FORMAL DIFFERENTIAL EQUATIONS

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A system of n homogeous linear differential equations

$$dx_i/dt = \sum_{j=1}^{n} a_{ij}(t)\, x_j\,,\ i = 1\,, \ldots, n,$$

can be represented by the matricial differential equation

$$du/dt = A(t)u,$$

where, for each value of t, u and $A(t)$ are linear transformations of an n-dimensional vector space V. For the initial conditions $u = u_0$ when $t = a$, the Picard's method of approximation yields the unique solution

$$u = Q_0(t)\, u_0 + Q_1(t)\, u_0 + \ldots, \tag{1}$$

where $Q_0(t)$ is always the identity transformation of V, and

$$Q_{p+1}(t) = \int_a^t A(s)\, Q_p(s)\, ds$$

for$p \geqq 0$. Instead of considering the sum of the above series, let us investigate the series itself. A term-by term differentiation of the series results the series

$$0 + A(t)\, Q_0(t)\, u_0 + A(t)\, Q_1(t)\, u_0 + \ldots \tag{2}$$

For any series of V-valued functions of t, $v_0(t) + v_1(t) + \ldots$, let us define

$$\frac{d}{dt}\,(v_0(t) + v_1(t) + \ldots) = v_0'(t) + v_1'(t) + \ldots$$

and

$$\widetilde{A}(t)\,(v_0(t) + v_1(t) + \ldots) = 0 + A(t)\, v_0(t) + A(t)\, v_1(t) + \ldots$$

Then, we observe that the series in [1] is precisely the solution of the differential equation

$$dw/dt = \widetilde{A}(t)\, w$$

with the initial condition $w = u_0 + 0 + 0 + \dots$ for $t = a$.

In this abstract, we shall consider only the formal aspects of the above differential equation in a generalized form.

A graded module M is a module defined by a family of submodules $M^{(p)}$, $p = 0, 1, \dots$, such that M is the direct sum $M^{(0)} + M^{(1)} + \dots$ If, moreover, M is an algebra with $M^{(p)} M^{(q)} \subset M^{(p+q)}$, then M is called a graded algebra. By taking the ideals $M^{(p)} + M^{(p+1)} + \dots$, $p = 0, 1, \dots$, as a system of neighborhoods of zero, M is given a topology, which is called the p-adic topology of M. Let $\widetilde{M}$ be the completion of M under this topology. Then every element of $\widetilde{M}$ is of the form

$$u = \pi_0 u + \pi_1 u + \dots + \pi_p u + \dots, \pi_p u \in M^{(p)}.$$

The homomorphism π_p from the module $\widetilde{M}$ onto the module $M^{(p)}$ is called the *p-th* projection of $\widetilde{M}$. All elements u of $\widetilde{M}$ with $\pi_0 u = \dots = \pi_{p-1} u = 0$ form an ideal $0^p(\widetilde{M})$ of $\widetilde{M}$; in particular, we write $0(\widetilde{M}) = 0^1(\widetilde{M})$. For simplicity, the ground field K of all modules will be taken to be either the field of all real numbers or that of all complex numbers, and $M^{(0)}$ will be assumed to be K. We shall call the algebra $\widetilde{M}$ a formal algebra over K. It may be easily seen that this conception is a generalization of formal power series algebras (commutative or noncommutative) and retains a good amount of essential properties of formal power series algebras.

For each element u of $0(\widetilde{M})$, exp u and log $(1 + u)$ may be defined in the obvious manner. If σ is a continuous homomorphism from the formal algebra $\widetilde{M}$ to another formal algebra $\widetilde{M}'$ over K, then $\sigma \exp u = \exp \sigma u$ and $\sigma \log(1 + u) = \log(1 + \sigma u)$.

Now we assume that each vector spade $M^{(p)}$ has, besides the topology induced by the p-adic topology of M, another topology so that we may talk about differentiation and integration of an $M^{(p)}$ —valued function of a real variable t. Moreover, the product $u\, v \in M^{(p+q)}$ is required to be a continuous function of $u \in M^{(p)}$

and $v \varepsilon M^{(q)}$. This new topology of $M^{(p)}$ will be called the linear topology of $M^{(p)}$. An $\widetilde{M}$-valued function $u(t)$ will be said to be continuous (piecewise continuous) if all $\pi_p u(t)$ are so in the vector space topology. The differentiation and integration of $u(t)$ are defined through the formulas: $\pi_p du(t)/dt = d\pi_p u(t)/dt$ and $\pi_p \int_p u(t)\, dt = \int \pi_p u(t)\, dt$, $p = 0, 1, \ldots$, if the respective right hand sides of the formulas exist.

Theorem. For any value of t in an interval I on the real line, let $A(t)$ be a linear transformation of $\widetilde{M}$ continuous in the p-adic topology with the following properties:

a) If $u(t)$ is a piecewids continuous $\widetilde{M}$-valued function over I, so is $A(t)\, u(t)$.

b) $A(t)\, 0^p(\widetilde{M}) \subset 0^{p+1}(\widetilde{M})$, $t \varepsilon I$, $p = 0, 1, 2, \ldots$

Then there exists one and only one continuous solution in $\widetilde{M}$ for the differential equation

$$dT/dt = A(t)\, T, \quad t \varepsilon I,$$

with the initial condition: $T = T_0$ for $t = t_0 \varepsilon I$.

The proof of the theorem can be carried out, in a straighforward fashion, by comparing $\pi_p dT/dt$ and $\pi_p A(t)\, T$ inductively on p.

Corollary. If $A(t) = \text{constant}$, then $T = (\exp(t - t_0) A)\, T_0 = T_0 + (t - t_0) A T_0 + (t - t_0)^2 A^2 T_0/2! + \ldots$ is the solution.

Example. Let $\widetilde{M}$ be the formal power series algebra of the non-commutative indeterminates X, Y, and let A be the linear transformation of $\widetilde{M}$ such that $Au = [u, X] = uX - Xu$ for any u in $\widetilde{M}$. Consider the differential equation $dT/dt = AT$ in $\widetilde{M}$ with the initial condition $T(0) = Y$.

The above corollary gives the solution

$$T(t) = (\exp tA)\, Y = Y + t\,[Y, X] + \frac{1}{2!} t^2\, [[Y, X], X] + \ldots;$$

while, by direct verification,

$$T(t) = (\exp -tX)\, Y\, (\exp tX)$$

is also a solution. Therefore the two solutions are equal, and, in particular, we have

$$(\exp - X)\ Y\ (\exp X) = (\exp A)\ Y,$$

which is a known result.

Now we confine ourselves to the case where $A(t)$, $A(t) \in O(M)$ for each value of t, is a piecewise continuous $\widetilde{M}$-valued function and where $A(t)\,T$ simply means the product. The solution of the differential equation $dT/dt = A(t)\,T$ with the initial condition $T = 1$ for $t = t_0$ will be denoted by $T = T(A;\, t_0, t)$. When there is no ambiguity, we simply write $T(t_0, t)$ for $T(A;\, t_0, t)$.

A good number of theorems in the theory of linear differential equations hold analogously in this case and may be proved in similar manners. For example, the formula

$$T(t_1, t)\ T(t_0, t_1) = T(t_0, t)$$

is a direct consequence of the uniqueness of the solution.

We confine ourselves once more to the case where $M = T(V) = T_0(V) + \ldots + T_p(V) + \ldots$ is the tensor algebra based on an n-dimensional vector space V, where $T_p(V)$ denotes the p-fold tensor product of V with itself. If we choose for V a base $X_1, \ldots, X_n$, then $\widetilde{T}(V)$ can be considered as the algebra of all formal power series of the noncommutative indeterminates $X_1, \ldots, X_n$. Furthermore, we demand that, for each value of t, $A(t) \in T_1(V) = V$, i.e. $A(t)$ is a linear form in $X_1, \ldots, X_n$.

For any $t_0, t_1, t_0 < t_1$, in the interval I, over which $A(t)$ is defined, a piecewise smooth path α is given in V through the formula

$$\alpha(t) = \int_{t_0}^{t} A(t)\, dt, \quad t_0 \leqq t \leqq t_1.$$

On the other hand, any picewise smooth path $\alpha(t)$ in V with $t_0 \leqq t \leqq t_1$ determines a piecewise continuous V-valued function $A(t) = d\alpha(t)/dt$, over the interval $[t_0, t_1]$.

We associate to the path α the element $\theta(\alpha) = T(d\alpha(t)/dt;\, t_0, t_1)$ of $\widetilde{T}(V)$. The following properties of $\theta(\alpha)$ may be verified in a straighforward fashions:

1. $\theta(\alpha)$ does not depend on the parametrization of α.
2. $\theta(\alpha)$ does not change when α is subject to a translation in V.

3. If α is a straight line segment in V, then $\theta(\alpha) = \exp(\alpha(t_1) - \alpha(t_0))$.
4. α is closed if and only if $\pi_1 \theta(\alpha) = 0$.
5. α is closed with its projection on any plane having zero algebraic area, if and only if $\pi_1 \theta(\pi) = 0$ and $\pi_2 \theta(\alpha) = 0$.
6. Let the product $\alpha\beta$ denote the piecewise smooth path obtained from α followed by the translation of another piecewise smooth path β. Then

$$\theta(\alpha\beta) = \theta(\beta)\ \theta(\alpha).$$

7. Let α^{-1} denote the piecewise smooth path obtained from α by reversing the sense. Then $\theta(\alpha^{-1}) = [\theta(\alpha)]^{-1}$.

For u, v in $\widetilde{T}(V)$, define $[u, v] = uv - vu$. Through this bracket multiplication, $V = T_1(V)$ generates a Lie algebra $L(V)$ in $T(V)$. $L(V)$ may be identified with the free Lie algebra based on V. An element u of $\widetilde{T}(V)$ is called a Lie element if $\pi_0 u = 0$ and $\pi_1 u \in L(V)$, $p = 1, 2, \ldots$

Theorem. log $\theta(\alpha)$ is a Lie element.

If we take α and β respectively to be the line segments tX_1, $0 \leqq t \leqq 1$, and tX_2, $0 \leqq t \leqq 1$, then $\log \theta(\alpha\beta) = \log \theta(\beta)\theta(\alpha) = \log(\exp X_2 \exp X_1)$ is a Lie element, which is a result due to Baker and Hausdorff.

Faithful representation theorem. Let α and β be two irreducible piecewise smooth path in V. Then α can be obtained fron β by a translation and a change of parameter if and only if $\theta(\alpha) = \theta(\beta)$. Here K is taken to be the field of all real numbers. Moreover, if $\alpha_1, \ldots, \alpha_r$ are irreducible piecewise smooth paths in V such that each one can not be obtained from any other by a translation and a change of parameter, the $\theta(\alpha_1), \ldots, \theta(\alpha_r)$ are linearly in dependent over K.

BIBLIOGRAPHY

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