to the product measure $\prod_m \exp(g_m(p^2 + \omega_m{}^2 q^2))\, dp_m dq_m$. (The arbitrary functions $g(z)$ must be normalized: $\iint \exp(g_m(p_m{}^2 + \omega_m{}^2 q_m{}^2)) \cdot dp_m\, dq_m = 1.$). The eigenvectors of $\Omega(H)$ with the eigenvalues $E = \Sigma\, n_m\, \omega_m$ are $\psi_E = \prod_m (\exp(i/2\; p_m\, q_m)) \cdot (p_m + \text{sign}(n_m) \cdot \omega_m\, q_m) |n_m|$. In order ψ_E is square integrable only if a finite number of the n_m is allowed to be different from 0.

The treatment of classical linear fields along these lines is straightforward. The measure becomes unique by the postulate of relativistic invariance. The construction of an invariant measure in non-linear field theory involves problems similar to those discussed in 2c.

ON THE UNITARY REPRESENTATIONS OF THE LORENTZ GROUP

por HANS JOOS

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In relativistic quantum mechanics the kinematical properties of closed systems are determined by the properties of the unitary representations of the inhomogeneous Lorentz group (iLG) and of its different sub-groups. In this paper we discuss the complete reduction of the representations of the homogeneous, proper Lorentz group (pLG), which are contained in the irreducible unitary representations of the iLG.

The pLG consists of the transformations of the space time coordinates $x^\mu, \mu = 0, 1, 2, 3,$

$$x^{\mu\prime} = \Lambda_\nu{}^\mu\, x^\nu \tag{1}$$

which conserve

$$x^2 = x^\mu\, x_\mu = x^\mu\, g_{\mu\nu}\, x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - (\vec{x})^2$$

and space and time orientation: $\det \Lambda = 1$, $\Lambda_0{}^0 > 0$. The iLG

(*) Contracted by the *Conselho Nacional das Pesquisas*, of Brasil. *Now at the Institut für theoretische Physik der Universität Hamburg.*

contains additionally the translations

$$x^{\mu'} = \Lambda_\nu{}^\mu x^\nu + a^\mu. \tag{2}$$

An unitary representation of the iLG associates with every transformation (Λ, a) a unitary operator $U(\Lambda, a)$, which satisfies

$$U(\Lambda, a) \,.\, U('\Lambda, 'a) = U(\Lambda \,.\, '\Lambda, a + \Lambda' a). \tag{3}$$

The iLG is 10-dimensional Lie group. In a unitary representation of the iLG the infinitesimal transformations are represented by hermitian operators U_i. If $U_{\mu\nu} = - U_{\nu\mu}$ corresponds to the infinitesimal «rotation» in the (μ, ν)-plane and if U_ν corresponds to the infinitesimal translations along the ν axis, the following commutation relations result from (3):

$$\begin{aligned} [U_{\mu\nu}, U_{\rho\tau}] &= i(g_{\nu\rho} U_{\mu\tau} - g_{\mu\rho} U_{\nu\tau} + g_{\mu\tau} U_{\nu\rho} - g_{\nu\tau} U_{\mu\rho}) \\ [U_{\mu\nu}, U_\rho] &= i(g_{\nu\rho} U_\mu - g_{\mu\rho} U_\tau) \quad ; \quad [U_\mu, U_\nu] = 0. \end{aligned} \tag{4}$$

As a consequence of these relations, the operators P and W:

$$\begin{aligned} P &= U_\mu U^\mu \\ W &= 1/2\, U_{\nu\mu} U^{\nu\mu} U_\rho U^\rho - U_{\nu\mu} U^{\rho\mu} U^\nu U_\rho \end{aligned} \tag{5}$$

are commuting with all $U_{\mu\nu}$, U_ν.

In an irreducible representation of the iLG P and W are constants, i. e. multiples of the unit operator, which are characteristic for the physically interesting representations (1). In relativistic quantum mechanics the U_ν resp. $U_{\mu\nu}$ are the operators of the 4-momentum resp. relativistic angular momentum. For the description of an elementary particle with rest mass μ (+) and spin s, W and P have meaning:

$$W = \mu^2 s(s+1), \qquad P = \mu^2.$$

We ask now for operators, which commute with the $U_{\mu\nu}$ only. Besides the P and W, there are only two other independent invariants

$$\begin{aligned} Q &= U_{\mu\nu} U^{\mu\nu} \\ R &= U_{01} U_{23} + U_{02} U_{31} + U_{03} U_{12}. \end{aligned} \tag{6}$$

(1) E. P. WIGNER, *Ann. Math.* Vol. 40, (1939), p. 149-204.
E. P. WIGNER, V. BARGMANN, *Nac. Ac. Scie.* Vol. 34 (1948), p. 211-223.

(+) $\hbar = c = 1$.

These invariants are characteristic for the irreducible, unitary representations of the pLG [2]. Our problem is identical with the determination of the spectral decomposition of the operators Q and R with $U_{\mu\nu}$ belonging to a given P and W.

We shall discuss this problem for $s=0$. In this case the hermitian operators $U_\nu, U_{\mu\nu}$ are equivalent to

$$U_\nu \psi(k) = k_\nu \psi(k); \quad U_{\mu\nu} \psi(k) = i\left(k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu}\right)\psi(k) \tag{7}$$

where

$$k_\nu, i\left(k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu}\right)$$

are self-adjoint operators with respect to the Hilbert space of the functions $\psi(k)$, which are defined on the momentum hyperboloid H^+: $k^2 = \mu^2, k^0 > 0$. The norm of of $\psi(k)$ is

$$\|\psi\|^2 = \int \psi^*(k)\, \psi(k)\, d\Omega,$$

$d\Omega$ invariant volume element on H^+. Immediately we get

$$R = 0. \tag{8}$$

In order to solve the simultaneous eigenvalue problem

$$U_{03}\psi = \alpha\psi, \quad U_{12}\psi = m\psi, \quad Q\psi = q\psi, \tag{9}$$

we introduce on H^+ cylindrical coordinates: $k_1 = u\cos\vartheta$, $k_2 = u \sin\vartheta$, $k_3 = v$, $k_0 = \omega = \sqrt{\mu^2 + u^2 + v^2}$. By straightforward calculations, we get as solution of (9)

$$\psi_{m,\alpha,\lambda} = N \,.\, e^{im\vartheta}\left(\frac{v+\omega}{\mu}\right)^{i\alpha} F\left(a, b;\, c;\, \frac{-u^2}{\mu^2}\right) \tag{10}$$

$F(a, b; c; z)$ is the hypergeometric function with the parameters $a = 1/2\,(m+1+i\alpha+i\lambda;$ $b = 1/2\,(m+1+i\alpha-i\lambda)$, $c = m+1$; λ is connected with the eigenvalue $q = 1+\lambda^2$; the normalization factor is

$$N = \frac{\Gamma(1/2(m+1+i\lambda+i\alpha))\,\Gamma(1/2(m+1+i\lambda-i\alpha))}{(2\pi)^{3/2}\,.\,\Gamma(i\lambda)\,\Gamma(m+1)}\,.$$

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(2) L. Gelfand, M. Neumark, J. Phys. (USSR) Vol. X, (1946), p. 93-94. Harish-Chandra, *Proc. Roy. Soc.* (London) A, Vol. 189, (1947), p. 372-401.
V. Bargmann, *Ann. Math.* Vol. 48, (1947), p. 568-640.

In the case $\mu^2=0$ the functions $\psi_{m,\alpha,\lambda}$ define a complete orthonormal system if:

$$m=0, \pm 1, \ldots, -\infty < \alpha < +\infty, \quad 0 \leqq \lambda < +\infty.$$

There is a degeneration in the Q-spectrum for $\mu^2=0$. In this case we get as solution of (8)

$$\psi_{m,\alpha,\lambda}=(2\pi)^{-3/2} e^{im\vartheta}(v+\omega)^{i\alpha} u^{i(\lambda-\alpha)-1}. \tag{10'}$$

These functions are only complete if λ varies over the whole real axis.

Introducing the following complex combinations of the $U_{\mu\nu}$

$$M_1^0=2^{-1}(U_{12}-iU_{03}) \quad ; \quad M_2^0=1^{-1}(U_{12}+iU_{03})$$

$$M_1^+=2^{-3/2}(U_{23}+iU_{31}-iU_{01}+U_{02}) ;$$

$$M_2^+=2^{-3/2}(U_{23}+iU_{31}+iU_{01}-U_{02}$$

$$M_1^-=2^{-3/2}(U_{23}-iU_{31}-iU_{01}-U_{02}) ;$$

$$M_2^-=2^{-3/2}(U_{23}-iU_{31}+iU_{01}+U_{02}) \tag{11}$$

we get instead of (4) the following commutation relations

$$[M_r^0, M_{r'}^\pm]=\pm M_r^\pm \delta_{rr'} \qquad [M_r^+, M_{r'}^-]=M_r^0 \delta_{rr'}$$

$$[M_r^0, M_{r'}^0]=[M_r^+, M_{r'}^+]=[M_r^-, M_{r'}^-]=0 \qquad r, r'=1,2. \tag{12}$$

Instead of the hermitian U, we have now

$$(M_1^+)^\dagger=M_2^-, \quad (M_1^-)^\dagger=M_2^+, \quad (M_1^0)^\dagger=M_2^0.$$

These relations indicate that the pLG and the complex orthogonal group are isomorphic. The functions $\psi_{m,\alpha,\lambda}$ defined in (10), are similarly related to the complex orthogonal group and H^+, as the spherical harmonics are related to the real orthogonal group and the sphere. This is shown by the following relations

$$M_r{}^+ \psi_{m,\alpha\cdot\lambda} = \pm (2)^{-3/2}(i\lambda + 1 + m \mp i\alpha)\, \psi_{m+1,\alpha \pm i\,\lambda}$$
$$M_r{}^- \psi_{m,\alpha\cdot\lambda} = \pm\ 2^{-3/2}\ (i\lambda + 1 - m \pm i\alpha)\, \psi_{m-1\cdot\alpha \mp i,\lambda}$$
$$M_r{}^0 \psi_{m,\alpha,\lambda} = \quad 2^{-1}\ (m \mp i\alpha)_m, \tag{13}$$

where the upper sigh correspond to $r=1$, the lower to $r=2$.

There are well known relations for the spherical harmonics, which are completely analogous to (13) ([3]).

In the discussion of our problem in the case with spin, $s=0$; eq. (13) play a important role, because these relations permit the generalization of the composition rule for orbital and spin angular momenta to the relativistic case ([4]).

([3]) for instance: Bethe, Handbuch der Physik Bd. XXIV/1 (1933), p. 551 et seq.

([4]) Hans Joos, to be published.

RESUMENES

I. SOBRE LA INVARIANCIA DE LAS ECUACIONES DE LA FISICA CLASICA

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Ya antes de la teoría de la relatividad era un postulado generalmente aceptado, que las ecuaciones de la física debían ser escritas en forma invariante con respecto al grupo de las transformaciones ortogonales, las que comprenden las rotaciones ortogonales y las reflexiones de las coordenadas. Era este postulado que conducía al cálculo vectorial.

La invariancia de las ecuaciones con respecto a las reflexiones tiene una implicación importante. Si encontramos una sustancia cuya estructura espacial no coincide con su imagen especular, la configuración especular también es compatible con las ecuaciones. Ya P. Curie mostró que para cada sustancia ópticamente activa también existe la sustancia opuesta. En el mundo orgánico encontramos seres con un solo sentido de espiralidad, pero este hecho puede ser explicado por fenómenos secundarios, sin abandonar la simetría de las leyes básicas.