RECENT DEVELOPMENTS IN FIELD THEORY

ABDUS SALAM (Imperial College, Londres)

I would like to give a brief review of the work done in Field Theory since the classic work of Schwinger, Feynman and Dyson. The results of the work have not been as spectacular as the work of the above authors and to an outsider it comes as a surprise to learn that any progress has been made at all. But it is a fact that the atmosphere in Field Theory has completely changed, the emphasis shifted and we feel that a real beginning to understand Field Theory has only just been made. In this respect, Field Theory is an astonishing subject. The more one understands it, the richer one finds the subjet and greater the rather sorry attempts people have made in the past.

Let us summarize the situation as it existed at the end of 1950. Schwinger had reformulated field theory so that the relativistic covariance was obvious at each stage, and he and Tomonaga were responsible for the introduction of what is called the interaction representation. Feynman had introduced his graphs and given his celebrated method of writing higher orders. Schwinger working to the low orders had given the fully covariant method of extracting finite parts from infinite coefficients of the perturbation theory, and shown thar these parts could be interpreted as mass and charge renormalization. Dyson had extended this to all orders and defined his famous infinite constant, the so called Z factors. Lamb Shift had been calculated and found to agree. Meson theory had been renormalized, calculated and there we met our Waterloo. The cry was: perturbation theory is false, out with it.

Perturbation theory must go, because may be the series diverges. But there was on the theoretical side the set of people who said may be the very existence of infinitiés is due to perturbation theory. If we could express the renormalization constants in something different from perturbation theory may be they turn out to be finite, and may be the theory is not as senseless as all that. It was really the fault of perturbation theory, but G. Källén did not believe this. Källén believed that the fault was more fundamental. The theory was fundamentally wrong; in a series of brilliant papers he tried to establish this. To my mind the papers are too heavily prejudiced; to me they do not carry conviction but in the process of proof (*Dans Mat. Fys. Medd.* Vol. 27.12 (1953)) Källén initiated, laid the foundations of much of the methodology which has subsequently been used by other authors, in particular by Lehmann, and Lehmann, Zimmerman and Symanzik (L Z S). The first important results of these investigations are the general expressions for the one-particle wave functions and for the renormalization constants in closed form. As *I* said, most of this work occurs already in Källén's papers; the clearest account is due to Lehmann, which *I* shall follow.

Before *I* do this let me set down in the form of axioms the faith of a modern field theorist. *I* am here following Wightman (Lille Conference Reports, June 1957).

I - Modern Field Theory works in terms of the Heisenberg representation where states are rays ψ of unit lengths in a Hilbert space H. There exists a state of minimum energy, the vacuum; and there are no negative energy states.

II - Gives a definition of field operators and existence of field entities with certain relativistic transformation properties.

III - Local Commutativity

$$[\phi(x), \phi(y)] = 0 \qquad \text{if} \qquad (x - y^2) < 0.$$

These 3 axioms specify the notion of a local field but do not guarantee that the theory has any content; the field heory must describe particle observables.

Vacuum expectation values are distributions in the sense of Schwartz:

$$F^{(n)}(x_1, \dots x_n) = \langle 0 \,|\phi(x_1) \dots \phi(x_n)|\, 0 \rangle.$$

$F^{(n)}$ is a Lorentz-invariant singular function (distribution in the sense of Schwartz).

Then (Wightman, Phys. Rev. 101, 860 (1956)):

Let $F^{(n)}$ be a denumerable set of analytic functions of complex variables: $n = 1, 2, \dots$ satisfying:

1) Relativistic invariance under Lorentz transformations (without time inversion).

2) Hermiticity:

$$<0|(\phi(x_1)\dots\phi(x_n))^*|0>^* = <0|\phi(x_n)\dots\phi(x_1)|0>^*.$$

3) Positive definiteness of the scalar product, that implies a set of inequalities connecting the boundary values of the $F^{(n)}$.

4) Local Commutativity: implies $F^{(n)}(z_{ij}^2) = F^{(n)}(P z_{ij}^2)$ with P any set of a certain set of permutations of the labels $i\,j$.

Let H be a Hilbert space. If, a represents an element of the inhomogeneous Lorentz Group, a vacuum state ψ, and a neutral scalar field ϕ such that the $n-$fold vacuum expectation value of ϕ is $F^{(n)}$, so one may study vacuum expectation values; then it is easy to show that $F^{(n)}$ is temperate in each variable separately. It is not yet proved that $F^{(n)}$ is temperate in all variables jointly. From Lorentz invariance one sees that:

$$F^{(n)} = F^{(n)}(x_1 - x_2, x_2 - x_3, \dots x_{n-1} - x_n) = F^{(n)}(\xi_1, \dots, \xi_{n-1}) =$$

$$= \int e^{-i\sum_1^{n-1} p_j \xi_j} \times G^{(n)}(p_1, \dots, p_{n-1})\, d^4 p_1, \dots, d^4 p_{n-1}$$

$G^{(n)}$ vanishes unless $p_j^2 = (p_j^0)^2 - \mathbf{p}_j^2 \geqq 0$, $(p_j^0 \geqq 0)$ by virtue of the assumptiòn that there exist no-negative energy states.

$G^{(n)}$ has the important property that its support contains only points $p_1 \dots p_{n-1}$ such that every p_j lies within or on the future cone

$$p_j^2 \geqq 0, \quad p_j^0 \geqq 0.$$

This means that it is possible to define the Laplace Transforms and so the resulting form is analytic in the tube whose points are

$z_1 \dots z_{n-1}$, $z_i = \xi_i - i\,\eta_i$ with η_i four vectors in the future light cone

$F^{(n)}(z_1 \dots z_{n-1}) = F^{(n)}(\wedge z_1 \dots \wedge z_{n-1})$ without time inversion

$F^{(n)}(z_1 \dots z_{n-1}) = F^{(n)}(-\bar{z}_1 \dots -\bar{z}_{n-1})$ with time inversion

$F^{(n)}(z_1 \dots z_{n-1}) = F^{(n)}(-\bar{z}_{n-1} \dots \bar{z}_1)$

($\wedge$ is a complex Lorentz transformation).

I am soon going to state the 4th axiom, but even before that, at this stage quite a lot can be done.

We can form quantities like:

Product like: 1) $<0|\phi(x), \phi(y)|0>$
2) $\vartheta(x-y)<0|[\phi(x)\,\phi(y)]|0>=R$ (Retarded)
3) $<0|T\,\phi(x), \phi(y)|0>=T$ (Time ordered).

R, T are common functions; we have at this stage no physical meaning attached to them; however we have all the tools necessary to express their properties.

Thus (Lehmann, Symanzik, Zimmermann): (N. Cim. 2 (1955) 425)

$$\Delta^{+\prime}(x-x')=<0|\phi(x)\,\phi(x')|0>$$
$$=\sum_p <0|\phi(x)|p><p|\phi(x')|0>, \quad \sum_p |p><p|=1|.$$

Since $\partial_\mu \phi(x)=i\,[\phi(x), P_\mu]$

$$<0|\phi(x)|p>=<0\,|\phi(0)|p>e^{ipx}=a_{op}\,e^{ipx}.$$

So $\Delta^{+\prime}(x-x')=\sum_p a_{op}\,a_{op}{}^*\,e^{ip(x-x')}$.

Define $\rho(p^2)=a_{op}\,a_{op}{}^*>0$

then $\Delta^{+\prime}(x-x')=-i\int \vartheta(p_0)\,\rho(p^2)\,e^{ip(x-x')}\,d^4p.$

Write $\rho(p^2)=\int_0^\infty \rho(K^2)\,\delta(p^2-K^2)\,d\,K^2$

then $\Delta^{+\prime}(x-x')=\int_0^\infty \Delta^+(x-x', K^2)\,\rho(K^2)\,d\,K^2$

where $\rho(K^2)>0$ ρ spectral functions.

Same proof for Feynman functions:

$$T=\Delta_F{}'(x-x')=<0|T\,\phi(x)\,\phi(x')|0>$$
$$=\int_0^\infty \Delta_F(x-x', K^2)\,\rho(K^2)\,d\,K^2$$
$$S_F{}'(x-x')=<0|T\,\overline{\psi}(x)\,\psi(x)|0>$$
$$S_F{}'(p)=-2\,i\int_0^\infty \frac{(i\gamma p-K)\rho_1(K^2)+\rho_2(K^2)}{p^2-k^2-i\varepsilon}\,d\,k^2$$
$$\rho_1\geqq 0 \qquad 0\leqq\rho_2\leqq 2\,K\,\rho_1(K^2).$$

The next problem that arises is: can we extend this parametric representation to vacuum expectation values of more than 2 operators? We shall see that this problem is closely connected with that of the most recent development: the development of dispersion theory. The great importance of all this has caused battles to rock and roll, and heads have fallen in the struggle and so *I* want to spend some time on all this.

Renormalization

In the conventional notation (Dyson)

$$\psi(x) = Z_2^{-1/2}\,\psi_v(x)$$

$$\phi(x) = Z_3^{-1/2}\,\phi_u(x)$$

$$M = M_0 + \delta M$$

$$m^2 = m_0^2 + \delta m^2 \qquad g = Z_1^{-1} Z_2 Z_3^{1/2} g_0$$

$$L = -\frac{1}{4} Z_2[\bar\psi(\gamma_\mu \partial_\mu + M)\psi] - \frac{1}{2} Z_3\{\partial_u \phi\, \partial_u + m^2 \phi^2) -$$

$$- i g Z_1 [\bar\psi \gamma_5 \psi \phi + Z_2 \delta M \bar\psi \psi + Z_3 \delta m^2 \phi^2]$$

then $$Z_2^{-1} = \int_0^\infty \rho_1(K^2)\, d K^2 \qquad 0 \leqq Z_3 < 1$$

$$Z_3^{-1} = \int_0^\infty \rho\ (K^2)\, d K^2 \qquad 0 \leqq Z_2 < 1.$$

There is no simple spectral representation por Z_1.

$$\delta m^2 = - Z_3 \int_0^\infty (K^2 - m^2)\,\rho\, d K^2 \leqq 0$$

$$\delta M = Z_3 \int_0^\infty [M - x)\,\rho_1 + \rho_2]\, d x^2.$$

Now in Electrodynamics (Ward): $Z_1 = Z_2$

So $\qquad e = Z_3^{1/2} e_0.$

So that $\qquad e \leqq e_0.$

This means that one physically measures also the charge of positrons surrounding the electron. No such result is known for g. Furthermore, Källén showed that Z_3 is intrinsically infinite.

T. D. Lee Model: Is a non trivial model that shows that infinities are intrinsic. (P. R. *95* (1954) 1329: The model consists of a particle with two states only: N and V, interacting with a Bose field K. From the Hamiltonian

$$H_0 = m_v \int \bar{\psi}_v \psi_v \, d\tau + m_N \int \bar{\psi}_N \psi_N \, d\tau +$$

$$+ \frac{1}{2} \int [\pi^2 + (\nabla \varphi)^2 + \mu^2 \varphi^2] \, d\tau$$

both mass and charge renormalization can be carried out; the constants can be computed and are infinite.

First let me say that the type of result everyone of us would like is the following:

$$< T \, \phi(x_1) \, \phi(x_2) \, \phi(x_3) >_0$$

$$= \int d K_1^2 \, d K_2^2 \, d K_3^2 \, . \, g(K_1, K_2, K_3) \, . \, \Delta_F(x_2 - x_3, K_1^2)$$

$$\Delta_F(x_3 - x_1, K_2^2) \, \Delta_F(x_1 - x_2, K_3^2).$$

Three approaches have been tried:

1) Nambu has studied individual orders in perturbation theory (Parametric Representation of General Green Functions, unpublished, and other papers by Nambu which have proved absortive).

2) Wightman and Källén have studied this question; their tool is the theorem I have stated

$$F^{(n)}(x_1, \ldots, x_{n-1}) = F^{(n)}[P(x_1, \ldots, x_n)].$$

When $\zeta_i = x_i - x_{i+1}$ are space-like.

Also both $F^{(n)}$ and $F^{(n)}(P(z))$ are analytic functions coinciding on the real environment.

Theorem by *Wightman and Hall* (Dansk. Mat. Fys. Medd.): A function f of n 4-vector variables $z_1, \ldots, z_n$ analytic in the tube defined by $z_j = \xi_j - i\eta_j$ (η_j real and in the future cone) and invariant under homogroup (orthochronous homogeneous Lorentz Group), is a function of the products $z_j \, . \, z_k$ and is analytic on the complex variety M_2 over which the scalar products vary when the vectors $z_1, \ldots, z_n$ vary over the tube. M_2 is composed of a few pieces of very simple analytic hypersurfaces. Then they find the explicit houndary of the union of permuted domains and it turns out not to be a holomorphy domain. Every function analytic in the union of the permuted M_2 is analytic in a certain larger domain which must be computed (the holomorphy envelope must be computed). Once this is done, use Bergmann-Weil formula to express $F^{(n)}$ anywhere in the interior of the domains as an integral over certain low dimensional subsets of the boundary of a certain kernel times the value of the function.

3) Third approach is that of *Schwinger* (Rochester, April 1957).

A wave function in its dependence upon the latest of all times contains only positive frequencies and in its dependence upon the earliest of all times contains only negative frequencies. Waves are always moving out of the region.

Replace boundary condition by regularity condition. Selection of out-going wave boundary conditions is equivalent to the requirement that the wave function defined as a function of space-time coordinate should remain a regular function when you make the time coordinate complex in a specific way and you never find an exponential which becomes unlimitedly large. The regularity reguirement is that when one takes all time coordinates and multiplies them by a complex number, the Green function remains regular (as a function of the time coordinate):

$$x^0 \longrightarrow x^0(1 - i\,\varepsilon) \qquad \varepsilon > 0$$

Example: two points:

$$x_1, x_2, x_1{}^0 > x_2{}^0;\; e^{-ip^0(x_1{}^0 - x_2{}^0)} \longrightarrow e^{-ip^0(x_1{}^0 - x_2{}^0)\,(1 - i\varepsilon)}.$$

Consider the space-time distance between two points:

$$(x - x')^2 \longrightarrow (x - x')^2 + i\,\varepsilon = (x - x')^2 - (x^0 - x'^0)^2\,(1 - i\,\varepsilon).$$

Then:

$$G(x_1\,x_2\,x_3) = G((x_1 - x_2)^2, (x_2 - x_3)^2, (x_3 - x_1)^2)$$

$$= \int d\lambda_1\,d\lambda_2\,d\lambda_3 \quad \exp \quad [-i\,\lambda_1(x_2 - x_3)^2 - i\,\lambda_2(x_3 - x_1)^2$$

$$- i\lambda_3(x_1 - x_2)^2]\, g(\lambda_1, \lambda_2, \lambda_3) \times$$
$$\times \exp. [- \varepsilon \{\lambda_1(x_2^0 - x_3^0)^2 + \lambda_2(x_3^0 - x_1^0)^2 + \lambda_3(x_1^0 - x_2^0)^2\}].$$

If the bracket has to remain greater than zero, then:

$$\lambda_1 + \lambda_2 > 0$$
$$\lambda_3 + \lambda_2 > 0$$
$$\lambda_3 + \lambda_1 > 0$$
$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 > 0.$$

Schwinger claims that these conditions after some manipulations lead him to structure of the form required plus abnormal cases (*Källén cases*)

$$\int d\, K_1^2\, d\, K_2^2\, d\, K_3^2 \quad g(K_1\, K_2\, K_3) \int d\, k_1\, d\, k_2\, d\, k_3$$
$$e^{ik_1(x_2-x_3)} \quad e^{ik_2(x_3-x_2)} \quad e^{ik_3(x_1-x_2)} \times [(k_1^2 - k_2^2 + K_2^2 - K_1^2 - i\,\varepsilon)$$
$$(k_2^2 - k_3^2 + K_3^2 - K_2^2 - i\,\varepsilon) \times (- k_3^2 + K_3^2 - i\,\varepsilon)]$$

$+ 2$ other terms.

It is not clear whether the outgoing wave boundary condition is satisfied in detail or not. Hence it is temporarily unaltered if abnormal terms can appear in Green's functions.

Dyson's lecture notes

While we are looking at this let me consider dispersion theory. For meson-nucleon scattering one may consider the vacuum expectation value

$$< 0 | T\, (\psi(x)\, \overline{\psi}(y)\, \phi(z)\, \phi(q) | 0 >$$

or alternatively

$$\int < p' | T\, (\phi(x)\, \phi(y) | p > e^{ikx}\, e^{-ik'y}\, d^n\, x\, d^n\, y, \;\; F(k \cdot p, p \cdot p').$$

I shall show later this is exactly the matrix-element for scattering. What we want is again a parametric representation for this quantify of the type we have been discussing. Once we know the parametric representation we know the position of the poles, considered as a function of k. p. for example.

In actual fact Goldberger considered not the T product but the R-product

$$\int (p' | \vartheta(x-x') [\phi(x), \phi(y)] | p) \, e^{ikx} e^{-ik'y} \, d^n x \, d^n y.$$

This contains the same informations. In $k \, . \, p$ plane all poles of M lie below the real axis (upper half-plane free of poles).

Proof has been given by

1) Symanzik
2) Jost and Lehmann

$\Big\} \; p = p'$.

3) Bogoliubov for restricted values of the masses.

One has the result:

$$\text{if } k \, . \, p = x, \qquad p \, . \, p' = y.$$

$$\mathrm{Re} \quad M(x, y) = P \int_{-\infty}^{\infty} \frac{Im\, M(x', y)}{x' - x} \, dx' \quad \text{for} \quad p = p'.$$

$Im\, M(x', y)$ is proportional to the total cross-section of $\pi - n$ scattering.

This then is a test formula based on I, II, III; of course for the physical interpretation we need Axiom IV.

Finally I come to Axiom IV: this is the axiom that relates our field operators to physical quantities and allows a description of the scattering-matrix. This is the axiom which essentially replaces the use of interaction representation in modern theory. This is the so called *Asymptotic condition in Field Theory* (L. Z. S.).

Axiom IV. Let ϕ be a neutral scalar field. Then ϕ satisfies the asymptotic condition if the limits

$$\lim_{x_0 \to -\infty} (\psi, \phi(x) \, \psi') = (\psi \, \phi^{in}(x) \, \psi')$$

$$\lim_{x_0 \to +\infty} (\psi, \phi(x) \, \psi') = (\psi, \phi^{out}(x) \, \psi')$$

exist, where

$$\phi^{in}, \phi^{out} \quad \text{satisfy} \quad \begin{cases} (\Box^2 - m^2) \, \phi^{in} = 0 \\ (\Box^2 - m^2) \, \phi^{out} = 0 \end{cases}$$

$$\phi^{out} = S^{-1} \, \phi^{in} \, S$$

then, by a *theorem* due to Lehmann-Zimmermann-Symanzik.

One can define ψ^{in}, ψ^{out} by applying ϕ^{in} and ϕ^{out} on ψ_0 and obtains the *Reduction formula;*

$$(\psi_0, T(x_1, \dots, x_n)\, \psi_{in}{}^{k_1 \dots k_n})$$
$$= \int (\Box_y{}^2 - \varkappa^2)\ (\psi_0, T(x_1, \dots, x_{n,y})\, \psi_{in}{}^{k_1 \dots k_{n-1}})\, e^{ik_n y}\, d^n y.$$

This formula does not need either causality or «crossing symmetry» for its proof.

Since S matrix is (ψ_{out}, ψ_{in}), so the S matrix-lement can be expressed in terms of a T-product.

The asymptotic condition now gives us the possibility of obtaining closed expressions for renormalization constants. (Lehmann).

$$< \psi_0\, \phi(x)\, \psi \mid \text{ (Particle state)} = < \psi_0\, \phi^{in}(x) \psi \mid \text{ (particle)}$$

we have

$$< \psi_0, \phi(x)\, \phi(y)\, \psi > \text{(deuteron)} = (\psi_0\, \phi^{in}(x)\, \phi^{in}(y)\, \psi)\ \text{(deuteron)}.$$

It is only possible to define asymptotic part. For completeness mention must be made of the:

Lee model (T.-D. Lee. Phys Rev *95* (1954) 1329): Interactiòn between two neutral fermion fields V and N and a neutral scalar boson field ϑ. The coupling constant (unrenormalized) in terms of the renormalized g_u:

$$g^2 = \frac{g_u{}^2}{1 - g_u{}^2 \sum\limits_{\omega} \frac{1}{2\omega\Omega} \frac{2}{(m_V - m_N - \omega)^2}}$$

g_u = renormalized coupling constant). If g_u does not vanish and remains finite, the unrenormalized coupling constant becomes

$$g = i\,\infty^{-1}.$$

1) cut-off necessary.

2) Even with a cut-off the denominator may be equal to zero, then g_u must be less than a critical number g_{crit}.

Landau and his school found same behaviour for electrodynamics.

Källén and Pauli investigated Lee model and, found that for $g > g_{crit}$ a «ghost state» appears for which indefinite metric is necessary, so that probabilities for the «ghost state» are counted negative. But this has no longer any connection with physics.