ON THE AUTOMORPHISMS OF THE LATTICE OF CLOSURE OPERATORS OF A COMPLETE LATTICE

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1.

OYSTEIN ORE [1] has stated that the group of automorphisms of the lattice of all closure operators definable over the lattice of subsets of a set S is isomorphic to the group of permutations of S.

The group of automorphisms of the lattice $\phi(L)$, whose elements are the closure operators definable over a complete lattice L (1), has been studied by PH. DWINGER [2]. However, the assertion, contained in [2], of the existence of an isomorphism between that group and the group of automorphisms of L, is not true, as one concludes from the following example: let L be the chain

$$a_1 < a_2 < \dots < a_n \quad (n > 2);$$

ot is clear that L has only one automorphism —the identity automorphism—, although the lattice $\phi(L)$, which is a Boolean algebra with $n - 1$ atoms, has $(n-1)!$ automorphisms.

In [3] we have introduced the notion of quasi-automorphism of a complete lattice L and we have shown that *the group of quasi-automorphisms of L is isomorphic to the group of automorphisms of $\phi(L)$*. We have obtained sufficient conditions to the isomorphisms between the group of automorphisms of L and the group of automorphisms of $\phi(L)$. From one of these conditions, we

(1) A closure operator φ of L is defined as an operator of L, satisfying the conditions: (i) $x \leq \varphi(x) = \varphi(\varphi(x))$, *for every* $x \in L$; (ii) *if* $x \leq y$, *then* $\varphi(x) \leq \varphi(y)$. It is known that, if L is a complete lattice, then ϕ (L) is a complete lattice relatively to the following partial order: $\varphi \leq \psi$, *if and only if* $\varphi(x) \leq \psi(x)$, *for every* $x \in L$.

have obtained a result which contains the ORE's theorem above, as a particular case (2).

In this note we improve the sufficient conditions obtained in [3] and we present some other results, namely, we show that if L is a complete lattice, then the group of automorphisms of $\phi(L)$ and of $\phi(\phi(L))$ are isomorphic.

2.

Let L be a complete lattice and h be a permutation of L. One says that h is a *quasi-automorphism of L*, if the following conditions hold:

(i) $h(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I'} h(x_i)$ and $h^{-1}(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I''} h^{-1}(x_i)$, *for every non-void family $\{x_i\}_{i \in I}$, of elements of L, and for some non-void subsets I' and I'' of I;*

(ii) $h(u) = u$, *where u is the last element of L.*

This notion arises naturally from the following observations:

1) If f is an automorphism of the complete lattice L and φ is a closure operator of L, then it is easy to see that the operator $\Psi = f \varphi f^{-1}$ is also a closure operator of L and that the operator π_f, defined by $\pi_f \varphi = \Psi$ is an automorphism of $\phi(L)$ (3).

2) The mapping $f \rightarrow \pi_f$, from the group of automorphisms of L into the group of automorphisms of $\phi(L)$, preserves the products and, if $f \neq g$, then $\pi_f \neq \pi_g$. This means that *the group of automorphisms of L is isomorphic to a subgroup of the group of automorphisms of $\phi(L)$*, namely, *the subgroup of automorphisms of the form π_f* (4).

3) Let us denote by φ_a the closure operator of L defined by

$$\varphi_a(x) = a, \text{ if } x \leq a \text{ and } \varphi_a(x) = u, \text{ if } x \nleq a.$$

One sees that φ_a is a dual atom of $\phi(L)$, i. e., φ_a is an element covered by the last element ω of $\phi(L)$. Now, if π is an automorphism of $\phi(L)$, one has $\pi(\omega) = \omega$ and $\pi\varphi_a = \varphi_{a'}$ for some element $a' \in L$.

(2) In [3], it is shown that, if L is a complete Boolean algebra, then the groups of automorphisms of L and of ϕ (L) are isomorphic.

(3) See [2].

(4) See theorem 1, [3].

One shows that the mapping h, defined by

$$h(u) = u, \text{ and } h(a) = a', \text{ if } a \neq u,$$

is a permutation of L satisfying the conditions (i) and (ii).

Since every automorphism of L satisfies the conditions (i) and (ii) (with $I' = I'' = I$), it seems natural to define a quasi-automorphism of L as any permutation of L satisfying these conditions.

3.

We know that the automorphisms of a lattice preserve the infimum of any two elements. For the quasi-automorphisms, the following holds:

THEOREM 1: *If h is a quasi-automorphism of a complete lattice L and if x_1 and x_2 are incomparable elements of L, then*

$$h(x_1 \wedge x_2) = h(x_1) \wedge h(x_2) \text{ and } h^{-1}(x_1 \wedge x_2) = h^{-1}(x_1) \wedge h^{-1}(x_2)$$

PROOF: Indeed, from condition (i), it follows that $h(x_1 \wedge x_2)$ is either $h(x_1)$ or $h(x_2)$ or $h(x_1) \wedge h(x_2)$. But, since h is a permutation of L, one has $h(x_1 \wedge x_2) = h(x_1)$, if and only if $x_1 \wedge x_2 = x_1$, i. e., if and only if $x_1 \leq x_2$. Since x_1 and x_2 are incomparable, one concludes that it is impossible to have $h(x_1 \wedge x_2) = h(x_1)$. By a similar argument, one sees that $h(x_1 \wedge x_2) \neq h(x_2)$. Analogously for h^{-1}.

Now, we can state the following.

THEOREM 2: *If x, y are elements of a complete lattice L, such that $x = y \wedge (\bigwedge_{i \in I} x_i)$, where $\{x_i\}_{i \in I}$ is a non-void family of elements of L, incomparable with y, then, for every quasi-automorphism h of L, one has $h(x) < h(y)$.*

PROOF: First, let us observe that $x < y$; indeed, one has $x \leq y$, but if $x = y$, then $y \leq \bigwedge_{i \in I} x_i$, hence $y \leq x_i$, contrarily to the hypothesis.

Now, one has either $\bigwedge_{i \in I} x_i < y$ or $\bigwedge_{i \in I} x_i \nless y$.

If $\bigwedge_{i \in I} x_i < y$, then

$$x = \bigwedge_{i \in I} x_i = y \wedge (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (y \wedge x_i),$$

hence

$$h(x) = h(\bigwedge_{i \in I} (y \wedge x_i)) = \bigwedge_{i \in I'} h(y \wedge x_i), \text{ with } 0 \neq I' \sqsubseteq I.$$

Since y and x_i are incomparable, one has, by theorem 1,

$$h(x) = \bigwedge_{i \in I'} (h(y) \wedge h(x_i)) = h(y) \wedge (\bigwedge_{i \in I'} h(x_i)).$$

From this it follows that $h(x) \leq h(y)$ and, since $h(x) \neq h(y)$, one has $h(x) < h(y)$.

If $\bigwedge_{i \in I} x_i \nless y$, one concludes that $\bigwedge_{i \in I} x_i$ and y are incomparable. In fact, if $\bigwedge_{i \in I} x_i \geq y$, then $x = y$, contrarily to the hypothesis. Hence, by theorem 1, one has $h(x) = h(y) \wedge h(\bigwedge_{i \in I} x_i)$, and from this follows $h(x) < h(y)$, since $h(x) \neq h(y)$.

Analogously one sees that $h^{-1}(x) < h^{-1}(y)$.

An automorphism is clearly a quasi-automorphism h such that, if $x < y$, then $h(x) < h(y)$. Therefore, the following holds:

THEOREM 3: *Let L be a complete lattice satisfying the condition: "if $x, y \in L$ and $x < y < u$, then there is in L a non-void family $\{x_i\}_{i \in I}$ such that each x_i is incomparable with y and $x = y \wedge (\bigwedge_{i \in I} x_i)$"; then every quasi-automorphism of L is an automorphism of L* (5).

In particular, one has

COROLLARY 1: *If the complete lattice L is dual atomistic* (6), *then every quasi-automorphism of L is an automorphism of L.*

Indeed, in this case, if $x < y < u$, one has $x = y \wedge (\bigwedge_{i \in I} x_i)$, where the elements x_i are the dual atoms which follow x and do not follow y.

Since the group of quasi-automorphisms of L is isomorphic to the group of automorphisms of $\phi(L)$ (7), one concludes the following:

COROLLARY 2: *If the complete lattice L is dual atomistic, then the groups of automorphisms of L and of $\phi(L)$ are isomorphic.*

(5) This theorem improves a result obtained in [3].

(6) We recall that a lattice is said to be dual atomistic, if each element is the infimum of the dual atoms following it.

(7) See [3], theorem 2.

We know that, if L is a complete lattice, then $\phi(L)$ is a dual atomistic [8] and complete lattice. From this it follows.

Corollary 3: *If L is a complete lattice, the groups of automorphisms of $\phi(L)$ and of $\phi(\phi(L))$ are isomorphic.*

Let us suppose that L is a complemented modular complete lattice and let x and y be elements of L such that $x < y < u$. If y' denotes a complement of y, one has

$$x = x \vee (y \wedge y') = y \wedge (x \vee y')$$

It is easy to see that the element $x \vee y'$ is incomparable with y. Indeed, one has not $y \leq x \vee y'$, otherwise it would be $x = y$, contrarily to the hypothesis; and one has not $y > x \vee y'$, otherwise it would be $y' \leq x$ and hence $y \vee y' \leq y \vee x = y$, that is to say, $y = u$, contrarily to the hypothesis.

Then, from theorem 3, it follows.

Corollary 4: *If L is a complemented modular complete lattice, then the groups of automorphisms of L and of $\phi(L)$ are isomorphic.*

We can improve theorem 3, by stating

Theorem 4: *Let L be a complete lattice satisfying the condition: "if $x, y \in L$ and $x < y < u$, then there are in L finite sequences*

$$y = y_0, y_1, y_2, \ldots, y_n = x \text{ and } t_1, t_2, \ldots, t_n,$$

such that

$$y_i = y_{i-1} \wedge t_i \text{ and } t_i = \bigwedge_{i \in I_i} x_j^{(i)}$$

where each $x_j^{(i)}$ is incomparable with y_{i-1}"; then the groups of automorphisms of the lattices L and $\phi(L)$ are isomorphic.

Proof: We know that these groups are isomorphic, if and only if every quasi-automorphism of L is an automorphism of L. Let h be a quasi-automorphism of L; by theorem 2, one has successively

$$h(x) = h(y_n) < h(y_{n-1}) < h(y_{n-2}) < \ldots < h(y_0) = h(y),$$

which proves the theorem.

(8) One shows that, if φ is a closure operator of a complete lattice L, then φ is the infimum of the closure operators φ_a, where a runs over the set of the elements closed under φ.

The theorems 3 and 4 give sufficient conditions in order to the groups of automorphisms of L and $\phi(L)$ be isomorphic. These conditions are not necessary; indeed, let us consider a lattice L isomorphic to $\mathbf{1} \oplus \mathbf{2}^2$, ordinal sum of a lattice constituted by one element and a Boolean algebra with two atoms; one sees that the groups of automorphisms of L and $\phi(L)$ are isomorphic and L does not satisfy the condition of theorem 3 nor the condition of theorem 4.

We have not been able to find a necessary and sufficient condition for the existence of an isomorphism between the grtups of automorphisms of the iattices L and $\phi(L)$.

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