GENERALIZED POTENTIAL OPERATORS

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The n-dimensional Hilbert transforms studied by Calderón and Zygmund [1] are convolution operators $\tilde{f} = f * h$ such that the kernel $h(x)$ satisfies the homogenous property $h(ax) = a^{-n} h(x), x \in E^n$, for all $a > 0$ and the integral of $h(x)$ taken over the set $1 < |x| < 2$ is zero. In the paper [2] the Calderón-Zygmund results have been extended to convolution operators $H_{0n} f = f * K$ with kernels K satisfying the homogenous property only for one value of a, for instance $a = 2 : K(2x) = 2^{-n} K(x)$. The operators H_{0n} include as special cases the operators of Fejer and ergodic type.

In this paper a similar generalization is done for the potential operators $\tilde{f}_{\gamma n} = f * |x|^{\gamma - n}$, $0 < \gamma \leqq n$, $x \in E^n$, where the kernel $h(x) = |x|^{\gamma - n}$ satisfies the homogenous property $h(ax) = a^{\gamma - n} h(x)$, for all $a > 0$. We consider operators of the form $H_{\gamma n} f(x) = f * K_{\gamma n}$ where the kernel satisfies the condition $K_{\gamma n}(2t) = 2^{\gamma - n} K_{\gamma n}(t), t \in E^n$. Such operators may be called generalized potential operators. We show that if $K_{\gamma n}$ satisfies certain conditions then the basic properties of the classical operators $\tilde{f}_{\gamma n}$, due to Hardy-Littlewood, Thorin, Sobolieff, Zygmund, Du Plessis and others, are true for these generalized operators $H_{\gamma n}$. We derive these results from general properties of linear operators, and we complete the results of Sobolieff in the case $f(t) \in L^1(E^n), x \in E^m, m \neq n$ (1).

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(1) Some of the results of this paper were presented at the 1957 meeting of the UMA in Bahia Blanca.

1. **Introduction.** We consider the n-dimensional Euclidean space $E^n = \{x, y, \dots t, \dots\}$, $x = (\xi_1, \dots \xi_n)$, $y = (\eta_1, \dots \eta_n) \dots$ and use the notations $x + y = (\xi_1 + \eta_1 \dots \xi_n + \eta_n)$, $(x, y) = \eta_1 \xi_1 + \dots + \xi_n \eta_n$, $|x|^2 = (x, x)$, $dx = d\xi_1 \dots d\xi_n$. If $E \subset E^n$, $|E|$ will denote the measure, and $\varphi(x, E)$ the characteristic function of E. By $L^p = L^p(E^n)$ we mean the set of measurable functions f such that

$$\|f\|_p = \left\{ \int_{E^n} |f(x)|^p \, dx \right\}^{1/p} < \infty. \tag{1}$$

If $f(x)$ is defined on E^n and if $E^m \subset E^n$, then the p-norm of the restriction of f to E^m will be denoted by

$$\|f\|_{p^{(m)}} = \left\{ \int_{E^m} |f(x)|^p \, dx \right\}^{1/p}. \tag{2}$$

We say that $h = Tf$ is an operator *of type* (p, s), or more precisely *of type* $(L^p(E^n), L^s(E^m))$, if for every $f \in L^p(E^n)$, $h = Tf$ is defined in E^m and satisfies

$$\|h\|_{s^{(m)}} \leqq M \, \|f\|_{p^{(n)}} \tag{3}$$

where the constant M is independent of f. The least value of M is the *norm* of T. If h is defined in E^m and if $a > 0$, we denote by $D(|h|, a)$ the (m-dimensional) measure of the set $E = \{x \in E^m;\ |h(x)| > a\}$. If $s < \infty$, we say with Zygmund [3] that T is of *weak type* (p, s) with constant M, if

$$D(|Tf|, a) \leqq (M \, \|f\|_{p^{(n)}} / a)^s \tag{4}$$

holds for any $a > 0$ and any $f \in L^p$. If $s = \infty$ then, by definition, weak type (p, s) is the same as type (p, s). The least value of M in (4) is the *weak norm* of T. If T is of type (or weak type) (p, s), we say also that T is of *type* P, where P is the point of the plane of coordinates $(1/p, 1/s)$; if $p \geqq 1, s \geqq 1$, then P is in the unit square, called the square of types. Given an operator T, one of the basic problems which arise is to determine the points P such that T is of type, or weak type P.

In the case of the operators $\tilde{\mathfrak{f}}_{\gamma n}$ we have the following results.

A) The n-dimensional Hilbert transform $\tilde{f}=\tilde{f}_{0n}=f*h$ is a singular integral defined as a Cauchy principal value. It was proved by Privaloff, Lusin, M. Riesz and Kolmogoroff in the case $n=1$, and by Calderón and Zygmund [1] in the general case, that if the kernel h satisfies some continuity condition, then $\tilde{f}$ exists as a principal value for any $f \in L^p, p \geqq 1$, and the operator $\tilde{f}=f*h$ is of type (p,p) if $1<p<\infty$, and of weak type $(1,1)$. Thus, $\tilde{f}_{0n}$ is of type P for every interior point P of the diagonal $A_0 B_0$ of the square of types, and of weak type at the end point B_0. (See figure 1).

B) Consider now the Riesz potentials or fractional integrals

$$\tilde{f}_{\gamma n}(x) = c_{\gamma n} \int_{E^n} f(t)\,|x-t|^{\gamma-n} \quad dt = f * |t|^{\gamma-n}, \tag{5}$$

$$0<\gamma \leqq n,\ c_{\gamma n} = \pi^{n/2}\, 2^{\gamma}\, \Gamma(\gamma/2)/\Gamma((n-\gamma)/2). \tag{5a}$$

Since the kernel $h=|t|^{\gamma-n}$ is non-negative the integral (5) is well defined for any $f \geqq 0$, so that (5) is not a singular integral. It was proved by Hardy-Littlewood in the case $n=1$, and by Thorin and Sobolieff in the case $n>1$, [4], (cfr. also [5]), that the operator (5) is of type (p,s) for every p,s such that

$$1/p - 1/s = \gamma/n, \qquad \gamma/n < 1/p < 1. \tag{6}$$

Hence, the operator (5) is of type P for any interior point of the segment AB obtained by translating the diagonal in γ/n. Zygmund [3] proved that this operator is of weak type at the end point $B=(1,(n-\gamma)/n)$.

C) Let $E^m \subset E^n$ be a subspace of E^n. Let in (5) t vary in E^n and x in E^m, then the operator (5) assigns to functions f defined on E^n, functions $\tilde{f}(x)$ defined on E^m. Sobolieff proved [6] that in this case the operator (5) is type $(L^p E^n)$, $L^s(E^m))$ for

$$1/p - (m/n)/s = \gamma/n, \quad \gamma/n < 1/p < 1, \quad (n-\gamma)/m < 1. \tag{7}$$

Hence here the operator is of type P in any interior point of the segment AB' obtained by translating and rotating the diagonal. Sobolieff really proved only a much weaker result, assuming that f and $\tilde{f}_{\gamma n}$ are considered on bounded domains and that $1/p - m/ns < \gamma/n$; he proposed the full result as a problem. The problem was solved by Ilin [7]. Without knowing the paper of Ilin we obtained the same result by a different method (2) which is given below for more general operators. Moreover, we proved that the operator is of weak type at the end point B'. Similar results hold if $E^m \supset E^n$ (cfr. [8]).

D) If the functions $f(t)$ and $\tilde{f}(x)$ are considered on bounded domains $D_n \subset E^n$, and $D_m \subset E^m$ (so that the integral in (5) is taken over D_n), then it was proved by Sobolieff and Kondrachieff ([6], cfr. [9]) that the operator (5) is of type $(L^p(D_n), L^s(D_m))$ for

$$1/p - (m/n)/s \leqq \gamma/n, \ \frac{n}{\gamma} \neq p, 1 \leqq s \leqq \infty, \quad n - \gamma < m, \tag{8}$$

that for

$$1/p - (m/n)/s < \gamma/n \tag{9}$$

the operator is completely continous, and if in addition $0 < 1/p < \gamma/n$ then (5) is a completely continous operator from $L^p(D_n)$ to $C(D_m)$.

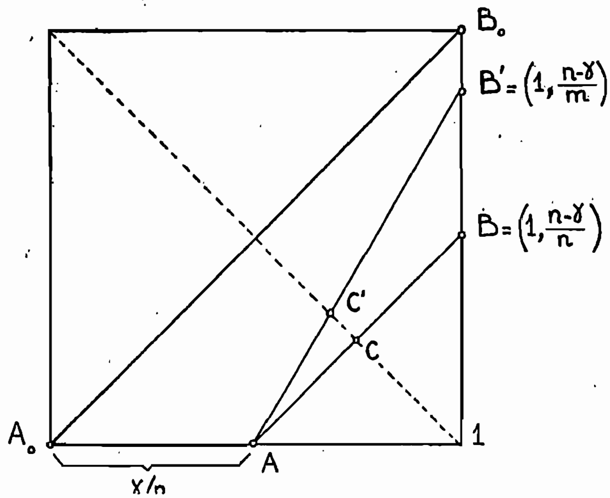
E) It was proved by Du Plessis [10] that if $f \in L^p, \gamma/n < 1/p < 1/2$ then the set where (5) is not finite is of zero β-capacity for $\beta > n - p\gamma$; if $1 \leqq p \leqq 2$ then this set is of zero β-capacity for $\beta = n - \gamma p$. Finally it was proved by Hardy-Littlewood in the case $n = 1$, and by Du Plessis in the case $n > 1$, that if $f \in Lip\,\beta$, $0 < \beta < 1$, then $\tilde{f}_{\gamma n} \in Lip\,(\alpha + \beta)$, $0 < \gamma + \beta < 1$; and if $f \in L^p$, $p > 1, 1/n + 1/p > \gamma/n > 1/p$, then $\tilde{f}_{\gamma n} \in lip\,(\gamma - n/p)$.

The results mentioned in A) have been extended in [2] to operators $H_{0n} f = f * K_{0n}$ where the kernel satisfies the homogenous condition only for $a = 2$. If for each kernel K we associate the kernel $k(t) = K(t)$ in $1 \leqq |t| < 2$ and zero other-

(2) See our communications to the 1957 meeting of the UMA.

wise, then $K = K_{0n}$ satisfies the homogeneous condition $K_{0n}(2t) = 2^{-n} K_{0n}(t)$ if and only if

$$K_{0n}(t) = \Sigma^{\infty}{}_{i=-\infty} 2^{-ni} k(2^{-i} t). \tag{10}$$



Thus, the results A) hold for kernels of form (10), provided $k(t)$ satisfies some continuity condition and its integral is zero.

The aim of this paper is to obtain a similar generalization for the operators $\tilde{f}_{\gamma n}, \gamma > 0$. Since the kernel $h(t) = |t|^{\gamma-n}$ satisfies the homogenous property $h(at) = a^{\gamma-n} h(t)$ for all $a > 0$, we consider generalized potential operators of the form

$$H_{\gamma n} f(x) = \int_{E_n} f(t) K_{\gamma n}(x-t)\, dt, 0 < \gamma \leqq n, \tag{11}$$

where the kernel satisfies the condition $K_{\gamma n}(2t) = 2^{\gamma-n} K(t)$. This is the same as to say that $K_{\gamma n}$ is of the form

$$K_{\gamma n}(t) = \Sigma^{\infty}{}_{i=-\infty} 2^{(\gamma-n)i} k(2^{-i} t). \tag{12}$$

We shall prove that properties B)—E) hold for operators (11) if k satisfies certain conditions. In the case $\gamma > 0$, the integral of k need not to be zero and we may take $k \geqq 0$.

Moreover, in [2] was given a further generalization of the Hilbert operators $\tilde{f}_{0n}$ which is as follows. Let $K = K_{0n}$ be of the form (10) and let $k_i(t) = 2^{-ni} k(2^{-i} t)$, so that $K_{0n} = \Sigma^{\infty}{}_{-\infty} k_i$

If the generating kernel k satisfies $k \in Lip(1,1)$, then the «generated» kernels k_i satisfy the conditions

$$\|k_i\|_1 \leqq c; \quad \|k_i\|_{(1,1)} \leqq 2^{-i} c; \quad \int_{E^n} k_i \, dt = 0, \tag{13}$$

where $\|k\|_{(1,\gamma)}$ denotes the least constant c such that $\int_{E^n} |k(x+h) - k(x)| \, dx \leqq c|h|^{\gamma}$, $(\gamma \leqq 1)$, and $k \in Lip(1,\gamma)$ means $\|k\|_{(1,\gamma)} < \infty$. (If $\gamma > 1$, $\gamma = \gamma' + \gamma''$, $\gamma' =$ integer, $\gamma'' \leqq 1$, we say that $k \in Lip(1,\gamma)$ if k has absolutely continous derivatives up to order γ' and if the derivatives of order γ' belong to $Lip(1,\gamma'')$). Conditions (13) imply that the k_i are «almost orthogonal», that is $\|k_{i+j} * k_i\|_1 \leqq 2^{-j} c$, if $j > 0$. It was proved in [2] that the properties A) hold for operators of the form $H_{0n} f = \Sigma f * k_i$ if the k_i satisfy (13) or the almost orthogonal conditions. Similarly if $K_{\gamma n}$ is of the form (12) and if $k \in Lip(1,1)$, then the corresponding k_i satisfy

$$\|k_i\|_1 \leqq 2^{i\gamma} c; \quad \|k_i\|_{(1,1)} \leqq 2^{(\gamma-1)i} c, \quad (\gamma < 1), \tag{14}$$

and the orthogonality condition $\|k_i * k_{i+j}\|_{(1,2\gamma)} \leqq 2^{-j\gamma} c$ holds. From the results of this paper and from our previous paper [11] it will follow that properties B)-D) hold for operators of the form $f * \Sigma k_i$ if the k_i satisfy (14) or the orthogonality conditions. (In [11] the method is only sketched very briefly, the details will be given in [12]).

We give diret proofs, based on general properties of linear operators, and do not assume known the properties of the classical potential operators.

2. General remarks on types. Let $L_0 = L_0(E^n)$ be the set of all step functions of E^n; by a step function we mean a linear combination of characteristic funcions of n-dimensional cubes. We say that the operator T is of type (or weack type) on L_0, if (3) (or (4)) holds for $f \in L_0$. Since L_0 is dense in all the $L^p, p < \infty$, any operator of type (p, s) on L_0 can be extended to an operator of type (p, s) on L^p. Our main purpose is to establish that $H_{\gamma n}$ is of type $P = (1/p, 1/s)$, for all interior points of a certain segment AB, and of weak

type at B. The following general properties show that it will be sufficient to establish the types only on L_0 and only for two special cases: $P=C$ and $P=B$. The case $P=C$ is easier to handle since in this case $s=p'=p/(p-1)$ and L^s is the dual of L^p (see section 3); and in the case $P=B$ we have $p=1$ and some special arguments can be used (see B) below and section 4).

A) In many cases is considered an operator T which is the limit of «good» operators T_ε. More precisely, let T_ε, $\varepsilon>0$, be a sequence of linear operators satisfying the following conditions:

(i) For each $\varepsilon>0, T_\varepsilon f$ assigns to any function $f \in L^p$, $p \geqq 1$, a function $T_\varepsilon f(x)$, finite for all x. (ii) If $f_n \in L^p$ (fixed p) and if $\|f_n\|_p \rightarrow 0$ as $n \rightarrow \infty$, then $\lim T_\varepsilon f_n(x)=0$, for each fixed $\varepsilon>0$ and for each x. (iii) $\lim T_\varepsilon f(x)=Tf(x)$ exists for almost all x and all $f \in L_0$. Thus the limit operator Tf is defined and is finite for all $f \in L_0$.

For instance, if $Tf=\tilde{f}_{\gamma n}=f * |t|^{\gamma-n}, h_\varepsilon(t)=|t|^{\gamma-n}$ if $\varepsilon<|t|<\varepsilon^{-1}$ and zero otherwise, and if $T_\varepsilon f=f * h_\varepsilon$, then each h_ε is a bounded integrable function and properties (i)-(iii) are satisfied.

Let Mf be the maximal operator of the sequence $T_\varepsilon f$, that is

$$Mf(x)=\sup_{\varepsilon>0}|T_\varepsilon f(x)|. \tag{15}$$

By (iii) $Mf(x)$ is finite for every $f \in L_0$ and almost all x, and $|Tf| \leqq Mf$. Mf is not a linear operator, but it is sublinear: $f=g+h$ implies $Mf(x) \leqq Mg(x)+Mh(x)$.

P r o p o s i t i o n 1. *Let T be a set of linear operators satisfying (i), (ii), (iii). a) If Mf is of type (of weak type) (p,s) on L_0, then Mf is of type (weak type) (p,s) on L^p, and (iii) holds for any $f \in L^p$.* b) *If Mf is of type (p,s) on L_0 then $T_\varepsilon f$ converges to Tf in the s-mean for $f \in L^p$; if M is of weak type (p,s) then $T_\varepsilon f$ converges in the s-mean only on bounded sets and only if $|f|^p \log(1+|f|)$ is integrable.*

Proposition 1 was proved in [2] (p. 119) in the case $p=s$, the proof in the general case is essentially the same (see [15]) and we shall not repeat it. Proposition 1 shows that under conditions (i)-(iii) *we may restrict ourselves to types on L_0.*

B) We say that Tf satisfies the *condition* (p, s) *of Kolmogoroff with constant* M', if for any $s' < s$, for any f, and for any bounded set $X \subset E^m$ it is true:

$$\left\{\int_X |Tf(y)|^{s'} dy\right\}^{1/s'} \leqq M'(s/(s-s'))^{1/s'} |X|^{1/s'-1/s} \|f\|_p. \tag{16}$$

Proposition 2. *If T satisfies inequality (16) for one value $s' < s$ and with constant M' then T is of weak type (p, s) with constant $M \leqq M'(s/(s-s'))^{1/s'}$. If T is of weak type (p, s) then T satisfies the Kolmogoroff condition with the same constant $M' = M$.*

This Proposition was proved in [2] (p. 67) in the case $s = p$; the proof is almost the same in the general case. It is important to observe that the proof of Proposition 2 gives somewhat more: if (4) is true for a *fixed* f and for all $a > 0$, then (16) is true for this f and for all $s' < s$ and X, and conversely. From this remark we deduce the following.

Proposition 3. *If $p = 1, 1 < s < \infty$, and if (4) is true for the characteristic functions of cubes, then T is of weak type $(1, s)$ on L_0.*

Proof. Any function $g \in L_0, g \geqq 0$, is of the form $g(x) = c_1 f_1(x) + \ldots c_l f_l(x)$, where f_i are characteristic functions of cubes. By hypothesis (4) is true for each f_i, and since (4) implies (16) for an individual f (16) is true for each f_i. Since $s > 1$, we may take $1 < s' < s$, and since $p = 1$,

$$\left\{\int_X |Tg(y)|^{s'} dy\right\}^{1/s'} \leqq \Sigma_i \left\{\int_X |c_i Tf_i(y)|^{s'} dy\right\}^{1/s'} \leqq$$
$$\leqq M(s/(s-s'))^{1/s'} |X|^{1/s'-1/s} \|\Sigma\, c_i f_i\|_1 =$$
$$= M(s/(s-s'))^{1/s'} |X|^{1/s'-1/s} \|g\|_1.$$

Hence (16) is true for $g(x)$, and therefore (4) is true for g.

Proposition 3, provides a very simple proof of the following theorem due to Zygmund [3] (Zygmund considered only the case $n = m$):

Proposition 4. *If $E^m \subset E^n$ and $m > n - \gamma$, then the potential operator (5) is of weak type $(L^1(E^n), L^s(E^m))$, with $s = m/(n-\gamma)$.*

Proof. Since $s > 1$, it is enough to prove that (4) holds if f = characteristic function of a cube Q. If c is the center and $l > 0$ the side of Q, then $\tilde{f}_{\gamma n}(x) \leqq$ the integral of $|t|^{\gamma-n}$ extended over the set $|t-(x-c)| < n\,l$. An easy computation shows that the last integral is $\leqq c\, l^n |x|^{\gamma-n}$. Therefore if x is such that $\tilde{f}_{\gamma n}(x) > a$ then $c\, l^n |x|^{\gamma-n} > a$, or $|x| < (c\, l^n/a)^{1/(n-\gamma)} = r$. Hence $D(\tilde{f}_{\gamma n}, a) \leqq$ volume of the sphere of radius $r \leqq (c_1 l^n/a)^{m/(n-\gamma)} \leqq (c_2 \|f\|_1/a)^{m/(n-\gamma)}$.

C) Let T be a linear (or sublinear) operator on L_0. The *Riesz-Thorin convexity theorem* asserts that if T is type P_1 and of type P_2, then T is of type P for any P of segment $P_1 P_2$; moreover, the corresponding norms M_1, M_2, M satisfy the inequality $M \leqq M_1{}^a M_2{}^{1-a}$ where a is the ratio in which P divides the segment. The Riesz-Thorin theorem is a particular case of the following *Convexity theorem for analytic operators* ([13], [14]): Assume that for each complex number $z, 0 \leqq Rz \leqq 1$, is given an operator T_z defined on L_0 such that 1) for fixed f, g of $L_0, (T_z f, g)$ is an analytic and (for instance) bounded function of z; 2) $\|T_{iu} f\|_{s_1} \leqq M_1 \|f\|_{p_1}$ and $\|T_{1+iu} f\|_{s_2} \leqq M_2 \|f\|_{p_2}$, $iu = (-1)^{1/2} u$, for all $f \in L_0$ and all real u, $(p_i, s_i \geqq 1$ for $i = 1, 2)$. Then for each $t, 0 < t < 1$, the operator T_t is of type (p, s), where $1/p = (1-t)/p_1 + t/p_2, 1/s = (1-t)/s_1 + t/s_2$, and $\|T_t\| \leqq M_1{}^{1-t} M_2{}^t$.

Another important generalization is the following *Theorem of Marcinkiewicz-Zygmund*: If T is of weak type P_1 and of weak type $P_2, P_i = (1/p_i, 1/s_i)$, $p_i \leqq s_i, s_1 \neq s_2$, then T is of type P in any interior point P of $P_1 P_2$, with norm M satisfying $M \leqq c M_1{}^{1-\alpha} M_2{}^{\alpha}$, where c depends only on the points P_i and tends to infinity as P approaches one of the end points. The theorem is not true if $s_1 = s_2$.

It is well known that a linear operator T is of type (p, s), $1 \leqq p, s, s < \infty$, if and only if $|(Tf, g)| \leqq \mathrm{M} \|f\|_p \|g\|_{s'}$ (this is not true if $s = \infty$). Therefore, for convolution operators $Tf = f * K$ is true the following property: if $T = f * K$ is of type $(L^p(E^n)$, $L^s(E^m)), p > 1, s \geqq 1$, then T is also of type $(L^{s'}(E^m), (L^{p'}(E^n))$. That is, if T is of type P it is also of type $P^* =$ the simmetric point to P with respect to the diagonal; this property is not true if $p = 1$.

Applying this remark and the Marcinkiewicz-Zygmund theorem to the operator $H_{\gamma n}$ we see that, in order to prove that $H_{\gamma n}$ is of type in the interior of AB and of weak type in B, *it is enough to prove that $H_{\gamma n}$ is of weak type at the end point B and of type at* $C=$intersection of AB with the diagonal $1/p+1/s=1$.

Let us still remark that if we are interested only in weak types and weak norms, then the Marcinkiewicz-Zygmund theorem takes the following perfected form: *If T is of weak type P_1 with weak norm M_1, and of weak type P_2 with weak norm M_2, then T is of weak type P in any point P of P_1P_2, with norm M satisfying $M \leqq 2M_1^{1-a}M_2^{a}$.* Here, as in the Riesz-Thorin theorem, P_i are arbitrary points in the square of types and the case $s_1=s_2$ is not excluded. The proof is essentially the the same, and much simpler, as that of Marcinkiewicz's theorem (see [15] and [12]).

Finally, the definition of weak type may be formulated in the following form, which is more similar to that of type. For any function $h(x)$ let us define the weak s-norm by

$$\{h\}_s=(\sup_{a>0}\{a^s D(|h|;a)\})^{1/s}. \tag{17}$$

Then T is of weak type (p,s) if $\{Tf\}_s \leqq M\,\|f\|_p$. Let $\{L^s\}$ be the set of all measurable functions h such that $\{h\}_s<\infty$. The weak norm $\{h\}_s$ is not a norm but it defines in $\{L^s\}$ a topology equivalent to that of a normed space. In fact, as C. Trejo showed in a forthcoming note, if V_d is the set $\{h\}_s \leqq d$, and if $[f]_s$ is the infimum of the numbers $d \geqq 0$ such that $f\in \mathrm{Conv}\{V_d\}$, then $[f]_s$ is a norm in $\{L^s\}$ which defines the same topology as $\{h\}_s$. Thus $[L^s]$ is a normed space, and a linear operator T is of weak type (p,s) if and only if T is a continous transformation from L^p to $[L^s]$. Proposition 4 asserts then that the potential operator (5) is a bounded operator from $L^p(E^n)$ to $[L^s(E^m)]$ for $s=m/(n-\gamma)$. Let us define the space $\{W_s^{(l)}\}$ as the set of all functions which admit generalized derivatives up to order l, and with norm $[f]_s+\Sigma\,[D^l f]_s<\infty$, where $D^l f$ denotes the generic $l-th$ order dervative. Then from Proposition 4 we may deduce the following generalization of the immersion theorem of Sobolieff [6]: if $k\geqq 0, k\geqq l-n$ and $s=n/(n-(l-k))$, then $W_1^{(l)}\subset[W_s^{k}]$. In particular if $n>l$,

$m \geqq n-l$ and $s=m/(n-l)$, then $W_1^l(E^n) \subset [L^s(E^m)]$. We shall return to this question in another paper.

3. Lemmas for types (p, p'). For each $p \geqq 1$ we write $p'=p/(p-1)$. Let $k_i(x), x \in E^n$, be a sequence of integrable kernels, so that $f * k_i$ is well-defined for each i. Let us see under which conditions the operator

$$Hf = \Sigma^{\infty}_{i=-\infty} f * k_i = f * K \tag{18}$$

will be of type (p, p'). For any N we put

$$K_N = \Sigma^N_{i=-N} k_i; \quad H_N f = f * K_N. \tag{18a}$$

We say that $k \in Lip(p, \gamma), 0 \leqq \gamma \leqq 1$, if $\|k(x+h)-k(h)\|_p \leqq M|h|^{\gamma}$ holds for any $h \in E^n$; the least value of M will be denoted by $\|k\|_{(p,\gamma)}$ (for $\gamma > 1$, see the definition on the end of the Introduction). The Fourier transform of f will be denoted by $\hat{f} = \mathrm{F} f$, $f = \mathrm{F}^* \hat{f}$, and the following well-known inequalities (see [16]) will be used $(1 < p \leqq 2)$:

$$\|\mathrm{F}^* \mathrm{f}\|_{p'} \leqq \|f\|_p, \tag{19}$$

$$\int_{E^n} |\hat{f}(u)|^p \, |u|^{n(p-2)} \, du \leqq c_p \int_{E^n} |f(x)|^p \, dx, \tag{20}$$

$$|\hat{f}(u)| \leqq c_{\gamma} \|f\|_{(1,\gamma)} |u|^{-\gamma}. \tag{21}$$

Lemma 1. *If for almost all $u \in E^n$ exists the limit*

$$\lim_{N \to \infty} \hat{K}_N(u) = h(u), \tag{22}$$

$$|\hat{K}_N(u)| \leqq M|u|^{-\gamma}, \text{ for all } N \text{ and almost all } u, \tag{23}$$

$$0 \leqq \gamma < n = \dim E^n, \quad p = 2n/(n+\gamma), \quad p' = 2n/(n-\gamma), \tag{24}$$

then: a) $\|H_N f\|_{p'} \leqq M c_p \|f\|_p$, where c_p does not depend of N; *b) for each $f \in L^p(E^n)$ the functions $H_N f$ converge in $L^{p'}$ to a limit $Hf \in L^{p'}(E^n)$ and $\|Hf\|_{p'} \leqq M c_p \|f\|_p$. c) $Hf = \mathrm{F}^*(\hat{f} h)$. Hence Hf is of type (p, p') = type C.* (See figure 1).

Proof. Consider first the case $f \in L_0$ so that all Fourier transforms will be well-defined. By (22) the sequence $|\hat{f}(u)\, h(u) - \hat{f}(u)\, \hat{K}_N(u)|^p$ converges to zero for almost all u, by (23) this sequence is dominated by the function $G(u) = (2M)^p\, |\hat{f}(u)|^p\, |u|^{-\gamma p} = (2M)^p\, |\hat{f}(u)|^p\, |u|^{n(p-2)}$, and by (20) G is integrable with integral $\leqq (2M)^p\, (c_p \|f\|_p)^p$. Therefore we may integrate term by term and we obtain that $\hat{f}(u)\, \hat{K}_N(u)$ converges in $L^{p'}$ to $\hat{f}(u)\, h(u)$ and $\|\hat{f} h\|_p \leqq (2Mc_p)\, \|f\|_p$. From (19) follows then that $f * K_N$ converges in $L^{p'}$ to $F^*(\hat{f} h) = Hf$ and $\|Hf\|_{p'} \leqq \|\hat{f} h\|_p \leqq c_p M\, \|f\|_p$. This proves the theorem for the case $f \in L_0$. If $f \in L^p$, we take $f_m \in L_0$ such that $\|f - f_m\|_p \to 0$. Then $H_N f_m$ converges in $L^{p'}$ to a function G_N, as $m \to \infty$. Since $\|G_N\|_{p'} = \lim \|H_N f_m\|_{p'} \leqq c_p M\, \|f\|_p$, and since $\|H_N f - H_N f_m\|_p \leqq \|K_N\|_1\, \|f - f_m\|_p \to 0$, $H_N f_m$ converges in L^p to $H_N f$, so that we must have $H_N f = G_N$. Hence $\|H_N f\|_{p'} \leqq c_p M\, \|f\|_p$. Thus $\|H_N\| \leqq c_p M$ and $H_N f$ converges on a dense subset L_0 of L^p, and by a known theorem this proves the lemma.

Remark 1: If $|k_i'(x)| \leqq k_i(x)$ and if the k_i satisfy (22), (23), then a), b), of lemma 1 are true for the operator $f * \Sigma k_i'$.

Lemma 2. *Let* $0 < \gamma < 1$. *If* $\|k_i\|_1 \leqq 2^{i\gamma} M$, $\|k_i\|_{(1,1)} \leqq 2^{-(1-\gamma)i} M$, *and if* $p = 2n/(n+\gamma)$, *then properties a), b) and c) of lemma 1 are true.*

Proof. From the hypothesis and (21) we obtain

$$|\hat{k}_i(u)| \leqq c\, M\, 2^{-(1-\gamma)i}\, |u|^{-1}, \quad \text{and} \quad |\hat{k}_i(u)| \leqq M\, 2^{i\gamma}. \tag{25}$$

Let us fix u and let r be such that $2^r \leqq |u|^{-1} < 2^{r+1}$. Then, using (25) and $1 - \gamma > 0$, we have

$$\begin{aligned} \Sigma_{i>r} |\hat{k}_i(u)| &\leqq c\, M\, |u|^{-1}\, \Sigma_{i>r}\, 2^{-(1-\gamma)i} \leqq \\ &\leqq cM\, 2^{-r(1-\gamma)}\, |u|^{-1}\, \Sigma_{i>0}\, 2^{-i(1-\gamma)} \leqq \\ &\leqq c'\, M (2^{-r}\, |u|^{-1})^{(1-\gamma)}\, |u|^{-\gamma} \leqq c''\, M\, |u|^{-\gamma}; \end{aligned}$$

$$\Sigma_{i \leq r} |\hat{k}_i(u)| \leqq M\, \Sigma_{i \leq r}\, 2^{i\gamma} \leqq cM\, |u|^{-\gamma}.$$

Hence (23) is true, and also (22) is true, since $\Sigma \hat{k}_i(u)$ converges absolutely for each $u \neq 0$.

Lemma 3. *Let* $1 \leqq \gamma < n$ *and let* $k_j^* = |k_j|^a$, $a = (n-1/2)/(n-\gamma)$. If $\|k_j^*\|_1 \leqq 2^{j/2} M$, $\|k_j^*\|_{(1,1)} \leqq 2^{-j/2} M$, *and if* k_j *vanishes outside of the set* $2^j \leqq |x| < 2^{j+1}$, *then a) and b) of Lemma 1 hold for* $p = 2n/(n+\gamma)$.

Proof. For any complex number z and for any fixed N we define the operator

$$H^*_z f = f * [\Sigma^N{}_{j=-N} |k_j^*(x)|^z]. \tag{26}$$

For any real number u we have $||k_j^*(x)|^{iu}| \leqq 1$ and by hypothesis $k_l^* k_j^* = 0$ if $l \neq j$, so that $|\Sigma\, |k_j^*(x)|^{iu}| \leqq 1$; hence

$$\|H^*_{ui} f\|_\infty \leqq \|f\|_1 \; (i\,u = (-1)^{1/2}\, u). \tag{27}$$

On the other hand, $||k_j^*(x)|^{1+iu}| \leqq k_j^*(x)$ and the k_j^* satisfy the hypothesis of lemma 2 with $\gamma = 1/2$. Hence by lemma 2 and Remark 1,

$$\|H^*_{1+iu} f\|_{r'} \leqq cM\, \|f\|_r,\, r = 2\,n/(n+1/2), \tag{28}$$

where c is independent of N. But it is easy to see that $H^*_z f$ is a bounded analytic operator in z, for $0 \leqq Rz \leqq 1$, hence from (27), (28) and from the convexity theorem for analytic operators it follows that H^*_t is of type (p, s) for $1/p = (1-t)/1 + t/r$, $1/s = (1-t)/_\infty + t/r', 0 < t < 1$. Letting $t = (n-\gamma)/n - 1/2)$ we obtain that H^*_t is of type (p, p') and $\|H_t^* f\|_{p'} \leqq cM\, \|f\|_p, c$ independent of N. Hence the operator $H'_N f = f * \Sigma^N{}_{-N} |k_i|$ satisfies $\|H'_N\, f\|_{p'} \leqq cM\, \|f\|_p$ and this proves part a) of the lemma. Part b) es easily deduced from the last inequality, observing that $|H_N f - Hf|$ is dominated by $|H'_N f - H' f|$, that $|H_N f| \leqq H'_N(|f|)$, and that $H'_N(|f|)$, is non-decreasing.

Consider now a subspace $E^m \subset E^n$, and let $E^n = E^m \times E^{n-m}$, $E^m = \{x_1\}, E^{n-m} = \{x_2\}, E^n = \{x\}, x = (x_1, x_2)$. If $K(x)$ is defined on E^n then the operator

$$Tf(x) = F(x) = f * K = \int_{E^n} f(t)\, K(x-t)\, dt \tag{29}$$

assigns to functions $f(x)$ defined on E^n functions $F(x)$ defined on E^n. If in (29) we let x vary only in E^m, that is if we consider the restriction $G(x_1)=F(x_1, 0)$ of F to E^m, then we obtain a second operator $T_1 f=G$ which assigns to functions f defined on E^n functions G defined on E^m. Finally if $g(x_1)$ is defined on E^m, we have a third operator

$$T_2\, g(x)=F_1(x)=\int_{E^m} g(t_1)\, K(x-t_1)\, dt_1 \tag{29a}$$

which assigns to functions g defined on E^m functions F_1 defined on E^n. Thus the convolution with K defines three diferent operators T, T_1, T_2. It was already observed in C) of Section 1, that if T_1 is of type $(L^p(E^n), L^s(E^m))$ then T_2 is of type $(L^{s'}(E^m), L^{p'}(E^n))$, provided $p>1, s\geqq 1$.

If we fix x_2 and consider $K(x_1, x_2)$ as a function of x_1, then the Fourier transform of this function will be denoted by $\mathrm{F}_1 K(u_1, x_2)$. Similarly is defined $\mathrm{F}_2 K(x_1, u_2)$. If K is in L^2 then we have

$$\mathrm{F}\, K(u)=\hat{K}(u_1, u_2)=\mathrm{F}_2\,[\mathrm{F}_1\, K(u_1,.)]\,(u_2)=\mathrm{F}_1\,\mathrm{F}_2\, K, \tag{30}$$

so that $\mathrm{F}_1 K(u_1, x_2)=\mathrm{F}^*{}_2\,\hat{K}(u_1, x_2)$. If $F=Tf, G=T_1 f, F_1=T_2\, g$ are the three above defined operators, then $\hat{F}(u)=\hat{f}\,(u)\,\hat{K}(u)$, but instead we have:

$$\hat{G}(u_1)=\mathrm{F}_1\, G=\int_{E^{n-m}} \mathrm{F}_1 f(u_1, x_2)\,\mathrm{F}_1\, K(u_1, -x_2)\, dx_2, \tag{31}$$

$$\hat{F}_1(u)=\mathrm{F}\, F_1(u_1, u_2)=\hat{g}\,(u_1)\,\hat{K}\,(u_1, u_2), \tag{31a}$$

$$|\mathrm{F}_1\, G(u_1)|\leqq\left\{\int_{E^{n-m}} |\mathrm{F}_1 f(u_1, x_2)|^p\, dx_2\right\}^{1/p} \cdot \left\{\int_{E^{n-m}} |\mathrm{F}_1\, K(u_1, x_2)|^{p'}\, dx_2\right\}^{1/p'}. \tag{31b}$$

Lemma 4. *If $0\leqq\gamma<n$, if $p=(n+m)/(m+\gamma)$, and if K satisfies*

$$\left\{\int_{E^{n-m}} |\mathrm{F}_1\, K(u_1, x_2)|^{p'}\, dx_2\right\}^{1/p'}<M\,|u_1|^{(n-m-p\gamma)/p}, \tag{32}$$

then $T_1 = f * K$ *is of type* $(L^p(E^n), L^{p'}(E^m))$ *and* $\|T_1 f\|_{p'}^{(m)} \leqq cM \|f\|_p^{(n)}$

Proof. Since $n - m - p\gamma = m(p-2)$, taking in account (32), (31b), (20) and (19),

$$\|T_1 f\|_{p'} = \|G\|_{p'} \leqq \|\widehat{G}\|_p \leqq M \left\{ \int_{E^m} |u_1|^{m(p-2)} \left[\int_{E^{n-m}} |\mathrm{F}_1 f(u_1, x_2)|^p \, dx_2 \right] du_1 \right\}^{1/p}$$

$$\leqq Mc_p \left\{ \int_{E^{n-m}} \left[\int_{E^m} |f(x_1, x_2)|^p \, dx_1 \right] dx_2 \right\}^{1/p} = Mc_p \|f\|_p.$$

Similarly is proved

Lemma 5. *If* $0 \leqq \gamma < n$, $p = (n+m)/(n+\gamma)$, *and if*

$$\left\{ \int_{E^{n-m}} |\widehat{K}(u_1, u_2)|^p \, du_2 \right\}^{1/p} < M \, |u_1|^{(n-m-p\gamma)/p}, \qquad (32a)$$

then $T_2 g = g * K$ *is of type* $(L^p(E^m), L^{p'}(E^n))$.

We denote by $\|g\|_{(r,s)}$ the least constant M such that $\|g(x+h) - g(x)\|_r \leqq M|h|^s$. With this notation we have,

Lemma 6. *Let* $E^n = E^m \times E^{n-m}$, $m < n < m + 2\gamma$, *and let* $p = (n+m)/(m+\gamma)$, $2 < p' < 2 + 1/m$. *If* $k_i(x)$ *vanishes outside of* $|x| < 2^i$, *and if*

$$\|k_i\|_{p'} \leqq 2^{-im/p'} c, \; \|k_i\|_{(p', 1/p')} \leqq 2^{-i(m+1)/p'} c, \qquad (33)$$

then $Hf = f * \Sigma k_i$ *is of type* $(L^p(E^n), L^{p'}(E^m))$.

Proof. Let us fix N. By lemma 4 it is enough to prove that $\mathrm{K} = \mathrm{K}_N$ satisfies condition (32). Let us fix u_1, and let $g(x_2) \in L^p(E^{n-m})$ be such that $\|g\|_p = 1$ and

$$J = \left\{ \int_{E^{n-m}} |\mathrm{F}_1 K_N(u_1, x_2)|^{p'} \, dx_2 \right\}^{1/p'} =$$

$$= \int_{E^{n-m}} \mathrm{F}_1 K_N(u_1, x_2) \, g(x_2) \, dx_2.$$

Then

$$J = \mathrm{F}_1 \left\{ \int_{E^{n-m}} K_N(x_1, x_2)\, g(x_2)\, dx_2 \right\} = \Sigma^N{}_{i=-N} \hat{L}_i(u_1), \qquad (34)$$

where

$$|L_i(x_1)| = \left| \int_{E^{n-m}} k_i(x_1\, x_2)\, g(x_2)\, dx_2 \right| \leqq$$

$$\leqq \|g\|_{p^{(n-m)}} \|k_i\|_{p'^{(n-m)}} = \|k_i\|_{p'^{(n-m)}}.$$

Since k_i vanishes for $|x_1| > 2^i$, we deduce from the above inequality and (33):

$$\int_{E^m} |L^i(x_1)|\, dx_1 \leqq \int_{|x_1|<2^i} \left\{ \int_{E^{n-m}} |k_i(x_1, x_2)|^{p'} dx_2 \right\}^{1/p'} dx_1 \leqq$$

$$2^{im/p} \left\{ \int_{E^m} \int_{E^{n-m}} |k_i(x_1, x_2)|^{p'} dx_1\, dx_2 \right\}^{1/p'} \leqq$$

$$\leqq 2^{im/p}\, c\, 2^{-im/p'} = 2^{\mu}\, c, \quad (34a)$$

where $\mu = m(2/p - 1) > 0$. Similarly, since $|L_i(x_1 + h) - L_i(x_1)| \leqq \|g\|_p \|k_i(x_1 + h, x_2) - k_i(x_1, x_2)\|_{p'^{(n-m)}}$, we obtain

$$\int_{E^m} |L_i(x_1 + h) - L_i(x_1)|\, dx_1 \leqq$$

$$\int_{|x_1|<2^i} \left\{ \int_{E^{n-m}} |k_i(x_1 + h, x_2) - k_i(x_1, x_2)|^{p'} dx_2 \right\}^{1/p'} dx_1$$

$$\leqq 2^{im/p} \|k_i\|_{(p',\, 1/p')} |h|^{1/p'} \leqq 2\, c\, 2^{i(\mu - 1/p')} |h|^{1/p'}. \qquad (34b)$$

Let r be such that $2^r \leqq |u_1|^{-1} < 2^{r+1}$, then since $1/p' - \mu > 0$, we obtain, using (34b) and (21), that

$$\Sigma_{i>r} |\hat{L}_i(u_1)| \leqq 2\, c\, \Sigma_{i>r} 2^{-i(1/p' - \mu)} |u_1|^{-1/p'} \leqq 2\, c |u_1|^{-\mu}.$$

Similarly from (34a) we obtain

$$\Sigma_{i<r} |\hat{L}_i(u_1)| \leqq 2\, c\, \Sigma_{i \leq r} 2^{\mu i} \leqq c\, |u|^{-\mu}.$$

Hence $\Sigma^N_{-N} |\hat{L}_i(u_1)| \leqq c|u|^{-\mu}$, $\mu = (p\gamma - n + m)/p$, so that in virtue of (34) K_N satisfies condition (32).

Since the condition $2 < p' < 2 + 1/m$ of lemma 6 implies $n - (n+m)/2 < \gamma < n - m(n+m)/(2m+1)$, lemma 6 does not apply to all $\gamma < n$. However:

L e m m a 7. *Let* $0 < \gamma < n$, $m \leqq n \leqq m + 2\gamma$, $p = (n+m)/(m+\gamma)$ *and let* $k_i^*(x) = |k_i(x)|^a$, $a = (n+m)/2(n-\gamma)$. *If* $k_i(x)$ *vanishes outside of the set* $2^i \leqq |x| < 2^{i+1}$ *and if*

$$\|k_i^*\|_2 \leqq 2^{-im/2}\, c;\ \|k_i^*\|_{(2,1/2)} \leqq 2^{-i(m+1)/2}\, c, \tag{35}$$

then $Hf = \Sigma f * k_i$ *is of type* $(L^p(E^n), L^{p'}(E^m))$. *Thus,* Hf *is of type* C' (see figure 1).

Proof. Using the convexity theorem for analytic operators, lemma 7 is deduced from lemma 6 in the same way as lemma 3 was deduced from lemma 2.

4. P s e u d o t y p e s $(1, r; d)$. In this section we consider a generalization of Riesz's convexity thearem in the case $p_2 = 1 \leqq s_2$, $1 \leqq p_1 \leqq s_1$. For $p_1 = s_1$, $p_2 = s_2 = 1$, this generalization reduces to one given in [2] (p. 77). In this section we consider only operators Tf defined on $L_0(E^n)$, so that f is defined on E^n and Tf on E^m. We shall use the following notations:

$S(f) =$ the support of $f =$ the set of points x where $f \neq 0$,

$m(f) =$ the minimum of $|f|$ on $S(f)$,

$$\mu(f; Q) = |Q|^{-1} \int_Q |f|\, dx$$

In the case $p_2 = 1$, Riesz's theorem says that if (i) T is of type $(1, r)$, and (ii) T is of type (p, s), then T is of type P for any interior point P of $P_1 P_2$, where $P_1 = (1/p, 1/s)$, $P_2 = (1, 1/r)$. In the theorem of Marcinkiewicz, type es replaced by weak type in both conditions (i), (ii). We shall now replace type by weak type only in (ii), and (i) will be replaced by another weaker condition, as follows. Condition (i) says that

$$\left\{ \int_{E^m} |Tf(x)|^r\, dx \right\}^{1/r} \leqq c\, \|f\|_1 \tag{36}$$

We consider the following weaker condition: for each function $f \in L_0$ there is a function h and a set $F \subset E^m$ such that

$$\left\{\int_{E^m-F} |T(f-h)(x)|^r dx\right\}^{1/r} \leqq c\,\|f\|_1 \qquad (37)$$

If $h=0$, and $F=0$, then (37) reduces to (36). Since (37) is always satisfied if $h=f$ or if $F=E^m$, we must impose further conditions upon h and F. In the case of Hilbert transforms and other singular integrals $Tf(x)$ is bad if x belongs to the support of f. This suggests to take $F=S(f)$. Since $|S(f)| \leqq \|f\|_1/m(f)$, it is natural to impose on F the more general condition $|F| \leqq c\,\|f\|_1/m(f)$, or still more generally, the condition

$$|F| \leqq [c\,\|f\|_1/m(f)]^{r/d}, d>0. \qquad (38)$$

Similarly the case of singular integrals suggests to impose on h the conditions:

$$|h(x)| \leqq c\,m(f), \text{ and } \|h\|_1 \leqq c\,\|f\|_1. \qquad (39)$$

The condition $|h| \leqq c\,m(f)$ is a very strict one. For this reason, instead of (38) and (39) we shall consider also conditions of the following type:

$$|F| \leqq c\,|Q|^{r/d}, d>0, Q \supset S(f), \qquad (40)$$

$$h(x)=0 \text{ if } x \in E^n - Q, \text{ and } |h(x)| \leqq$$
$$\leqq c\,\mu(f; Q) = c|Q|^{-1}\,\|f\|_1. \qquad (41)$$

Definition 1: We say that the operator T, defined on $L_0(E^n)$, is of *pseudo type* * $(1, r; d)$ with constant c, if for each $f \in L_0$ there is a set $F \subset E^m$ and a funtion $h \in L_0(E^n)$ such that (37), (38) and (39) are satisfied.

Definition 2: We say that T is of *pseudo type* $(1, r; d)$, or more precisely of pseudo type $((L^1(E^n), L^r(E^m); d)$, with constant c, if for each $f \in L_0(E^n)$ and for each cube $Q \supset S(f)$, there is a set $F \subset E^m$ and a function $h \in L_0(E^n)$ such that condition (37), (40) and (41) are satisfied.

We have then the following generalizations of Riesz's convexity theorem.

Theorem 1. *Let* $1 \leqq p \leqq s, 1 \leqq r, d = (s-r)ps^{-1}/(p-1) > 0$, *and let* T *be a linear (or sublinear) operator defined on* $L_0(E^n)$. *If* T *is of weak type* (p, s) *and of pseudo type** $(1, r; d)$, *then* T *is of weak type* $(1, r)$, *and therefore of type* P *for each interior point of* $P_1 P_2, P_1 = (1/p, 1/s), P_2 = (1, 1/r)$.

Remark: The constant d has the following meaning: If ϑ is the argument of the vector $0P_2$ and φ the argument of $P_1 P_2$, then $d = \operatorname{tg} \varphi / \operatorname{tg} \vartheta$. Therefore the condition $d > 0$ says that theorem 1 is true if the point P_2 is «above» P_1 in the square of types.

Theorem 2. *If* $d \leqq r$ *and if* T *is of pseudo type* $(1, r; d)$, *then* T *is also of pseudo type** $(1, r; d)$.

Theorem 2a. *Let* $d = (s-r)\, ps^{-1}/(p-1), d \leqq r, 1 \leqq p \leqq s, 1 \leqq r$. *If* T *is of weak type* (p, s) *and of pseudo type* $(1, r; d)$, *then* T *is of weak type* $(1, r)$, *and therefore of type* P *for each interior point of* $P_1 P_2$.

The proofs of these theorems are based on the following two lemmas.

Lemma 8. *Assume that for any* $f \in L_0, \|f\|_p \leqq 1$, *is true that*

$$D(|Tf|\,;\,(m(f))^{1/d}) \leqq c(\|f\|_p)^l/(m(f))^{r/d}, \tag{42}$$

where c, l, d *are positive constants independent of* f. *Then for any* $f \in L_0, \|f\|_p \leqq 1$, *and any* a, $0 < a < m(f)$, *it is true that*

$$D(|Tf|\,;\,a^{1/d}) \leqq 2^r c(\|f\|_p)^l/(a)^{r/d}. \tag{43}$$

Proof. We may assume $\|f\|_p < 1$. Let S, S' be two sets such that $S \subset S(f), S' \cap S(f) = 0, |S| < \varepsilon, |S'| < \varepsilon$, and let g be defined by: $g = f$ in $E^n - (S \cup S')$, $g = a/2^d < m(f)$ in $S \cup S'$. If ε is small enough we have $\|g\|_p \leqq 1$, and since $m(g) = a/2^d$, $m(f-g) \leqq a/2^d$, we have by hypothesis,

$$D(|Tf|\,;a^{1/d}) \leqq D(|Tg|\,;a^{1/d}/2) + D(|T(f-g)|\,;a^{1/d}/2) \leqq$$
$$\leqq D(|Tg|\,;a^{1/d}/2) + D(|T(f-g)|\,(m(f-g))^{1/d}) \leqq$$
$$\leqq 2^r c(\|f\|_p)^l/a^{r/d} + c(\|f-g\|_p)^l/(m(f-g))^{r/d}.$$

Let ε tend to zero. Then $\|f-g\|_p$ tends to zero, and since $m(f-g) = \inf\{m(f) - a/2^d;\, a/2^d\} =$ fixed positive number, we obtain (43).

Lemma 9. *Let* $d = (s-r)\,ps^{-1}/(p-1), 1 \leqq p \leqq s, 1 \leqq r$. *If* T *satisfies condition* (42) *and if* T *is of weak type* (p,s) *then* T *is of weak type* $(1,r)$.

Proof. We have to prove that $D(|Tf|\,;a) \leqq M(\|f\|_1/a)^r$, and we may assume $\|f\|_1 = 1$. Let $g(x) = f(x)$ if $|f(x)| \geqq a^d$ and zero otherwise, so that $h(x) = f(x) - g(x)$ is $f(x)$ if $|f(x| < a^d$ and zero otherwise. From $|Tf| \leqq |Tg| + |Th|$ we have

$$D(|Tf|\,;a) \leqq D(|Tg|\,;a/2) + D(|Th|\,;a/2), \tag{44}$$

and using the hypothesis and lemma 8, and that $m(g) \geqq a^d$, we have

$$D(|Tg|\,;a/2) \leqq c_1(\|g\|_p)^l/a^r \leqq c_1(\|f\|_p)^l/a^r = c_1/a^r,$$
$$D(|Th|\,;a/2) \leqq (c\|h\|_p/a)^s =$$
$$= c\left\{\int_{E_n} (|h(x)|\,a^{-d})^{p-1}\,|h(x)|\,a^{-p-d\,(1-p)}\,dx\right\}^{s/p} \leqq$$
$$c\left\{\int_{E_n} |h(x)|\,a^{-p+d(p-1)}\,dx\right\}^{s/p} \leqq c(\|f\|_1)^{s/p}\,a^{-r} = c\,a^{-r}.$$

Hence $D(|Tf|\,;a) \leqq (c+c_1)\,a^{-r}$.

Proof of Theorem 1. Given f let us put $m(f) = a$, $b = d^{-1}$. By lemma 9 it is sufficient to prove that $D(|Tf|\,;a^b) \leqq c(\|f\|_1)^l$ a^{-br}, and we may assume $\|f\|_1 \leqq 1$. By hypothesis, (37), (38) and (39) are true. Let h be the function of (39) and let $g = f - h$, so that (44) is true. Using (39) and that T is of weak type (p,s), we have

$$D(|Th|\,;a^b/2) \leqq (c\,a^{-b}\,\|h\|_p)^s = c\left[\int_{E_n} |h(x)|^p\,a^{-bp}\,dx\right]^{s/p} =$$

$$c\left[\int_{E^n}(|h(x)|\,a^{-1})^{p-1}\,(|h(x)|\,a^{-1+p-pb})\,dx\right]^{s/p}$$

$$\leqq c_1\left[\int_{E^n}|h(x)|\,a^{-rpb/s}\,dx\right]^{s/p}\leqq$$

$$\leqq c_1\,a^{-br}\,(\|f\|_1)^{s/p}. \tag{45}$$

If
$$G=\{x;\ |Tg(x)|>a^b/2\}$$

we have

$$D(|Tg|\,;\,a^b/2)\leqq|G\cap(E^m-F)|+|F|.$$

By (38),

$$|F|\leqq c(\|f\|_1)^{br}\,a^{-br},$$

and by (37) we have

$$|G\cap(E^m-F)|\,a^{br}\leqq 2^r\int_{E^m-F}|Tg|^r\,dx\leqq(2\,c\|f\|_1)^r.$$

Therefore

$$D(|Tg|\,;\,a^b/2)\leqq c\,(\|f\|_1)^{br}\,a^{-br}+c_1\,(\|f\|_1)^r\,a^{-b^r}. \tag{46}$$

From (44), (45) and (46) we obtain the desired inequality $D(|Tf|\,a^b)\leqq c\,(\|f\|_1)^l a^{-br}$ with $l=\inf(r, br, s/p)$.

Proof of Theorem 2. The proof of this theorem is quite similar to that of therem 4, p. 77, of [2], so that we only sketch briefly the main steps. By hypothesis, given a cube $Q\supset S(f)$ there is a function h and a set F satisfying (37), (40) and (41). For any point $x_i\in S(f)$ there is a cube $Q_i=Q(x_i)$, with center in x_i, such that

$$\mu(Q_i;\,f)=3/4\,m(f)=a, \tag{47}$$

$$|Q_i|\leqq 4/3\,(m(f))^{-1}\int_{Q_i}|f|\,dx. \tag{47a}$$

By hypothesis, to each set $E_i\subset Q_i$ there correspond a set $F_i\subset E^m$ and a function h_i such that

$$\left\{\int_{Em-Fi} |T(\varphi_i f - h_i)|^r dx\right\}^{1/r} \leqq c\int_{Ei} |f|\, dx = c\|f\,\varphi_i\|_1, \quad (48)$$

$$|F_i| \leqq c\,|Q_i|^{r/d}, h_i = 0 \text{ in } E^n - Q_i, |h_i| \leqq c\,\mu(Q_i, f\,\varphi_i) \leqq c\,a\,3/4, \quad (48a)$$

where φ_i is the characteristic function of E_i. As in [2] we shall see that $S(f)$ may be covered by a finite number of these Q_i, and E_i, in such a way that any point of E belongs to at most 4^n cubes Q_i and the E_i are disjoint. Writing $h = \Sigma\, h_i$, $F = \bigcup F_i$, we have $f = \Sigma\, f\,\varphi_i$ and

$$\left\{\int_{Em-Fi} |T(f-h)|^r dx\right\}^{1/r}$$

$$\leqq \Sigma_i \left\{\int_{Em-Fi} |T(f\,\varphi_i - h_i))|^r dx\right\}^{1/r} \leqq$$

$$c\,\Sigma_i \int_{Ei} |f|\, dx \leqq c\|f\|_1 \quad (49)$$

Taking in account that $r/d \geqq 1$, that each point belongs to at most 4^n cubes and (47a), we obtain

$$|F| \leqq \Sigma\, |F_i| \leqq c\,\Sigma\,|Q_i|^{r/d} \leqq c\,(\Sigma\,|Q_i|)^{r/d} \leqq$$

$$c\left(a^{-1}\,\Sigma_i \int_{Qi} |f|\, dx\right)^{r/d} \leqq$$

$$c\left(a^{-1}\,4^n \int_{UQi} |f|\, dx\right)^{r/d} \leqq c_1(\|f\|_1)^{r/d}\,(m(f))^{-r/d}.$$

Similarly we obtain that $\|h\|_1 \leqq c\,\|f\|_1$, so that F and h satisfy conditions (38) and (39), as well as condition (37) = (49), hence T is of pseudo type $*\,(1, r;\, d)$.

Proof of Theorem 2a. This is a direct consequence of theorems 1 and 2.

Finally, lemmas 2 and 6 have the following correspondents for pseudo types:

L e m m a 10. *Let* $Hf = \Sigma_{-\infty}^{\infty} f * k_i$, $k_i \in L^1(E^n)$, *and* $r \geqq 1$. *If* $\|k_i\|_{(r,1/r)} \leqq 2^{-i/r}\, c, i = \pm 1, \pm 2, \ldots;\ k_i(x) = 0$ *in* $|x| > 2^{i+1}$, *then*:

a) Hf is of pseudo type $(L^1(E^n), L^r(E^n); r)$. b) If $E^m \subset E^n$, *then Hf is of pseudo type $(L^1(E^m), L^r(E^n); mr/n)$.*

Proof. Let $f \in L_0(E^n)$ and let $Q \supset S(f)$; we may assume that center of Q is the origin 0. Let j be an integer such that $|x| < 2^j$ implies $x \in Q$, and such that $|x| > n^{1/2}\, 2^j$ implies $x \in E^n - Q$, so that $x \in E^n - 2\, n^{1/2}\, Q$ implies $|x| > 2^{j+1}\, n^{1/2}$ (mQ is the cube of same center as Q and m times the side. Let $h(t)$ be defined by

$$h(x) = \mu(Q, f) \text{ if } x \in Q,\ h(x) = 0 \text{ if } x \in E^n - Q, \tag{50}$$

so that h satisfies condition (41). If $g = f - h$, we have

$$\int_{E^n} g = \int_Q g = 0, \qquad \|g\|_1 \leqq 2\|f\|_1. \tag{51}$$

Let $F = 2n^{1/2}\, Q$. F satisfies condition (40) with $d = r$. Since $g = 0$ in $E^n - Q$, and $k_i = 0$ in $|x| > 2^{i+1}$, we have that if $x \in E^n - F$ and $i < j$ then

$$g * k_i(x) = \int_Q g(y)\, k_i(x - y)\, dy = 0\,.\ (x \in E^n - F). \tag{52}$$

Hence for any $x \in E^n - F$, $Hg(x) = H(f - h)(x) = \Sigma_{i>j}\, g * k_i(x)$, and by (51),

$$Hg(x) = \Sigma_{i>j} \int_Q g(y)\, [k_i(x - y) - k_i(x)]\, dy. \tag{53}$$

Applying to (53) the integral inequality of Minkowski and using the hypothesis, we have

$$\left\{\int_{E^n - F} |Hg(x)|^r\, dx\right\}^{1/r} \leqq$$

$$\leqq \Sigma_{i>j} \int_Q |g(t)| \left\{\int_{E^n - F} |k_i(x - t) - k_i(x)|^r\, dx\right\}^{1/r} dt \leqq$$

$$\Sigma_{i>j} \int_Q |g(t)|\, 2^{-i/r}\, c\, |t|^{1/r}\, dt \leqq c\, \Sigma_{i>j}\, 2^{-i/r}\, n^{r/2}\, 2^{j/r} \|g\|_1$$

$$\leqq c_1\, 2^{j/r} \|g\|_1\, \Sigma_{i>j}\, 2^{-i/r} \leqq c\|g\|_1 \leqq c\|f\|_1.$$

Hence condition (37) is also satisfied. This proves part a). Part b) is proved in the same way.

If k is defined on $E^n = E^m \times E^{n-m} = \{(x_1, x_2)\}$, we say that $k \in Lip(p, r, E^m)$ if for any $h = (h_1, h_2) \in E^n$,

$$\left\{\int_{E^m} |k(x_1 + h_1, x_2 + h_2) - k(x_1, x_2)|^p \, dx_1\right\}^{1/p} \leqq M|h|^{1/r},$$

and the least value of M denoted by $\|k\|_{(p,r)}^{(m)}$.

Then the same proof give us.

Lemma 10a. *Let* $Hf = \Sigma f * k_i, k_i \in L^1(E^n)$, *and let* $E^m \subset E^n, r \geqq 1$. If $\|k_i\|_{(r,1/r)}^{(m)} \leqq 2^{-i/r} c, i = \pm 1, \pm 2, \ldots$, *then* Hf *is of pseudo type* $(L^1(E^n), L^r(E^m); nr/m)$.

5. Type properties of $H_{\gamma n}$. Let $k(x) \in L^1(E^n)$ be a fixed kernel defined on E^n, let $0 < \gamma \leqq n$, and let

$$k_i(x) = 2^{-i(n-\gamma)} k(2^{-i} x), \quad i = 0, \pm 1, \pm 2, \ldots \tag{54}$$

$$K_{\gamma n}(x) = \Sigma_{i=-\infty}^{\infty} k_i(x) = \Sigma_{-\infty}^{\infty} 2^{-i(n-\gamma)} k(2^{-i} x), \tag{54a}$$

$$H_{\gamma n} f = f * K_{\gamma n} = \Sigma f * k_i, \tag{55}$$

so that

$$K_{\gamma n} f(2^j x) = 2^{-j(n-\gamma)} K_{\gamma n} f(x). \tag{55a}$$

We assume that the *generating kernel* k satisfies the following conditions:

α) $k \geqq 0$, and $k = 0$ outside of the set $1 \leqq |x| < 2$.

β) $|k(x)|^a \in L^1(E^n) \cap Lip(1, 1, E^n)$ for $a = 1, a = n/(n-\gamma)$ and $a = (n - 1/2)/(n - \gamma)$.

In dealing with subspaces E^m we shall also assume the following conditions:

γ) If $E^m \subset E^n$ then $|k(x)|^a \in L^1(E^n) \cap Lip(1, 1, E^n)$ for $a = (n + m)/(n - \gamma)$.

δ) If $E^m \subset E^n$ then $|k(x)|^a \in Lip(1, 1, E^m)$ for $a = m/(n-\gamma)$.

In most theorems below it will be enough to assume that $k(x)$ vanishes outside of a compact set and that $|k| \in L^1(E^n) \cap Lip(1, 1; E^n)$, but for sake of simplicity we shall stick to conditions α) - δ).

If $k(x) = |x|^{\gamma - n}$ for $1 \leqq |x| < 2$ and zero otherwise, then $K_{\gamma n}(x) = |x|^{\gamma - n}$ and $H_{\gamma n} f$ reduces to the classical potential operator $\widetilde{f}_{\gamma n}$. It is easy to see that in this case $k(x)$ is bounded and conditions α) - δ) are satisfied.

From (54) and α) - δ) we obtain the following properties of the *generated* kernels:

α_1) $k_i(x) = 0$ outside of $2^i \leqq |x| < 2^{i+1}$.

β_1) $\|k_i\|_1 \leqq 2^{i\gamma} c$, $\|k_i\|_{(1,1)} \leqq 2^{-(1-\gamma)i} c$.

β_2) $\|k\|_{(r,1/r)} \leqq 2^{-i/r} c$, for $r = n/(n - \gamma)$.

β_3) $\| |k|^a \|_1 \leqq 2^{i/2} c$, $\| |k|^a \|_{(1,1)} \leqq 2^{-i/2} c$, for $a = (n - 1/2)/(n - \gamma)$.

γ_1) $\|k_i\|_r \leqq 2^{-im/r}$ for $r = (n + m)/(n - \gamma)$, $m < n$.

γ_2) $\|k_i\|_{(r,1/r)} \leqq 2^{-i(m+1)/r}$ for $r = (n + m)/(n - \gamma)$, $m < n$.

δ_1) $\|k_i\|_{(r,1/r)}^{(m)} \leqq 2^{-i/r} c$ for $r = m/(n - \gamma)$, $m < n$.

From α_1) - δ_1), from lemmas 2, 3, 6, 7, 10, 10a), theorems 1, 2, and taking in account Proposition 1, we obtain the following theorems:

Theorem 3. *If $0 < \gamma \leqq n$, then $H_{\gamma n}$ is of type $(L^p(E^n), L^s(E^n))$, for any p, s satisfying*

$$1/p - 1/s = \gamma/n, \quad 1 < p < n/\gamma. \tag{56}$$

For $p = 1$, $H_{\gamma n}$ is of weak type $(L^1(E^n), L^{n/(n-\gamma)}(E^n))$. Thus, $H_{\gamma n}$ is of type P for every interior point P of AB, and of weak type at B.

Theorem 4. *If $0 < \gamma \leqq n$, $E^m \subset E^n$ and $m < n < m + \gamma$, then $H_{\gamma n}$ is of type $(L^p(E^n), L^s(E^m))$ for any p, s such that*

$$1/p - (m/n)\, 1/s = \gamma/n, \quad (1 < p < n/\gamma) \tag{57}$$

For $p=1, H_{\gamma n}$ *is of weak type* $(L^1(E^n), L^{m/(n-\gamma)}(E^m))$. *Thus* $H_{\gamma n}$ *is of type in* AB' *and weak type at* B'.

Theorem 5. *If* $0<\gamma\leqq n, m<n<m+\gamma$, *and if* $E^m\subset E^n$, *then* $H_{\gamma n}$ *is of type* $(L^p(E^m), L^s(E^n))$ *for*

$$1/p-(n/m)\,1/s=(\gamma+m-n)/m, \quad (n/(n-\gamma)<s<\infty). \tag{58}$$

For $p=1, H_{\gamma n}$ *is of weak type* $(L^1(E^m), L^{m/(m+\gamma-n)}(E^n))$.

In all cases the series (55) *is convergent for almost all* x *of* E^n *or of* E^m. *If* $1<p<n/\gamma$ *then this series converges also in* $L^s(E^n)$, *or in* $L^s(E^m)$. *For* $\gamma<1$, *we have in addition that* $H_{\gamma n}$ *is a multiplier transform:* $H_{\gamma n}f=F^*(h\hat{f})$, *where* $h(u)=\lim\Sigma^N_{i=-N}\hat{k}_i(u), N\to\infty$, *and* $|h(u)|\leqq c|u|^{-\gamma}$.

For the case of bounded domains we have the following theorems.

Theorem 6. *Let* $D_n\subset E^n$, $D_m\subset E^m$ *be bounded domains,* $E^m\subset E^n$, *and let* $0<\gamma<n$, $m\leqq n<m+\gamma$. *Then: a)* $H_{\gamma n}$ *is of type* $(L^p(D_n, L^s(D_m))$, *that is* $H_{\gamma n}$ *is a bounded transformation from* $L^p(D_n)$ *to* $L^s(D_m)$, *for any* p, s, *such that*

$$1/p-(m/n)\,(1/s)\leqq\gamma/n, \; (1<p<\infty, \; p\neq n/\gamma, 1\leqq s\leqq\infty), \tag{59}$$

and for

$$p=1, \; s<m/(n-\gamma), \text{ and } p=n/\gamma, \; s<\infty. \tag{59a}$$

b) *If*

$$1/p-(m/n)\,1/s<\gamma/n, \; 1<p<\infty, \; p\neq n/\gamma, 1\leqq s\leqq\infty, \tag{60}$$

then $H_{\gamma n}$ *is a completely continous operation from* $L^p(D_n)$ *to* $L^s(D_m)$.

Proof. a) Now $f(t)$ is defined on D_n, so that we may consider that $f(t)=0$ for $t\in E^n-D_n$, and $H_{\gamma n}f(x)$ is considered only for $x\in D_m$. Hence, since $k_i=0$ for $|t|>2\,2^i$, there is number $i_0=0$, so that $H_{\gamma n}$ is now of the form

$$H_{\gamma n}f=\Sigma_{i=-\infty}^{0}f*k_i=f*K'_{\gamma n} \tag{61}$$

where $$K'_{\gamma n} = \Sigma_{-\infty}{}^{0} k_i. \tag{61a}$$

Since $\gamma > 0$, we obtain from β_1) and β_2)

$$\|K'_{\gamma n}\|_1 \leqq \Sigma_{-\infty}{}^{0} \|k_i\|_1 \leqq c\, \Sigma_{-\infty}{}^{0} 2^{\gamma i} \leqq c_1;$$
$$\|K'_{\gamma n}\|_r \leqq c\, \Sigma_{-\infty}{}^{0} 2^{-i(n-\gamma)} 2^{in/r} \leqq c_2 \quad \text{if} \quad r < n/(n-\gamma).$$

Hence we have now

$$K'_{\gamma n} \in L^1(E^n);\ K'_{\gamma n} \in L^r(E^n) \ \text{ if } r \leqq n/(n-\gamma). \tag{62}$$

If p, s satisfy (57) then, by theorem 4, $H_{\gamma n}$ is of type $(L^p(D_n), L^s(D_m))$, and since D_m is a bounded domain, $H_{\gamma n}$ is of type (p, s') for all $s' \leqq s$. This proves part a) for $p < n/\gamma$. For $p > n/\gamma$ we have $\|H_{\gamma n} f\|_\infty \leqq \|f\|_p \|K'_{\gamma n}\|_{p'}$, and since $p' < n/(n-\gamma)$, we obtain from (62) that $H_{\gamma n}$ is of type (p, ∞) and hence also of type (p, s) for any $s \leqq \infty$.

b) Let p, s satisfy (60), so that $1/p - m/(ns) - \gamma/n = -d$, $d > 0$. For any N let $H_{\gamma n} = H_N + R_N$ where

$$H_N f = \Sigma^0{}_{-N} f * k_i,\ R_N f = f * K^N{}_{\gamma n} \quad , \tag{63}$$

$$K^N{}_{\gamma n}(x) = \Sigma_{-\infty}{}^{-N-1} k_i(x) = \Sigma_{-\infty}{}^{-N-1} 2^{N(n-\gamma)} k_{i+N}(2^N x) = 2^{N(n-\gamma)} K'_{\gamma n}(2^N x). \tag{63a}$$

Each k_i is a «good» kernel, hence $H_N f$ is a completely continous operation from $L^p(D_n)$ to $L^s(D_m)$, so that it is enough to prove that $\|R_N\| \to 0$. If $g(x)$ is defined by $g(x) = f(2^{-N} x)$, then from (63a) we obtain $R_N f(x) = 2^{-N\gamma}(g * K')(2^N x) = 2^{-N\gamma} H_{\gamma n} g(2^N x)$; hence by part a) already proped, we have

$$\|R_N f\|_s = 2^{-N\gamma} 2^{-Nm/s} \|H_{\gamma n} g\|_s \leqq 2^{-N\gamma - Nm/s} c\|g\|_p = 2^{-N\gamma - Nm/s + Nn/p} c\|f\|_p = 2^{-Nnd} c\|f\|_p,$$

and therefore

$$\|R_N\| \leqq |2^{-nd}|^N c \to 0 \text{ for } N \to \infty.$$

Similarly, for $p > n/\gamma$ we have:

Theorem 6a. *If* $D_n \subset E^n$, $D_m \subset E^m$, *are bounded domains,* $E^m \subset E^n$, $0<\gamma<n$, *and if* $n/\gamma<p<\infty$, *then* $H_{\gamma n}$ *is a completely continuos operation from* $L^p(D_n)$ *to* $C(D_m)$.

Remark. The classical kernel $k(x)=|x|^{\gamma-n}$ satisfies also the following condition: If $E^m=\{y\}$, $E^{n-m}=\{z\}$, $E^n=E^m \times E^{n-m}=\{(y,z)\}$, $m<n$, then there is a kernel $k^*(y)$ such that $k(x) \leqq c K_\delta^*(y)$, $x=(y,z)$, $\delta=m-n+\gamma>0$. If this condition is satisfied, then theorem 4 holds for any $m<n$ if $1<p<s$, $p<n/\gamma$ (cfr. [12]).

Notations: The rest of this section is devoted to capacity properties of $H_{\gamma n}$, and *only Borel* sets, functions and measures will be considered. μ will denote a non-negativa measure in E^n. If $\mu(E^n)=1$ then μ is a distribution; if $\mu(E^n-S)=0$ then μ is concentrated in S. If K, L, N are generating kernels (that is Borel functions satisfying conditions α), β)) then $K_{\gamma n}$, $L_{\gamma n}$, $N_{\gamma n}$, denote the corresponding kernels defined as in (54a). We shall write K_γ instead of $K_{\gamma n}$ and $\varphi(A)$ instead of $\varphi(x, A)$.

Given a function $N(x)$, we say that the set S is of *zero N-capacity* (cfr. [15]), and write $c(S,N)=0$, if for any distribution μ concentrated in S,

$$V(N,\mu)=\sup_{t \in E^n} \int_{E^n} |N(t-x)| \, d\mu(x)=\sup(|N| * \mu)=\infty .$$

The following properties are easily verified:

a) If $|M| \leqq |N|$ in $S' \subset S$ and $|N(x)| \leqq c$ in $S-S'$, then $c(S,M)=0$ implies $c(S,N)=0$.

b) $c(A_i,N)=0$, $A \subset \bigcup_1^\infty A_i$ imply $c(A,N)=0$.

c) If $f \in L^1(E^n)$ and $S=\{x;\ |N*f(x)|=\infty\}$, then $c(S,N)=0$.

d) If the generating kernel K satisfies $|K(x)| \leqq c$ and if $0<\gamma<\delta<n$, then $c(S,K_\delta)=0$ implies $c(S,K_\gamma)=0$ (because then $|K_\delta(x)| \leqq |K_\gamma(x)|$ in $|x| \leqq 1$, and $|K_\delta| \leqq c$ in $|x| \geqq 1$).

If $K_\gamma=|x|^{\gamma-n}$ then the notion of zero K_γ —capacity reduces to the classical notion of zero γ —capacity, and theorems 7, 7a, below reduce (taking in account properties c) and d)) to theorems of Du Plessis [10]. However, theorem 7b, is probably new even in the case $K_\gamma=|x|^{\gamma-n}$ (cfr. [15]).

Lemma 11. *Let K, N, L be three generating kernels such that $L=(K)^r . N$, and $|N|^2 \geqq c N, c>0$, let $V=V(N_\gamma, \mu)$ and let $\delta=\gamma-\gamma r/p'$. Then for any set A,*

$$\|\varphi(A)\,(K_{\gamma/p} * \mu)\|_r \leqq c_1 V^{1/r'} \|\varphi(A)\,(L_\delta * \mu)\|_1,$$

where $1<r<\infty$, and c_1 is a fixed constant.

Proof. By hypothesis, $2^{-i(n-\gamma/p)} K(2^{-i} x) \leqq$

$$c_2\, 2^{-i\gamma/p'} K(2^{-i} x)\, 2^{-i(n-\gamma)} N(2^{-i} x),$$

hence

$$\int K_{\gamma/p}(t-x)\, d\mu(x) \leqq c_2 \int K_{n-\gamma/p'}(t-x)\, N_\gamma(t-x)\, d\mu(x) \leqq$$

$$c_3 \left\{\int |K_{n-\gamma/p'}(t-x)|^r\, N_\gamma(t-x)\, d\mu(x)\right\}^{1/r} \{V\}^{1/r'}.$$

Observing that

$$|K_{n-\gamma/p'}(t)|^r = [K^r]_{n-\gamma r/p'}(t),\ [K^r]_{n-\gamma r/p'}(x)\, N_\gamma(x) = L_\delta(x)$$

and integrating over A the last inequality we obtain the desired inequality.

Lemma 12. *Let $1<r<2, |K|^2 \geqq c|K|, c>0, L=|K|^{r'/2}$, $N(x)=|K(x)|^{r'/(r'-2)}$, $W=V(N_\gamma, \mu)$. Then*

$$(\|K_{\gamma/r} * \mu\|_{r'})^{r'} \leqq W^{r'-2} (\|L_{\gamma/2} * \mu\|_2)^2.$$

Proof. Let $a=n+(r'-2)(\gamma-n)/r'$, $b=n+(\gamma-2n)/r'$. Then we have:

$$2^{-i(n-\gamma/r)} K(2^{-i} x) \leqq c\, 2^{i(\gamma-n)(r'-2)/r'} N(2^{-i} x)\, 2^{-i(2n-\gamma)/r'} K(2^{-i} x),$$

$K_{\gamma/r}(x) \leqq c\, N_a(x)\, K_l(x)$, and hence

$$\left|\int K_{\gamma/r}(t-x)\, d\mu(x)\right|^{r'}$$

$$\leqq c \left|\int |N_a(t-x)|^{r'/(r'-2)}\, d\mu(x)\right|^{r'-2} \left|\int |K_b(t-x)|^{r'/2}\, d\mu(x)\right|^2.$$

Taking in account that

$$(N_a)^{r'/(r'-2)} = [N^{r'/r'-2}]_\gamma,\ (K_b)^{r'/2} = (K^{r'/2})_{\gamma/2},$$

and integrating in t, we obtain the desired inequality.

The following lemma is easily verified and we omit the proof.

Lemma 13. *If* $(K_{\gamma/2} * K'_{\gamma/2})(t) \leqq c\, K_\gamma(t)$, *where* $K'(x) = K(-x)$, *then* $[\|(K_{\gamma/2} * \mu)\|_2]^2 \leqq c\|(K_\gamma * \mu)\|_{1,\mu} \leqq c_1 V(K_\gamma, \mu)$.

Theorem 7. *Let* $f \geqq 0, f \in L^p(E^n), S = \{x; |K_{\gamma/p} * f(x)| = \infty\}$, *and let* $L = K^{p'/2}$, $N = K^{p'/(p'-2)}$, $K^2 \geqq c\,K$, $c > 0$, a) *If* $(L_{\gamma/2} * L_{\gamma/2})(t) \leqq c\, L_\gamma(t)$, *then* S *is of zero* L_γ — *capacity for* $1 < p < 4/3$, *and of zero* H_γ — *capacity for* $4/3 \leqq p \leqq 2$. b) *If* $p = 1$, *then* S *is of zero* K_γ — *capacity.*

Proof. a) We may assume that S is a bounded set, and let μ be concentrated in S, then by lemmas 12 and 13,

$$\infty = \int_S (K_{\gamma/p} * f(x))\, d\mu \leqq \int_{E^n} f(t) \left[\int_S K_{\gamma/p}(t-x)\, d\mu(x) \right] dt \leqq$$

$$\|f\|_p\, \|K_{\gamma/p} * \mu\|_{p'} \leqq \|f\|_p\, (W^{p'-2}\, \|L_{\gamma/2} * \mu\|_2{}^2)^{1/p'} \leqq$$

$$[c(W)^{(p'-2)}\, V]^{1/p'}\, \|f\|_p, \qquad (64)$$

where $V = V(L_\gamma, \mu)$, $W = V(N_\gamma, \mu)$. If $p < 4/3$ then $L \geqq c_1 N$, $V \geqq c_1 W$, and if $p \geqq 4/3$ then $W \geqq c_1 V$.

b) This follows directly from property c).

Lemma 14. *Let* $K_{\gamma/p} = K'_{\gamma/p} + K''_{\gamma/p}$, *where* $K'_{\gamma/p}$ *is defined as in* (61a), $0 < \gamma < n$, $1 < p < \infty$, *and let* μ *be a distribution.* a) *If* $K \in L^{p'}$ *then* $K'_{\gamma/p} * \mu \in L^{p'}$. b) *If* $K \in L^1$ *then* $K_\varepsilon'' * \mu \in L^1$ *for any* $\varepsilon > 0$.

Proof. a) We have

$$\|K'_{\gamma/p} * \mu\|_{p'} = \|\Sigma_{i=0}^{\infty} 2^{(\gamma/p-n)i} \int K(2^{-i}(x-t))\, d\mu(t)\|_{p'} \leqq$$

$$\leqq \Sigma_{i=0}^{\infty} 2^{(\gamma/p-n)i} \int \left[\int_{E^n} |K(2^{-i}(x-t))|^{p'}\, dx \right]^{1/p'} d\mu(t)$$

$$\leqq \Sigma_{i=0}^{\infty} 2^{(\gamma/p-n)i}\, 2^{ni/p'}\, \|K\|_{p'} = \|K\|_{p'}\, \Sigma_{i=0}^{\infty} 2^{i(\gamma-n)/p} < \infty.$$

b) The proof is the same as in a) and we will not repeat it.

Theorem 7a. *Suppose that the generating kernel satisfies the condition* $|K|^2 \geqq cK, c>0, K \in L^{p+1}$. *If* $f \in L^p, 2<p<\infty$, *and if* $S=\{x; |K_{\gamma/p} * f(x)|=\infty\}$, *then* S *is of zero* K_γ*— capacity for all* $\delta, 0<\delta<\gamma$.

Proof. Let $\delta<\gamma$, $\gamma/p=\delta/r$, $r'-p'=\varepsilon>0$. We may assume that S is a bounded set; let the distribution μ be concentrated in S. Since $K''_{\gamma/p}$ vanishes outside of $|x| \leqq 1$, $K''_{\gamma/p} * \mu$ vanishes outside of a bounded set A, so that $K''_{\gamma/p} * \mu = \varphi(A) K''_{\gamma/p} * \mu$, hence by lemma 11 (with $N=K$) and by b) of lemma 14 (with $K=L$),

$$\|K''_{\gamma/p} * \mu\|_{p'} = \|\varphi(A) K''_{\delta/r} * \mu\|_{r'-\varepsilon} \leqq$$

$$\leqq c|V|^b \|L''_a * \mu\|_1 \leqq c_1 |V|^b, \qquad (65)$$

where $L=K^{p+1} \in L^1$ and $V=V(K_\delta, \mu)$, since $a=\delta-\delta(r'-\varepsilon)r' > 0$. Since $|K|^{p'} \leqq c_1 |K|^{p+1} \in L^1$, we have by a) of lemma 14,

$$\|K'_{\gamma/p} * \mu\|_{p'} < \infty. \qquad (65a)$$

Since $\|K_{\gamma/p} * \mu\|_{p'} \leqq \|K'_{\gamma/p} * \mu\|_{p'} + \|K''_{\gamma/p} * \mu\|_{p'}$, from the second inequality of (64), (65) and (65a) we obtain

$$V(K_\delta * \mu) = \infty.$$

Let $E^n = E^m \times E^{n-m}$. We say that S *is of* N*-capacity in* E^m, if $S \subset E^m$ and if $\sup_{t \in E^m} |N * \mu(t)| = \infty$ for any μ concentrated in S. For simplicity let us assume that $K_\gamma(x) = |x|^{\gamma-n}$, $H_{\gamma n} f = f * K_\gamma = \tilde{f}_{\gamma n} = f_\gamma$. Then we have:

Theorem 7b. *Let* $0<\gamma p<n$, $f \in L^p(E^n)$, $\delta = \inf\,(\gamma - (n-m)/p; \gamma)$, $s=\inf\,(n,m)$. *If* $1 \leqq p \leqq 2$, *then the set* $S=\{x \in E^m; |K_\gamma * f(x)|=\infty\}$ *is of zero* $K_{\delta p}$ *— capacity in* E^s. *If* $2<p<\infty$ *then* S *is of zero* $K_{\varepsilon p}$ *— capacity in* E^s, *for any* $0<\varepsilon<\delta$.

Proof. If $m=n$ we have the theorem of Du Plessis, already proved. If $m>n$, we have

$$|f_\gamma(x)| \leqq \|f\|_p \, \||(x_1-t, x_2)|^{\gamma-n}\|_{p'},$$

where the norm is taken in the variable t, and $x=(x_1, x_2)$. If $x_2 \neq 0$, the last norm is finite: In fact, we may take $p>1$, so that we have $\gamma<n$ and $(\gamma-n)\,p'<n$. If $x_2=0$, then $f_\gamma(x)=\int f(t)\,|x_1-t|^{\gamma-n}\,dt$ and we obtain the preceding case.

Let now $m<n$. Then

$$|f_\gamma(x)| \leqq \int_{E^{n-m}} dt_2 \int_{E^m} |f(t_1, t_2)|\,|(t_1-x, t_2)|^{\gamma-n}\,dt_1 \leqq$$

$$\left\{ \int_{E^m} dt_1 \left[\int_{E^{n-m}} |f(t_1, t_2)|^p\,dt_2 \right]^{1/p} \right\}.$$

$$\left[\int_{E^{n-m}} |(t_1-x, t_2)|^{(\gamma-n)p'}\,dt_2 \right]^{1/p'} \leqq \int_{E^m} \|f(t_1, t_2)\|_p\,|t_1-x|^{\delta-m}.$$

$$\left[\int_{E^{n-m}} |(h, u_2)|^{(-n)p'}\,du_2 \right]^{1/p'} dt_1,$$

where $|h|=1$, $t_2=|t_1-x|\,u_2$. Since $(\gamma-n)\,p'<m-n$, the last integral is finite. Since $g(t_1)=\|f(t_1, t_2)\|_p \in L^p(E^m)$, and $\|f\|_p{}^{(n)}=\|g\|_p{}^{(m)}$, we obtain

$$|f_\gamma(x)| \leqq c \int_{E^m} g(t)\,|t-x|^{\delta-m}\,dt,$$

and we obtaiin again the precedent case, already proved.

Remark: We observe that the hypothesis $K \in Lip(1, 1)$ of β) *was not used in theorems 7 - 7b*). Similar theorems hold in the case $E^n \cap E^m = E^t$, even if E^m is not contained in E^n (cfr. [15]).

6. Lipschitz properties of $H_{\gamma n}$. Now we shall generalize some theorems due to Hardy and Littlewood [17] (cfr. [10] and [15]). In this section we consider a *fixed* generating kernel $k(x)$ which is assumed to satisfy condition α) of § 5 and to belong to $L^1(E^n) \cap Lip(1, 1, E^n)$ (some additional conditions are assumed in theorems 8a and 8b). $K_{\gamma n}, k_i$ denote the generated kernels defined by (54), (54a), and we write K_γ instead of $K_{\gamma n}$, and Hf instead of $H_{\gamma n} f$. The set $\{y \in E^n;\, a < |y-x| < b\}$ is denoted by (x, a, b) and $E^n-(x, a, b)$ by $(x, a, b)'$.

We say that $f \in Lip\,\beta(E^n)$ if $|f(x+h)-f(x)| \leqq c|h|^\beta = 0(|h|^\beta)$, and that $f \in lip\,\beta$ if $|f(x+h)-f(x)| = o(|h|^\beta)$.

Let $\Sigma_{-\infty}{}^0 k_i$ be denoted by K_γ^-, $\Sigma_0{}^\infty k_i$ by K_γ^+ and $\Sigma_{-\infty}{}^N k_i$ by K_γ^{N-}. Then we have the following properties of the generated kernels, which are easily deduced from the definition (54) and α_1), β_1).

a) $\|K_\gamma^+(t+h) - K_\gamma^+(t)\|_1 = 0(|h|)$.

b) $\| |t|^\beta |K_\gamma^-(t+h) - K_\gamma^-(t)| \|_1 = 0(|h|^{\beta+\gamma}), \beta > 0$.

(For proof, take $N<0$ such that $2^N \leqq |h| < 2^{N+1}$ and split K_γ^- into $K_\gamma^{N-} + (K_\gamma^- - K_\gamma^{N-})$; in estimating K_γ^{N-} use condition $\|k_i\|_1 \leqq 2^{i\gamma} c, k_i = 0$ in $|x| > 2^i$; in estimating $K_\gamma^- - K_\gamma^{N-}$ use condition $\|k_i\|_{(1,1)} \leqq 2^{(\gamma-1)i} c$ (cfr. the proof of lemma 2).

c) By similar splittings is proved that if $k \in Lip(q',1)$ and if N is such that $2^{-N-1} \leqq |h| \leqq 2^{-N}$, $|h| < 1/3$, and if $2^{-M} \leqq \delta < 1-3h$, then:

$$c_1) \left\{ \int_{(0.3h,\delta)} |K_\gamma^-(x+h) - K_\gamma^-(x)|^{q'} dx \right\}^{1/q'} \leqq$$

$$\Sigma^\infty{}_{i=0} 2^{(n-\gamma-n/q')i} \left\{ \int_{(0,|h|2^i,\delta 2^i)} |k(x+2^i h) - k(x)|^{q'} \right\}^{1/q'} \leqq$$

$$\leqq \Sigma_{i=0}{}^{N+1} 2^{(n-\gamma-n/)q'i} \|k(x+2^i h) - k(x)\|_{q'} = 0(|h|^{\gamma-n/q}).$$

$$c) \quad \Sigma^\infty{}_{i=-\infty} \left\{ \int_{(0,0,4h)} |k_i(x)|^{q'} dx \right\}^{1/q'} =$$

$$= \Sigma_{-\infty}{}^{-N-2} \left\{ \int_{(0,0,4|h|)} |k_i|^{q'} \right\}^{1/q'} \leqq$$

$$\leqq |2^{-N}|^{\gamma-n/q} \|k\|_{q'} \leqq \|k\|_{q'}(|h|^{\gamma-n/q}).$$

$$c) \quad \Sigma^\infty{}_{i=-\infty} \left\{ \int_{(0,\delta,1)} |k_i(x+h) - k_i(x)|^{q'} dx \right\}^{1/q'} \leqq$$

$$\leqq \Sigma^0{}_{-M} 2^{i(\gamma-n/q)}.$$

Theorem 8. *Let* $0<\gamma<1$, $\alpha = inf(\beta+\gamma, 1)$. *If* $f \in L^1(E^n) \cap Lip\,\beta(E^n), 0<\beta$, *then* $Hf \in Lip\,\alpha(E^n)$.

Proof. From the hypothesis we have $|f(x)| < c < \infty$. We have

$$|Hf(x+h) - Hf(x)| = \left| \int_{E^n} f(x-t)\,[K_\gamma(t+h) - K_\gamma(t)]\,dt \right| \leqq$$

$$\left| \int_{(0,0,1)'} f(x-t)\,[K_\gamma^+(t+h) - K_\gamma^+(t)]\,dt \right| +$$

$$\left| \int_{(0,0,1)} (f(x-t) - f(x))\,[K_\gamma^-(t+h) - K_\gamma^-(t)]\,\mathrm{d}t + \right.$$

$$\left. + |f(x)| \left| \left[\int_{(h,0,1)} K_\gamma^-(t)\,dt - \int_{(0,0,1)} K_\gamma^-(\mathrm{t})\,dt \right] \right| \right|,$$

and $|f(x+t) - f(x)| = 0(|t|^\beta)$, hence using a) and b) we obtain $|Hf(x+h) - Hf(x)| = 0(|h|^{\beta+\gamma} + |h|)$.

Theorem 8a. *Let* $k \in L(q',1)$, $q>1$, $1+n/q > \gamma > n/q$, $0<\gamma<n$. *If* $f \in L^q$ *then* $Hf \in lip(\gamma - n/q)$.

Proof. We have

$$|Hf(x+h) - Hf(x)| \leqq \int_{(0,0,1)'} |f(x-t)|\,|K_\gamma^+(t+h) - K_\gamma^+(t)|\,dt +$$

$$\left[\int_{(0,0,3h)} + \int_{(0,3h,\delta)} + \int_{(0,\delta,1)} \right] [|f(x-t)|\,|K_\gamma^-(t+h) - K_\gamma^-(t)|\,dt].$$

We take $N>1$ such that $2^{-N-1} \leqq |h| < 2^{-N}$ and δ such that $2^{-M} \leqq \delta < 1 - 3|h|$ and

$$\int_{(0,\delta)} |f(x-t)|^q\,dt \leqq \varepsilon.$$

Applying Hölder's inequality and using $c_1)$, $c_2)$, $c_3)$ we obtain $|Hf(x+h) - Hf(x)| = o(|h|^{\gamma - n/q}) + 0(|h|) = o(|h|^{\gamma - n/q})$.

Theorem 8b. *Let* $0<\gamma<1$, $k \in L^\infty(E^n)$. *If* $f \in L^p(E^n)$, $1 \leqq p < \infty$, *then* $Hf \in lip\,(p,\gamma)$.

Proof. Let

$$g_1(x) = \int_{(x,0,c|h|)} f(t)\,K_\gamma(x-t)\,dt;$$

$$g_2(x)=\int_{(x,c|h|,d|h|)} f(t)\, K_\Upsilon(x-t)\, dt,$$

$$g_3(x)=\int_{(x,0,d|h|)'} f(t)\, K_\Upsilon(x-t)\, dt;$$

$$\triangle Hf(x)=Hf(x)-Hf(x-h)=\triangle g_1+\triangle g_2+\triangle g_3.$$

It is sufficient to prove that, given $\varepsilon>0$, for sufficiently small constant c and sufficiently large d, it is true that (for small h tending to zero):

$$\int |\triangle g_3|^p\, dx \leqq \varepsilon |h|^{p\Upsilon}, \tag{66}$$

$$(\|\triangle g_1\|_p)^p=\int |\triangle g_1|^p\, dx \leqq \varepsilon |h|^{p\Upsilon}, \tag{67}$$

$$\|\triangle g_2\|_p = o\,(|h|^\Upsilon). \tag{68}$$

We first prove (68). Let $A=(x, c|h|, d|h|), A_0=(0, c|h|, d|h|)$. If $t\in A, u=x-t$, and $|h|$ small, then $\Sigma_{-\infty}^{\infty} |k_i(u)| \leqq 2^{n-\Upsilon} M\, |u|^{\Upsilon-n}$, hence

$$|\triangle g_2(x)|=\left|\int_A [f(t)-f(t-h)]\, \Sigma\, k_i(x-t)\, dt\right| \leqq$$

$$\leqq 2^{n-\Upsilon} M \int_A |f(t)-f(t-h|\; |x-t|^{\Upsilon-n}\, dt \leqq$$

$$M_1\, d|h|^{\Upsilon-n}\, [(d-c)\; |h|^n]^{1/p'} \left\{\int_A |f(t)-f(t-h)|^p\, dt\right\}^{1/p} =$$

$$=M(c,d)\; |h|^{\Upsilon-n/p} \left\{\int_A |f(t)-f(t-h)|^p\, dt\right\}^{1/p};$$

$$(\|\triangle g_2\|_p)^p \leqq M(c,d)\; |h|^{\Upsilon p-n} \int_{E_n} dx \int_A |f(t)-f(t-h)|^p\, dt=$$

$$= M|h|^{\Upsilon p-n} \int_A dx \int_{E_n} |f(t)-f(t-h)|^p\, dt = o\,(|h|^\Upsilon).$$

Now we prove (67). Let $B=(0,0,c|h|)$. Since $|K_\Upsilon(u)| \leqq$

$M|u|^{\Upsilon-n}$ if $u \in B$ and $|h|$ is small, we have for sufficiently small c,

$$\|g_1(x)\|_p \leqq \| \int_B |f(x-u)| \, |K_\Upsilon(u)| \, du\|_p \leqq$$

$$\leqq M \| \int_B |f(x-u)| \, |u|^{\Upsilon-n} du\|_p \leqq$$

$$M \int_B |u|^{\Upsilon-n} \|f\|_p \, du \leqq M_1(c|h|)^\Upsilon \leqq \varepsilon |h|^\Upsilon.$$

Similarly $\|g_1(x-h)\|_p \leqq \varepsilon |h|^\Upsilon$.

Now let us prove (66). Let $D=(0,0,(d+1)\,|h|), D'=(h, d|h|, (d+1)\,|h|), 2^{-N} \leqq (d+1)\,|h| < 2^{-N+1}$. We have

$$|\triangle g_3(x)| \leqq \left| \int_D f(x-u) \, \Sigma_{-\infty}{}^\infty [k_i(u-h) - k_i(u)] \, du \right| +$$

$$\int_D |f(x-u)| \, \Sigma_{-\infty}{}^\infty |k_i(u)| \, du +$$

$$+ \int_{D'} |f(x-u)| \, \Sigma_{-\infty}{}^\infty |k_i(u)| \, du = R_1 + R_2 + R_3.$$

If d is sufficiently large, then

$$\|R_1\|_p \leqq \left\{ \int \left| \int_D \Sigma |k_i(u-h) - k_i(u)| \, |f(x-u)| \, du \right|^p dx \right\}^{1/p} \leqq$$

$$\leqq \int_D \Sigma \, |k_i(u-h) - k_i(u)| \, \|f(x-u)\|_p \, du$$

$$\leqq \|f\|_p \, \Sigma^\infty{}_{i=N} \, 2^{i(\Upsilon-n)} |h| \leqq M |d|^{\Upsilon-n} \, |h|^\Upsilon \leqq \varepsilon |h|^\Upsilon.$$

Let $D''=(0, d|h|, (d+1)\,|h|)$, then applying Minkowki's integral inequality,

$$(\|R_2\|_p)^p \leqq \int dx \left[\int_{D''} |f(x-u)| \, \Sigma_{-\infty}{}^\infty |k_i(u)| \, du \right]^p \leqq$$

$$\leqq M_1(d|h|)^{(\Upsilon-n)p} \int_{E_n} \left[\int_{D''} |f(x-u)| \, du \right]^p dx$$

$$\leqq M_2(d|h|)^{(\Upsilon-n)p} \, |h^{np/p'} d^{(n-1)p/p'} \int_{E_n} dx \int_{D''} |f(x-u)|^p \, du \leqq$$

$$M_2\, d^{p(\gamma-n)+(n-1)(p-1)}|h|^{p\gamma-n}\int_{E^n}|f(x)|^p\,dx\;|(x, d|h|, (d+1)|h|)|=$$

$$=M_3\, d^{p(\gamma-1)}\,|h|^{p\gamma}\leqq \varepsilon|h|^{p\gamma},\ \text{since}\ \gamma<1.$$

Similarly we see that $(\|R_3\|_p)^p \leqq \varepsilon|h|^{\gamma p}$.

Theorem 8c. *Let* $1 \leqq p < \infty$, $0 < \gamma < 1-\alpha$, $\alpha > 0$, $k(x) = k(-x)$. *If* $f \in L^p \cap Lip(p, \alpha)$, *then* $Hf \in Lip(p, \gamma+\alpha)$. *If* $f \in lip(p, \alpha)$ *then* $Hf \in lip(p, \gamma+\alpha)$.

Proof. We prove the first part of the theorem; the other part is proved in the same way. Let $A = (0, 0, 3|h|)$, $B = E^n - A$, $K(t) = K_\gamma(t)$, then

$$|Hf(x+h) - Hf(x-h)| = |\triangle Hf(x)| =$$

$$\left|\int_{E^n} f(x+u)\,K(u-h)\,du - \int_{E^n} f(x-u)\,K(u-h)\,du\right| =$$

$$= \left|\int_B (f(x+u) - f(x-u))\,[K(u-h) - K(u)]\,du + \right.$$

$$+ \int_A [f(x-u) - f(x+u)]\,K(u)\,du +$$

$$\left. + \int_A [f(x+u) - f(x-u)]\,K(u-h)\,du\right| = |J_1 + J_2 + J_3|.$$

Applying the integral inequality of Minkowski, and letting

$2^{-N} < 3|h| \leqq 2^{-N+1}$, $N > 1$, $\|J_1\|_p \leqq$

$$\leqq \left\{\int_{E^n}\left|\int_B |f(x+u) - f(x-u)|\,|K(u-h) - K(u)|\,du\right|^p dx\right\}^{1/p} \leqq$$

$$\leqq \int_B |K(u-h) - K(u)|\;\|\triangle f\|_p\,du \leqq$$

$$\leqq M\int_B |u|^\alpha\,|K(u-h) - K(u)|\,du \leqq$$

$$\leqq M\int |x|^\alpha\,\Sigma^\infty{}_{i=-\infty}\,2^{(\gamma-n)i}\,|k((x-h)/2^i) - k(2^{-i}x)|\,dx$$

$$\leqq M_1\,\Sigma^\infty{}_{-N}\,2^{\alpha i + in(\gamma-n)}\,|2^{-i}h| =$$

$$= M_1(\Sigma^1{}_{-N} + \Sigma_0{}^\infty)\,|h| \leqq\ M_3|h|^{\gamma+\alpha}$$

$$\|J_2\|_p \leqq \int_A |K(u)| \left\{ \int_{En} |f(x+u)-f(x-u)|^p dx \right\}^{1/p'} du \leqq$$

$$\leqq M \int |u|^\alpha \Sigma_{-\infty}^{-N+1} 2^{i(\gamma-n)} |k(2^{-i} u)| \, du \leqq$$

$$\leqq M_2 \Sigma_{-\infty}^{-N+1} 2^{(\gamma-n+\alpha)i} \int k(2^{-i} u) \, du \leqq M_3 |h|^{\alpha+\gamma}.$$

$$\|J_3\|_p \leqq M \int_{(0,0,3|h|)} |K(u-h)| \, |u|^\alpha du \leqq M \int_{(0,0,4|h|)} |K(u)| \, |u|^\alpha du$$

and the last integral is estimated in the same fashion as J_2.

Remark. It was proved in [2] that in the case of $\gamma=0$ the properties of Hilbert transforms hold also for «ergodic Hilbert transforms», that is for operators of the form

$$H_{0n} f(x) = \int_{En} f(\sigma_t x) K_{0n}(t) \, dt,$$

where $\{\sigma_t x\}$ is a group of measure-preserving transformations of an abstract measure space $X=\{x\}$. The corresponding extension for $\gamma>0$ is not clear and the few results we obtained are mostly of negative character (cfr. [15]). Hence it is an open problem which are the type and capacity properties of the «ergodic potential operators».

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PROFESOR CHARLES EHRESMANN

En la Asamblea General de la Unión Matemática Argentina celebrada en Córdoba el día 20 de octubre de 1959, se nombró Miembro Honorario de la sociedad al Profesor Charles Ehresmann.

Nacido el 19 de Abril de 1905 en Strasbourg el profesor Ehresmann, después de haber sido profesor en la Universidad de su ciudad natal, es actualmente (desde 1954) profesor de la Sorbonne en París. Su especialidad es la Geometría Diferencial, de cuyas tendencias modernas es uno de los más caracterizados fundadores. Sus numerosas publicaciones pueden clasificarse en los siguientes grupos, cuyos títulos genéricos indican los campos abarcados, en todos los cuales ha contribuído con aportes significativos y de algunos de ellos ha sido el creador: Topología de ciertos espacios homogéneos, Grupos de Lie, Estructuras locales y espacios localmente homogéneos, Espacios fibrados, Variedades casi complejas, Variedades foliadas, Conexiones infinitesimales en un espacio fibrado, Pseudogrupos de Lie.

Actualmente, de unos años a esta parte, está interesado principalmente en la fundamentación de la geometría diferencial. Sus ideas y resultados a este respecto han sido expuestos de manera sistemática en el curso que bajo el título de ''Estructuras locales y geometría diferencial'' ha dado en la Facultad de Ciencias de la Universidad de Buenos Aires durante los meses de agosto a noviembre de 1959.