A DERIVATION OF THE MAIN RESULTS OF THE THEORY OF H^p SPACES

by MARY and GUIDO WEISS
(Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires)

§ 1. *Introduction.*

The space H^p, $p > 0$, is the vector space of all analytic functions $F(z)$ defined in the open unit disc, $|z| = r < 1$, such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta < \infty. \qquad [1]$$

The basic result in the theory of these spaces is the following theorem:

Theorem A.

If $F \in H^p$, $p > 0$, *then*

$$\lim_{r \to 1} F(re^{i\theta}) = F(e^{i\theta}) \qquad [i]$$

exists and is finite a.e. (*);

$$\lim_{r \to 1} \int_0^{2\pi} |F(re^{i\theta}) - F(e^{i\theta})|^p \, d\theta = 0. \qquad [ii]$$

It is obvious that $H^{p_1} \subset H^{p_2}$ if $p_2 < p_1$. Thus, the first result of theorem A (which does not involve the index p) becomes more general as p tends to 0. In fact, limits of this type exist for a class of functions that includes all the H^p spaces, $p > 0$. This is the *Nevanlinna class, N*, of all analytic functions $F(z)$, defined in the open unit disc, such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta < \infty,$$

(*) More generally, it can be shown that "non-tangential" limits exist, but we shall not deal with them here. The reader is referred to "Trigonometric Series" by A. Zygmund (Cambridge University Press, 1959) for their definition, as well as for all the unproved assertions that are made in this paper.

where

$$\log^+ x = \log x \text{ if } x \geq 1 \text{ and } \log^+ x = 0 \text{ if } x < 1.$$

More precisely, we have the theorem:

Theorem B.

If $F \in N$ *then* $\lim_{r \to 1} F(re^{i\theta}) = F(e^{i\theta})$ *exists and is finite a.e.*

When $p > 1$, theorem A is an elementary result in the theory of Fourier series. For, in this case, inequality [1] implies that $F(re^{i\theta})$ is the Poisson integral of a function in $L^p(0, 2\pi)$ and, from this, both conclusions of theorem A follow easily. The other cases, when $0 < p \leq 1$, can be reduced to this situation (that is, the case $p > 1$) by means of a basic decomposition theorem from which theorem B also follows. This decomposition theorem is the following:

Theorem C.

If $F \in N$ *and* F *is not identically zero then* $F = BG$ *where*

$B(z)$ *and* $G(z)$ *are analytic in the open unit disc;* [i]

z *is a zero of multiplicity* ν *of* B *if and only if it is a zero of multiplicity* ν *of* F; [ii]

$|B(z)| \leq 1$ *for* $|z| < 1$; [iii]

$\lim_{r \to 1} B(re^{i\theta}) = B(e^{i\theta})$ *exists and* $|B(e^{i\theta})| = 1$, *a.e.*; [iv]

there exists a finite regular measure μ *such that* [v]

$$G(re^{i\theta}) = exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\phi} + re^{i\theta}}{e^{i\phi} - re^{i\theta}}\, d\mu(\phi)\right\}.$$

Furthermore, if $F \in H^p$, $0 < p$, *with* $\sup_{0 \leqq r < 1} \int_0^{2\pi} |F(re^{i\theta})|^p\, d\theta \leqq m < \infty$

then $G \in H^p$ *with* $\sup_{\leqq r < 1} \int_0^{2\pi} |G(re^{i\theta})|^p\, d\theta \leq m$.

We shall not show how theorem C enables us to reduce theorem A to the easier case $p > 1$, since this reduction is well known. The main idea of this reduction, however, is simple: the last part of theorem C tells us that the function G carries the "size" of F and part [ii] (or [v]) tells us that it has no zeroes (all the zeroes of F are carried by B). Thus, if, say, $p = 1$, we can form a square root of G and express F as a product, $[B\sqrt{G}]\,[\sqrt{G}]$, of two functions in H^2 for which the result on the existence of boundary values is easily derived.

Theorem C is generally proved by examining the zeroes of F and showing that they are so distributed that the function $B(z)$ can be constructed as a (possibly infinite) product, the *Blaschke product*, with each zero, ζ, of F corresponding to a factor B having ζ as its only zero. It is the purpose of this paper to show that theorem C can be derived by a different method in which we do not need to examine the behavior of the zeroes of F.

§ 2. *Proof of Theorem C.*

Since $F \not\equiv 0$ there exists a sequence $\{r_n\}$, with $\frac{1}{2} \leq r_n < 1$ and $\lim_{n \to \infty} r_n = 1$, such that $F(z)$ is not zero when $|z| = r_n$.

Let us put $F_n(z) = F(r_n z)$ and

$$G_n(re^{i\theta}) = exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\phi} + re^{i\theta}}{e^{i\phi} - re^{i\theta}} \log |F_n(e^{i\phi})|\, d\phi\right\} =$$

$$= exp\left\{\int_0^{2\pi} P(r, \theta - \phi) \log |F_n(e^{i\phi})|\, d\phi + i\int_0^{2\pi} Q(r, \theta - \phi) \log |F_n(e^{i\phi})|\, d\phi\right\},$$

where

$$P(r, \theta - \phi) = \frac{1 - r^2}{2\pi(1 - 2r\cos(\theta - \phi) + r^2)}$$

is the Poisson kernel and

$$Q(r, \theta - \phi) = \frac{r \sin(\theta - \phi)}{\pi(1 - 2r\cos(\theta - \phi) + r^2)}$$

is the conjugate Poisson kernel.

It is an elementary result in the theory of Fourier series that $\lim_{r \to 1} G_n(re^{i\theta}) = G_n(e^{i\theta})$ exists and, moreover,

$$|G_n(e^{i\theta})| = |F_n(e^{i\theta})|. \qquad [2]$$

Let us observe that, since G_n is an exponential, $G_n(z)$ is never zero in the interior of the unit disc. Thus $B_n(z) = F_n(z)/G_n(z)$ is analytic in the interior of the unit disc. By the maximum modulus theorem and equality [2], therefore, we obtain the fact that

$$|B_n(z)| \leqq 1 \text{ if } |z| < 1 \text{ and } |B_n(e^{i\theta})| \equiv 1. \qquad [3]$$

Since $\log |G_n(z)|$ is harmonic, the mean value theorem for harmonic functions assures us that

$$\log |G_n(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log |G_n(re^{i\phi})|\, d\phi, \quad 0 \leqq r \leqq 1. \qquad [4]$$

Now suppose that 0 is a zero of F of order k, $0 \leqq k < \infty$; thus, $F(z) = z^k F_0(z)$ with $F_0(0) \neq 0$. Since G_n is never zero and $F_n = B_n G_n$, the function B_n also has a zero of order k at the origin. By [3] we see that $|B_n(z)/(r_n z)^k| \leqq 1/r_n^k$, because $|B_n(z)| \leqq 1$ and $B_n(z)/z^k$ is analytic. But, $B_n(z)\, G_n(z) = F_n(z) = (r_n z)^k F_0(r_n z)$ or, equivalently, $F_0(r_n z) = G_n(z)\left[\frac{B_n(z)}{(r_n z)^k}\right]$; consequently, $|F_0(0)| \leqq |G_n(0)|/r_n^k$. Using this inequality, equality [4] and the equalities

$$\log |G_n(z)| = \log^+ |G_n(z)| - \log^- |G_n(z)|$$
$$|\log |G_n(z)|| = \log^+ |G_n(z)| + \log^- |G_n(z)|,$$

we obtain, for $0 \leqq r \leqq 1$,

$$\log r_n^k |F_0(0)| \leqq \log |G_n(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log |G_n(re^{i\phi})|\, d\phi =$$

$$= 2\frac{1}{2\pi}\int_0^{2\pi} \log^+ |G_n(re^{i\phi})|\, d\phi - \frac{1}{2\pi}\int_0^{2\pi} |\log |G_n re^{i\phi})||\, d\phi \leqq$$

$$\leqq 2\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_0^{2\pi} P(r, \theta-\phi) \log^+ |F_n(e^{i\theta})|\, d\theta\right\} d\phi -$$

$$-\frac{1}{2\pi}\int_0^{2\pi} |\log |G_n(re^{i\phi})||\, d\phi \leqq$$

$$\leqq 2m - \frac{1}{2\pi}\int_0^{2\pi} |\log |G_n(re^{i\phi})||\, d\phi, \; (**)$$

(**) The fact that $\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_0^{2\pi} P(r, \theta-\phi) \log^+ |F_n(re^{i\theta})|\, d\theta\right\} d\phi \leqq \leqq \frac{1}{2\pi}\int_0^{2\pi} \log^+ |F_n(re^{i\theta})|\, d\theta \leqq m$ follows immediately from Fubini's theorem and the well-known fact about the Poisson kernel: $\int_0^{2\pi} P(r, \theta)\, d\theta = 1$.

where

$$m = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta .$$

This shows

$$\frac{1}{2\pi} \int_0^{2\pi} |\log | G_n(re^{i\phi}) || \, d\phi \leqq 2m - \log | F_0(0) | + \log \frac{1}{(r_n)^k} \leqq$$

$$\leqq 2m - \log | F_0(0) | + \log 2^k, \text{ for } 0 \leqq r \leqq 1 . (***) \qquad [5]$$

The right hand side does not depend on n; this means that the sequence $\{f_n(\phi)\} = \{\log | G_n(e^{i\phi}) |\}$ is uniformly bounded in $L^1(0, 2\pi)$. Hence, there exists a regular measure μ and a subsequence, which, after a relabelling we can assume to be the entire sequence $\{f_n(\phi)\}$, such that

$$\lim_{n \to \infty} \int_0^{2\pi} f_n(\phi) \, g(\phi) \, d\phi = \int_0^{2\pi} g(\phi) \, d\mu(\phi),$$

for each continuous g.

Taking, in particular, $g(\phi) = P(r, \theta - \phi) + i Q(r, \theta - \phi)$ and keeping in mind that $f_n(\phi) = \log | G_n(e^{i\phi}) | = \log | F_n(e^{i\phi}) |$ we obtain, for each $z = re^{i\theta}$ of the unit disc $|z| < 1$,

$$\log G_n(re^{i\theta}) = \int_0^{2\pi} \{P(r, \theta - \phi) + i Q(r, \theta - \phi\} \log | F_n(e^{i\phi}) | \, d\phi \longrightarrow$$

$$\longrightarrow \int_0^{2\pi} \{ P(r, \theta - \phi) + i Q(r, \theta - \phi) \} \, d\mu(\phi), \text{ as } n \longrightarrow \infty .$$

If we put $G(re^{i\theta}) = exp \left\{ \int_0^{2\pi} \{P(r, \theta - \phi) + i Q(r, \theta - \phi)\} \, d\mu(\phi) \right\}$, then $\lim_{n \to \infty} G_n(re^{i\theta}) = G(re^{i\theta})$. Let us also observe that G is analytic and, being an exponential, is never zero.

By [1], we see that the sequence $\{B_n\}$ is uniformly bounded by 1. Thus, there exists a subsequence, which, again, we may suppose to be the original sequence $\{B_n\}$, and a function $B(z)$ such that $\lim_{n \to \infty} B_n(z) = B(z)$ uniformly in each disc $|z| \leqq \rho < 1$.

(***) This is the only place where the artificial assumption $r_n \geqq \frac{1}{2}$ is used — we see immediately that any number greater than 0 (but, of course, less than 1) could have been used.

Since

$$F_n(z) \longrightarrow F(z) \text{ as } n \longrightarrow \infty$$

we have

$$F(z) = \lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} B_n(z)\, G_n(z) = B(z)\, G(z),$$

where $G(z)$ has no zeroes and $|B(z)| \leqq 1$ for $|z| < 1$ (since $|B_n(z)| \leqq 1$ for $|z| < 1$). Moreover, we see that G is of the form given in part [v] of theorem C.

By Fatou's theorem $\lim_{r\to 1} B(re^{i\theta}) = B(e^{i\theta})$ exists *a.e.* We still have to show that $|B(e^{i\theta})| = 1$ *a.e.*

Since $F_n(z) = B_n(z)\, G_n(z)$ and $F_n(z) = F(r_n z) = B(r_n z)\, G(r_n z)$ we have $B(r_n e^{i\theta}) = B_n(e^{i\theta})\, G_n(e^{i\theta})/G(r_n e^{i\theta})$ and, consequently,

$$|B(r_n e^{i\theta})| = \frac{|G_n(e^{i\theta})|}{|G(r_n e^{i\theta})|}. \qquad [6]$$

Setting $H(z) = \dfrac{G_n(z)}{G(r_n z)}$, since H is analytic, we obtain

$$H(0) = \frac{1}{2\pi}\int_0^{2\pi} H(re^{i\theta})\, d\theta, \quad 0 \leqq r \leqq 1.$$

Thus,

$$|H(0)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |H(re^{i\theta})|\, d\theta, \qquad [7]$$

which, together with [6], yields

$$\left|\frac{G_n(0)}{G(0)}\right| \leqq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{G_n(e^{i\theta})}{G(r_n e^{i\theta})}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} |B(r_n e^{i\theta})|\, d\theta.$$

Letting $n \longrightarrow \infty$, we obtain $B(r_n e^{i\theta}) \longrightarrow B(e^{i\theta})$ *a.e.*,

$$\int_0^{2\pi} |B(r_n e^{i\theta})|\, d\theta \longrightarrow \int_0^{2\pi} |B(e^{i\theta})|\, d\theta$$

(since the last convergence is majorized by 1) and

$$\left|\frac{G_n(0)}{G(0)}\right| \longrightarrow 1.$$

Consequently,

$$1 \leqq \frac{1}{2\pi}\int_0^{2\pi} |B(e^{i\theta})|\, d\theta.$$

But $|B(e^{i\theta})| \leqq 1$ *a.e.* since $|B(re^{i\theta})| \leqq 1$ when $r < 1$. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} |B(e^{i\theta})|\, d\theta = 1$$

and this could not hold if $|B(e^{i\theta})| < 1$ for θ in a subset of $[0, 2\pi]$ of positive measure. This shows that $|B(e^{i\theta})| = 1$ *a.e.*

The only thing that remains to be proved is the last part of theorem C. Using Jensen's inequality and the fact that

$$\int_0^{2\pi} P(r, \theta - \phi)\, d\phi = 1$$

we have

$$|G_n(re^{i\theta})|^p = exp\left\{p\int_0^{2\pi} P(r, \theta - \phi) \log |F_n(e^{i\phi})|\, d\phi\right\} =$$

$$= exp\left\{\int_0^{2\pi} P(r, \theta - \phi) \log |F_n(e^{i\phi})|^p\, d\phi\right\} \leqq$$

$$\leqq \int_0^{2\pi} P(r, \theta - \phi)\, |F_n(e^{i\phi})|^p\, d\phi, \quad 0 \leqq r < 1;$$

thus, by Fubini's theorem,

$$\int_0^{2\pi} |G_n(re^{i\theta})|^p\, d\theta \leqq \int_0^{2\pi} |F_n(e^{i\phi})|^p \left\{\int_0^{2\pi} P(r, \theta - \phi)\, d\theta\right\} d\phi =$$

$$= \int_0^{2\pi} |F_n(e^{i\phi})|^p \,.\, 1\, d\phi = \int_0^{2\pi} |F(r_n e^{i\phi})|^p\, d\phi \leqq m\,.$$

Since $\lim_{n\to\infty} G_n(z) = G(z)$ the last part of the theorem follows from Fatou's lemma.

§ 3. *Uniqueness of the Decomposition.*

As was stated in the introduction, theorem C is usually proved by constructing the function carrying the zeroes of F as an infinite product, the Blaschke product. It is logically possible that our function $B(z)$ may not be such a product. If, for example, we started out with $F(z) = F(re^{i\theta}) = e^{-[P(r,\theta) + iQ(r,\theta)]}$, then we would have $F \varepsilon N$ and $F(z) \neq 0$ when $|z| < 1$. On the other hand,

$$\lim_{r\to 1} P(r, \theta) = 0 \quad a.e.$$

(in fact, this limit is 0 if $\theta \neq 0$) and, consequently,

$$| F(e^{i\theta}) | = e^0 = 1 \quad a.e.$$

Thus, at first sight, F could be identified with either B *or* G in our statement of theorem C. We shall show that the proof we have given does yield, in fact, the classical Blaschke product (up to a constant factor of absolute value 1).

Let $F = \widetilde{B}\widetilde{G}$ be the decomposition of F obtained by the above mentioned construction of the Blaschke product $\widetilde{B}$. We shall make use of the well-known fact

$$\lim_{r \to 1} \int_0^{2\pi} | \log | \widetilde{B}(re^{i\theta}) || \, d\theta = 0. \qquad [8]$$

Since $\widetilde{G}$ is never 0 we have, for each $z = re^{i\theta}$ of the unit disc $r < 1$,

$$\gamma \widetilde{G}(r_n re^{i\theta}) = exp \left\{ \int_0^{2\pi} [P(r, \theta - \phi) + i Q(r, \theta - \phi)] \log | \widetilde{G}(r_n e^{i\phi}) | \, d\phi \right\}, \qquad [9]$$

where γ is a constant of absolute value 1.

We have, using (9),

$$G_n(re^{i\theta}) = exp \left\{ \int_0^{2\pi} [P(r, \theta - \phi) + i Q(r, \theta - \phi)] \log | F(r_n e^{i\phi}) | \, d\phi \right\} =$$

$$= exp \left\{ \int_0^{2\pi} [P(r, \theta - \phi) + i Q(r, \theta - \phi)] \log | \widetilde{B}(r_n e^{i\phi}) \widetilde{G}(r_n e^{i\phi}) | \, d\phi \right\} =$$

$$= \gamma \widetilde{G}(r_n re^{i\theta}) \, exp \left\{ \int_0^{2\pi} [P(r, \theta - \phi) + i Q(r, \theta - \phi)] \log | \widetilde{B}(r_n e^{i\theta}) | \, d\phi \right\}.$$

Now, keeping r fixed and letting $n \longrightarrow \infty$ (and, thus, $r_n \longrightarrow 1$), we have, by [8]

$$exp \left\{ \int_0^{2\pi} [P(r, \theta - \phi) + i Q(r, \theta - \phi)] \log | \widetilde{B}(r_n e^{i\phi}) | \, d\phi \right\} \longrightarrow 1.$$

But, we know that

$$\lim_{n \to \infty} G_n(re^{i\theta}) = G(re^{i\theta}) \; (****)$$

and, clearly,

$$\lim_{n \to \infty} \widetilde{G}(r_n re^{i\theta}) = \widetilde{G}(re^{i\theta}) \,.$$

Thus, we have shown:

$$G(z) = \gamma \widetilde{G}(z)$$

for each z with $|z| < 1$. Since

$$B(z)\, G(z) = F(z) = \widetilde{B}(z)\, \widetilde{G}(z) \,,$$

we must have

$$B(z) = \gamma^{-1} \widetilde{B}(z)$$

and our assertion concerning the uniqueness of our decomposition is proved.

(****) We are tacitly assuming that we have relabelled the sequence $\{G_n\}$, for, in reality, only a subsequence has a limit — see the argument in the last section.