I$ is its corresponding ideal, $F$ and $I$ are disjoint.* Proof: Suppose $F$ and $I$ not disjoint,

then there is an element $a \in F$ such that $a \notin I$. If $a \in I$, $a = f_1 - f_2$, $(f_1, f_2 \in F)$, and by $F_1$), $a - f_1 + f_2 \in F$, but $a - f_1 + f_2 = 0$ hence $0 \in F$ and $F$ can not be proper.

6 - *Duality*. — Let $R$ be a ring with two-sided unity $1$. Then, theorem 4 holds. With the elements of the ser $\mathcal{R}$ [if $R = (\mathcal{R}, +, .)$] we wish to build a new ring $R^*$ such that its ideals be formed with the elements of the filters of $R$ and conversely.

Since $1 \in F$ for every filter $F$ (left-, right- and two-sided-) by theorem 3, we can replace condition $F_1)'$ by.

$F'_1$) *For every* $f_1, f_2 \in F$, $f_1 - f_2 + 1 \in F$.

Let us remember conditions for ideals:

$I_1$) *If* $i_1, i_2 \in I$, *then* $i_1 - i_2 \in I$.

$I_2^L$) *If* $i \in I$, *then* $ri \in I$ *for every* $r \in R$ (*left ideals*).

$I_2^R$) *If* $i \in I$, *then* $ir \in I$ *for every* $r \in R$ (*right ideals*).

If we compare conditions $I_1$) and $F'_1$) we see that both connect two variable elements of each set with a new element obtained from them by known operations.

It is known that $I_1$) states, by an «inverse operation», that I is a subgroup of the group defined by the «direct operation» $(+)$ on $R$.

We shall see that $F_1'$) states a similar property.

First we shall prove that the operatión $a +^* b = a + b - 1$ makes $\mathcal{R}$ an abelian group.

$G_1$) $a +^* b$ is defined for every $a, b \in \mathcal{R}$.

$G_2$) $(a +^* b) +^* c = a +^* (b +^* c)$.

$G_3$) $a +^* b = b +^* a$.

$G_4$) $a +^* 1 = a$ for every $a \in \mathcal{R}$, hence $1$ is the «neutral» element of $(\mathcal{R}, +^*)$.

$G_5$) For every $a \in \mathcal{R}$ there is an «invere element», that is, the equation $a +^* x = 1$ has always solution.

Proofs of $G_1 - G_4$ are trivial.

We can prove $G_5$ by proving the existence of an «inverse operation» $-^*$, defined as follows:

$$(a -^* b) +^* b = a$$

Hence

$$(a -^* b) + b - 1 = a$$

$$a -^* b = a - b + 1.$$

Then, $(\mathcal{R}, +^*)$ is abelian group and condition $F'_1$) says that the elements of a filter in $R$ form a subgroup of $(\mathcal{R}, +^*)$.

We shall define a new binary operation on $\mathcal{R}$ which must be associative and distributive over $+^*$.

$F_2$) [$F_2^L$) and $F_2^R$)] leads us to it, in view of $I_2$) [$I_2^L$) and $I_2^R$)] respectively. $I_2$) says that any ideal is «absorbent» for the operation «multiplication» (from the left- or right-hand, respectively).

$F_2$) says also that a filter is absorbent respect to the combination $a+f-af$ or $a+f-fa$.

This induce us to define the new operation

$$a \times^* b = a + b - ab$$

and we can prove easily that:

$A_1$) $a \times^* b$ is defined for every $a, b \in \mathcal{R}$
$A_2$) $(a \times^* b) \times^* c = a \times^* (b \times^* c)$
$A_3$) $a \times^* (b +^* c) = (a \times^* b) +^* (a \times^* c)$
$A'_3$) $(a +^* b) \times^* c = (a \times^* c) +^* (b \times^* c)$
$A_4$) $a \times^* 0 = a$ for every $a \in \mathcal{R}$.

Then, $R^* = (\mathcal{R}, +^*, \times^*)$ is a ring in which the set of elements of each filter of $R$ has the properties:

$I_1^*$) If $a, b \in I^*$, then $a -^* b \in I^*$ (see $F'_1$)).
$I_2^{L*}$) If $f \in I^{L*}$, then $r \times^* f \in I^{L*}$ (for left-filters). See $F_2^L$).
$I_2^{R*}$) If $f \in I^{R*}$, then $f \times^* r \in I^{R*}$ (for right-filters). See $F_2^R$).

If we wish to write the operations of $R = (\mathcal{R}, +, .)$ in terms of those of $R^* = (\mathcal{R}, +^*, \times^*)$, we arrive to:

$$a + b = a +^* b -^* 0$$
$$ab = a +^* b \, (a -^* (a \times^* b).$$

(Observe that $0$ is the $\times^*$-unity of $R^*$!!).

This shows that:

1) $(R^*)^* = R$.

2) The filters of $R^*$ are the ideals of $R$.

Property 1) shows that the operation * on rings with unity is involutorial.

Referred to commutative rings with unity, Foster and Bernstein (1) have proved certain proporties of the operation * which can be stated also for general rings using in general the same proofs.

The most important of them are:

$B_1$) $R$ and $R^*$ are isomorphic, using the transformation

$$x \leftrightarrow 1 - x$$

Since $0 -^* x = 1 - x$, then $(1 - x)^* = 1 - x$.

$R$ and $R^*$ are called *dual rings*.

$B_2$) If $R$ is a Boolean ring, $R^*$ is the classical dual Boolean ring.

(1) FOSTER and BERNSTEIN, *Symmetric approach to commutative rings, with duality theorem: Boolean duality as special case*. Duke Mathematical Journal, vol. 11 (1944), pp. 603-616.