ON THE CLIFFORD AND JORDAN-WIGNER ALGEBRAS

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SUMMARY:

The theory of the Clifford algebras is discussed in connection with the finite-dimensional Jordan-Wigner algebras and the affine algebra G_n of Schönberg. The relations of G_n with the n-dimensional antisymmetric tensors allow to see in a simple way the origin of some of the properties of the Clifford algebras and the spinors, especially the connection between 2p-dimensional spinors and p-dimensional antisymmetric tensors. On the other hand the theory of G_n is further extended. The comparison with the Jordan-Wigner algebra leads to relations between the Clifford algebras and the functions of two-valued variables. The spinors are shown to be functions of the sub-sets of finite sets and the Jordan-Wigner algebra is discussed in relation with such sets.

The null vectors in the Clifford algebra.

1. The Clifford algebra is a metric vector calculus of a real n-dimensional flat space endowed with a non-degenerate metric quadratic form $g_{jk} x^j x^k$, the x denoting cartesian coordinates. To any vector $\vec{V}$ corresponds an element γ_V of the Clifford algebra, the correspondence being characterized by two conditions

$$\text{I)} \quad c_1 \vec{V}_1 + c_2 \vec{V}_2 \longrightarrow c_1 \gamma_{V_1} + c_2 \gamma_{V_2} \qquad [1\text{ a}]$$

(the c denote numbers)

$$\text{II)} \quad \gamma^2_V = g_{jk} V^j V^k 1_C \qquad [1\text{ b}]$$

(1_C denotes the unity of the Clifford algebra)

It follows from the conditions (I) and (II) that

$$\gamma_{V_1}\gamma_{V_2} + \gamma_{V_2}\gamma_{V_1} = 2\, g_{jk} V_1^j V_2^k 1_C \qquad [2]$$

The V^j denote the cartesian components of the contravariant vector $\vec{V}$. *The Clifford algebra $C_n^{(g)}$ of the n-dimensional flat space* *endowed with the metric g_{jk} can be defined as the associative algebra with a unity 1_C generated by the symbols γ_V of the contravariant vectors $\vec{V}$ that satisfy the conditions* (I) *and* (II). We may obviously take as generators the symbols of n linearly independent contravariant vectors. We shall denote by γ_j the symbol of the *j-th* basic vector $\vec{I}_j$ of the cartesian coordinate system. The n symbols γ_j can be taken as the generators of $C_n^{(o)}$, the multiplication rule being

$$\gamma_j \gamma_k + \gamma_k \gamma_j = 2\, g_{jk}\, 1_C \qquad [3]$$

as a consequence of equation [2]. It will be convenient to take $C_n^{(o)}$ over the field of the complex numbers

We shall denote by $\vec{W}$ a generic null vector: $g_{jk} W^j W^k = 0$. *The symbols γ_W are nilpotent*: $\gamma^2_W = 0$. Let $\vec{W}_1, .., \vec{W}_r$ be r mutually orthogonal and linearly independent null vectors. The unity 1_C and the r symbols $\gamma_{W_1}, .., \gamma_{W_r}$ generate a r-dimensional Grassmann algebra, which is simply the Grassmann algebra of the r-dimensionall totally null vector space of the linear combinations of the r null and mutually orthogonal vectors $\vec{W}_1, .., \vec{W}_r$. In the euclidean case the $\vec{W}$ are always complex vectors, but in the case of an indefinite metric there are also real null (or isotropic) vectors.

It is well known that when n is even, $n = 2\,p$, the maximal number of linearly independent mutually orthogonal null vectors is p. The totally null (or totally isotropic) linear space defined by p linearly independent and mutually orthogonal null vectors is the linear space of the vectors of a maximal flat space of the null hypercone of the n-dimensional space, whose equation is

$$g_{jk} x^j x^k = 0.$$

When n is odd, $n = 2\,\mathrm{p} + 1$, the maximal number of linearly independent and mutually orthogonal null vectors is also p and the vector space they generate is the linear space of the vectors of a maximal flat space on the hypercone $g_{jk} x^j x^k = 0$. *Thereby when $n > 1$, there is a Grassmann algebra of the order* $[n/2]$, *associated to any maximal flat space on the hypercone $g_{jk} x^j x^k = 0$, generated by the symbols of any set of* $[n/2]$ *linearly independent vectors of the flat space,* $[n/2]$ *denoting the largest integer contained in* $n/2$. We shall from now on write $[n/2] = p$.

Let the $\vec{J}_\alpha$ be a set of $2\,[n/2] = 2\,p$ mutually orthogonal real vectors, such that $g_{jk} J_\alpha^j J_\alpha^k = e_\alpha$, with $e_\alpha = 1$ or -1. We shall introduce $2\,p$ null vectors $\vec{W}_a$ and $\vec{W}_a^+$

$$2\,\vec{W}_a = \sqrt{e_a}\,\vec{J}_a - \sqrt{-e_{a+p}}\,\vec{J}_{a+p} \quad , \quad 2\,\vec{W}_a^{+} = \sqrt{e_a}\,\vec{J}_a + \sqrt{-e_{a+p}}\,\vec{J}_{a+p} \qquad [4]$$
$$(a = 1,\ 2,\ldots,p)$$

We have obviously

$$\vec{W}_a \,.\, \vec{W}_b = 0 \quad , \quad \vec{W}_a^+ \,.\, \vec{W}_b^+ = 0 \quad , \quad 2\,\vec{W}_a \,.\, \vec{W}_b^+ = \delta_{a,b} \qquad [5]$$
$$(a,\ b = 1,\ldots,p)$$

the point indicating the inner product. *The linear combinations of the $\vec{W}_a$ constitute a p-dimensional totally isotropic vector space and the linear combinations of the $\vec{W}_a^+$ another such vector space. When the space is euclidean, $\vec{W}_a^+$ coincides with the complex conjugate $\vec{W}_a^*$ of $\vec{W}_a$, since $e_\alpha = 1$ for all α.* The $2\,p$ null vectors $\vec{W}_a$ and $\vec{W}_a^+$ are linearly independent. When $n = 2\,p$, the $2\,p$ vectors $\vec{W}_a$ and $\vec{W}_a^+$ constitute a basis of the linear space of the vectors $\vec{V}$. We shall call null basis of a $2\,p$-dimensional metric space a set of $2\,p$ null vectors $\vec{W}_a$, $\vec{W}_a^+$ defined in the above way.

When $n = 2\,p$, we can express any vector $\vec{V}$ as a linear combination of the vectors of a null basis: $\vec{V} = \omega^a{}_V\,\vec{W}_a + \omega_V{}^{+a}\,\vec{W}_a^+$, with summation over the still indices a. We shall denote the null vector $\omega^a{}_V\,\vec{W}_a$ by $\vec{\omega}_V$ and the null vector $\omega_V{}^{+a}\,\vec{W}_a^+$ by $\vec{\omega}_V^+$. *Thus any $\vec{V}$ is the sum of two null vectors $\vec{\omega}_V$ and $\vec{\omega}_V^+$, the $\vec{\omega}_V$ belonging to a totally isotropic p-dimensional vector space and the $\vec{\omega}_V^+$ to another such vector space.* We have $g_{jk} V^j V^k = \Sigma\, \omega_V{}^a\, \omega_V{}^{+a}$.

In the $2\,p$-dimensional euclidean space the null vectors $\vec{\omega}_V$ and $\vec{\omega}_V^+$ of a real vector $\vec{V}$ are complex conjugate. *Hence there is a one-one linear correspondence between the real vectors $\vec{V}$ and the vectors $\vec{\omega}$ of the p-dimensional totally isotropic vector space of the complex linear combination of the $\vec{W}_a$. The euclidean geometry in the* *real space is associated to the unitary geometry of the* $\vec{\omega}$ *— space since* $g_{jk} V^j V^k = \Sigma \mid \omega_V{}^a \mid^2$. *In the case of* $p = 1$ *we get the well, known correspondence between the real vectors of the euclidean plane and the complex numbers.*

We shall call maximally indefinite metric of a $2\,p$-dimensional flat space the metric whose canonical quadratic form has p positive and p negative squares. In this case, p of the e_α are equal to 1 and the other p are equal to -1. We may numerate the $\vec{J}_\alpha$ in such a way that the e_a be 1 and the e_{a+p} be -1. *Thus all the* $\vec{W}_a$ *and* $\vec{W}_a{}^+$ *are real and we get two real totally isotropic p-dimensional vector spaces* of $\vec{\omega}_V$ *and* $\vec{\omega}_V{}^+$ *associated to the linear space of the real* $\vec{V}$. For other types of indefinite metric there are no real p-dimensional totally isotropic vector spaces. When the metric is maximally indefinite and n is divisible by 4, $p = 2\,q$, we may numerate the $\vec{J}_\alpha$ in such a way that $e_1 = e_2 = \ldots = e_q = e_{p+1} = \ldots = e_{p+q} = 1$ and $e_{q+1} = \ldots = e_p = e_{p+q+1} = \ldots = e_{2p} = -1$, in order that $\vec{W}_r{}^+ = \vec{W}_r{}^*$ for $r = 1,\ldots,q$ and $\vec{W}_s{}^+ = -\vec{W}_s{}^*$ for $s = q+1,\ldots,p$. Thus for a real $\vec{V}$, $\omega_V{}^{+r} = \omega_V{}^{r*}$ and $\omega_V{}^{+s} = -\omega_V{}^{s*}$ and $g_{jk} V^j V^k = = \sum_r \mid \omega_V{}^r \mid^2 - \sum_s \mid \omega_V{}^s \mid^2$, so that the indefinite metric of the $\vec{V}$-space is now associated to an indefinite hermitian metric in the linear space of the complex vectors $\vec{\omega}$. It is easily seen in a similar way that the indefinite metric whose canonical quadratic form contains $2\,m_+$ positive squares and $2\,m_-$ negative ones can be associated to the hermitian metric of the complex $\vec{\omega}$ — space $\sum_b \mid \omega_V{}^b \mid^2 - \sum_c \mid \omega_V{}^c \mid^2$ with $b = 1, \ldots, m_+$ and $c = m_+ + 1, \ldots, m_+ + m_-$.

The elements γ_{W_a} and $\gamma_{W_a{}^+}$ will be denoted by W_a and $W_a{}^+$, for the sake of simplicity. It follows from the equations [5] that

$$W_a W_b + W_b W_a = 0 \ , \ W_a{}^+ W_b{}^+ + W_b{}^+ W_a{}^+ = 0,$$
$$W_a W_b{}^+ + W_b{}^+ W_a = \delta_{a,b}\, 1_C \qquad [6]$$

When $n = 2\,p$, the W_a and $W_a{}^+$ are a set of generators of the Clifford algebra $C_{2p}{}^{(o)}$ of our $2\,p$-dimensional metric space. *We may also interpret them as the generators of an associative algebra of a pair* *of p-dimensional* $\vec{\omega}$ *and* $\vec{\omega}^+$ *vector spaces, in which the symbols* ω *and* ω^+ *have the multiplication rules*

$$\omega_{\mathrm{I}}\omega_{\mathrm{II}} + \omega_{\mathrm{II}}\omega_{\mathrm{I}} = 0 \ , \ \omega_{\mathrm{I}}^+\omega_{\mathrm{II}}^+ + \omega_{\mathrm{II}}^+\omega_{\mathrm{I}}^+ = 0 \qquad [7\ \mathrm{a}]$$

$$\omega_{\mathrm{I}}\omega_{\mathrm{II}}^+ + \omega_{\mathrm{II}}^+\omega_{\mathrm{I}} = \Sigma\, \omega_{\mathrm{I}}^a\omega_{\mathrm{II}}^{+a}\, 1_C \qquad [7\ \mathrm{b}]$$

By taking $\omega = \omega^a W_a$ and $\omega^+ = \omega^{+a} W_a^+$, we see that the rules [7 a] and [7 b] are a consequence of [6]. Actually we have

$$\omega = \gamma_\omega \quad \text{and} \quad \omega^+ = \gamma_{\omega^+}.$$

We may interpret the $\vec{\omega}^+$ *as the covariant vectors of a p-dimensinal space whose contravariant vectors are the* $\vec{\omega}$. *Thus the* ω, ω^+ *algebra appears as the algebra* G_p *of a p-dimensional affine space introduced by Schönberg* (1). The interpretation of the $\vec{\omega}^+$ as covariant vectors of a p-dimensional space is quite natural, since any $\vec{\omega}_{\mathrm{I}}^+$ defines a linear functional on the $\vec{\omega}$, namely that which associates to $\vec{\omega}$ the number $\vec{\omega} \,.\, \vec{\omega}_{\mathrm{I}}^+$, and any linear functional can be obtained from one and only one vector of the $\vec{\omega}^+$ — space by means of the inner product of the n-dimensional metric.

The possibility of reinterpreting the Clifford algebra of a $2\,p$-dimensional real space as a kind of algebra of a p-dimensional complex space is particularly natural in the case of the $2\,p$-dimensional euclidean metric, in which there is a one-one correspondence between the real $\vec{V}$ and the complex $\vec{\omega}$. *The euclidean* C_{2p} *can be identified to an algebra of the unitary geometry of the complex* $\vec{\omega}$ — *space. Now* W *and* W_a^+ *are the symbols of complex conjugate vectors and it is convenient to replace the* W_a^+ *by new symbols* W_a^*. Thus the multiplication rules [6] for the euclidean case become

$$W_a W_b + W_b W_a = 0 \ , \ W_a^* W_b^* + W_b^* W_a^* = 0,$$

$$W_a W_b^* + W_b^* W_a = \delta_{a,b}\, 1_C \qquad [8]$$

The multiplication rules [8] are analogous to those of the creation and anihilation operators of the quantum theory of a system of particles obeying the Fermi statistics, developed by Jordan and Wigner (2). Now there is only a finite number p of w_a, whereas there are infinite W_a in the Jordan-Wigner formalism. Thereby it is reasonable to call the W_a, W_a^* algebra a p-dimensional Jordan-

Wigner algebra. The original Jordan-Wigner formalism may be viewed as a geometric algebra of a separable complex Hilbert space of infinite dimensionality.

The choice of the totally isotropic $\vec{\omega}$ *and* $\vec{\omega}^+$ *vector spaces depends on the choice of a null basis* $\vec{W}_a$, $\vec{W}_a{}^+$, *but it does not depend on that of the basic vectors* $\vec{I}_j$ *of the cartesian frame of coordinates, when the* $\vec{J}_a$ *are not related to the vectors* $\vec{I}_j$. When $n = 2\,p$, it is possible to take a set of $\vec{W}_a$ and $\vec{W}_a{}^+$ as basic vectors $\vec{I}_j$. This is particularly convenient in the case of the maximally indefinite metrics in which we may take $2\,p$ real $\vec{W}_a$ and $\vec{W}_a{}^+$, as we have already seen.

The affine algebra G_n.

2. Let the $\vec{I}_j$ and $\underset{\rightarrow}{I^i}$ be the basic contravariant and covariant vectors of a n-dimensional affine space. We shall denote the contravariant vectors by $\vec{V}$ and the covariant ones by $\underset{\rightarrow}{U}$. Thus $\vec{V} = V^i \vec{I}_j$ and $\underset{\rightarrow}{U} = U_j \underset{\rightarrow}{I^i}$. The invariant $V^i U_j$ will be denoted by $<V, U>$. The elementary vector calculus involves an affine product of contravariant vectors $\vec{V} \wedge \vec{V}'$ and a metric product $\vec{V} \times \vec{V}' = g_{jk} V^i V'^k$. That calculus does not distinguish the contravariant and covariant vectors. Such a procedure is not inconvenient as long as one deals only with an euclidean space and uses only orthogonal cartesian coordinates.

The Grassmann algebra is an affine calculus of contravariant vectors, closely related to the ordinary vector calculus. In the Grassmann algebra the generators are the symbols I_j of the $\vec{I}_j$ and a unity (1) with the anticommutative multiplication rule $I_j I_k + I_k I_j = 0$. It is also possible to build a Grassmann algebra of the covariant vectors generated by a unity (1) and the symbols I^i of the $\underset{\rightarrow}{I^i}$, with the rule $I^i I^k + I^k I^i = 0$. Schönberg (1) introduced the algebra G_n of the contravariant and covariant vectors, with a unity (1), generated by the $2\,n$ elements I_j and I^i with the multiplication rules

$$I_j I_k + I_k I_j = 0 \ , \ I^i I^k + I^k I^i = 0 \ , \ I_j I^k + I^k I_j = \delta_j{}^k (1) \qquad [1]$$

In the algebra G_n the vectors $\vec{V}$ and $\underset{\rightarrow}{U}$ are represented by symbols V and U with the multiplication rules

$$V\,V' + V'\,V = 0 \ , \ U\,U' + U'\,U = 0 \ ,$$
$$V\,U + U\,V = \ < V,\ U > (1) \qquad [2]$$

Let us consider the direct sum of the $\vec{V}$ and $\underset{\rightarrow}{U}$ linear spaces. We may take the $\vec{I}_j$ and $\underset{\rightarrow}{I^i}$ as a basis of the sum space. Thus any vector $\vec{\Omega}$ of the sum-space can be represented as follows:

$$\vec{\Omega} = V^i\,\vec{I}_j + U_j\,\underset{\rightarrow}{I^i} = \vec{V} + \underset{\rightarrow}{U}.$$

The affine geometry of the n-dimensional space leads naturally to the introduction of a metric in the $\vec{\Omega}$ *— space, for which the square of the length of* $\vec{\Omega}$ *is* $<V,\ U>$ *and the inner product of* $\vec{\Omega}$ *and* $\vec{\Omega}'$ *is* $^1/_2\ (<V,\ U'> + <V',\ U>)$. *The* $\vec{V}$ *and* $\underset{\rightarrow}{U}$ *may be regarded as special kinds of null vectors of the* $\vec{\Omega}$ *— space constituting two totally isotropic n-dimensional vector spaces. The* $2\,n$ *vectors* $\vec{I}_j$ *and* $\underset{\rightarrow}{I^i}$ *constitute a null basis of the* $\vec{\Omega}$ *— space endowed with the above* $<V,\ \vec{U}>$ *metric.*

Schönberg pointed out (1) that the affine algebra G_n is the Clifford algebra C_{2n} of the real $2\,n$-dimensional $\vec{\Omega}$ — space endowed with the maximally indefinite metric $<V,\ U> \ = V^i\,U_j$. The approach to the theory of C_{2p} given in section 1 is obviously suggested by the theory of G_n as a C_{2n}. There, the $2\,p$-dimensional $\vec{V}$ — space was essentially treated as the direct sum of the two totally isotropic $\vec{\omega}$ and $\vec{\omega}^+$ vector-spaces. When the vectors $\vec{W}_a$ and $\vec{W}_a^+$ of a null basis are taken as basic cartesian vectors of the $2\,p$-dimensional cartesian space, the expression of the square of the length of V is $\Sigma\, \omega_V{}^a\, \omega_V{}^{+a}$, analogous to the expression $V^i\,U_j$ of the square of the length of a $2\,n$-dimensional vector $\vec{\Omega}$.

G_n may be regarded as the basic elementary vector calculus of the n-dimensional affine geometry. It involves only the affine vector product, as the Grassmann algebra, but it deals symmetrically with the covariant and contravariant vectors, whilst the

Grassmann algebra deals only with one of the two kinds of vectors. The symmetry with respect to the two kinds of vectors gives to G_n an extremely simple algebraic structure: *G_n and the algebra of the $2^n \times 2^n$ matrices taken over the same field of characteristic zero are isomorphic.* Schönberg (3) showed that G_n is the algebra of the linear operators on the vectors ξ of the linear space of dimensionality 2^n obtained by the direct sum of the linear spaces of the covariant antisymmetric tensors of the orders 0, 1, 2,.., n of the n-dimensional affine space. It follows from the symmetry of G_n with respect to the covariant and contravariant vectors that it is also the algebra of the linear operators of the vector space obtained by the direct sum of the linear spaces of the antisymmetric contravariant tensors of the orders 0, 1,...,n of the n-dimensional affine space. We shall denote by η the vectors of the latter 2^n — dimensional space.

When G_u is treated as the algebra of the linear operators on the ξ, the ξ appear as contravariant spinors and the η as covariant spinors of the $2\,n$-dimensional $\vec{\Omega}$ — space endowed with the maximally indefinite metric $<V,\ U>$. When G_u is regarded as the algebra of the linear operators on the η, they appear as the contravariant spinors and the ξ as the covariant spinors of the $\vec{\Omega}$ — space with the metric $<V,\ U>$.

There is a set of n commutable idempotent elements $N_j = I^j I_j$ associated to the basic cartesian vectors. The $\hat{N}_j = I_j I^j$ are also idempotent and commutable. The elements $P = \hat{N}_1 \ldots N_n$ and $Q = N_1 \ldots N_n$ do not depend on the choice of the $\vec{I}_j$ and are also idempotent. It is easily seen that $I_j P = 0$, $P I^j = 0$, $I^j Q = 0$ and $Q I^j = 0$. Let us introduce the elements $P_{j_1,\ldots,j_r} = P\, I_{j_r} I_{j_1}$ and $P^{j_1,\ldots,j_s} = I^{j_1} \ldots I^{j_s} P$; we shall take $j_1 < j_2 \ldots < j_s$ or j_r. It is easily seen that $P_{(j)}\, P^{(k)} = \delta^{(k)}_{(j)}\,(1)$. By means of the $P_{(j)}$ and $P^{(k)}$ we build the 2^{2n} elements $P^{(k)}_{(j)}$ with all the ordered sets of values (j) and (k) : $P^{(k)}_{(j)} = P^{(k)}\, P_{(j)}$. It is easily seen that

$$P^{(k)}_{(j)}\, P^{(i)}_{(h)} = \delta^{(i)}_{(j)}\, P^{(k)}_{(h)} \qquad [3]$$

It follows from the orthogonality relations [3] that the 2^{2n} elements $P^{(k)}_{(j)}$ are linearly independent and can be taken as a basis of G_n. *This choice of the basis gives to G_n the form of the algebra of the linear operators on the above ξ spinors.* In order to give to G_n the form of the algebra of the linear operators on the η it suffices to take as basis of G_n the 2^{2n} elements $Q^{(k)}_{(j)} = Q_{(j)}\, Q^{(k)}$ with $Q_{j_1,\dots,j_r} = I_{j_1} \dots I_{j_r}\, Q$ and $Q^{k_1,\dots,k_s} = Q\, I^{k_s} \dots I^{k_1}$. Since

$$Q^{(k)}_{(j)}\, Q^{(i)}_{(h)} = \delta^{(k)}_{(h)}\, Q^{(i)}_{(j)} \qquad [4]$$

the 2^{2n} elements $Q^{(k)}_{(j)}$ corresponding to all the 2^{2n} different pairs of ordered sets (j), (k) are linearly independent.

Let us consider the left ideal $G_n P$ constituted by the products ΓP of the elements Γ of G_n by the idempotent P. $G_n P$ is a vector space of dimensionality 2^n having as basis the 2^n elements $P^{(j)}$ with all the distinct ordered sets (j). *$G_n P$ is essentially the direct sum of the linear spaces of the covariant antisymmetric tensors of all orders, endowed with a product that renders it an algebra.* The Γ may be regarded as the linear operators on the elements $\Gamma' P$ of $G_n P$ that transform $\Gamma' P$ into $\Gamma\, \Gamma' P$. *Thus G_n appears as the algebra of the linear operators on the ξ.*

The linear space $G_n Q$ admits as a basis the set of the 2^n elements $Q_{(j)}$ corresponding to all the distinct ordered sets (j). *$G_n Q$ is essentially the direct sum of the linear spaces of the antisymmetric contravariant tensors of all orders, endowed with a product that renders it an algebra.* The elements of G_n can be associated to linear operators on the vectors of $G_n Q$. Thus G_n appears as the algebra of the linear operators on the η.

Matrix representations of G_n and C_{2p}.

3. The representation of the elements of G_n by linear operators on the ξ, discussed in section 2, gives a natural representation of the elements of G_n over the real numbers by real $2^n \times 2^n$ matrices. *In that representation the matrix of I_j is the transpose of that of I^j and the matrices of the n commutable idempotent elements $N_j = I^j I_j$ are diagonal.* The matrices of the elements $J_j = I_j + I^j$ are therefore symmetrical and those of the $J_{j+n} = I_j - I^j$ are antisymmetrical. The J_α are the symbols of 2 n-dimensional orthogonal vectors $\vec{J}_\alpha$.

The generators w_a and $w_a{}^+$ of a $C^{(g)}_{2p}$ over the complex numbers introduced in section 1 have the same multiplication rules as the generators I_a and I^a of a G_p. We may therefore represent any C_{2p} over the complex field by $2^p \times 2^p$ complex matrices, in such a way that for all a the matrices of w_a and $w_a{}^+$ be both real and mutually transposed, hence adjoint, by means of the above representation of G_p. In the present representation of $C^{2p}_{(g)}$ the matrices of the real V are not always hermitian when $g_{jk} V^i V^k > 0$ and anti-hermitian when $g_{jk} V^i V^k < 0$. The matrices of the elements $w_a{}^+ w_a$ are diagonal and real.

The present representation of G_n gives also a representation of the n-dimensional Jordan-Wigner algebra over the real numbers by $2^n \times 2^n$ real matrices, which is analogous to the well known representation of the quantum field theory of the fermions, in which the occupation-number operators of the fermion states are diagonalized.

The above vectors $\vec{J}_\alpha$, $\vec{J}_j = \vec{I}_j + \vec{I^i}$ and $\vec{J}_{j+n} = \vec{I}_j - \vec{I^i}$, constitute an orthonormal basis of the 2 n-dimensional $\vec{\Omega}$ — space with the metric $< V, U >$. The n-vectors $\vec{J}_{\alpha_1,\ldots,\alpha_n} = \vec{J}_{\alpha_1} \wedge \vec{J}_{\alpha_2} \wedge \ldots \wedge \vec{J}_{\alpha_n}$ are associated to the elements $J_{\alpha_1,\ldots,\alpha_n} = J_{\alpha_1} \ldots J_{\alpha_n}$. The set of those elements constitutes a symbolic antisymmetric tensor of the order n of the $\vec{\Omega}$ — space. It is easily seen that the square of any of those elements is either (1) or — (1). and that

$$J_{\alpha_1,\ldots,\alpha_n} I_j J^{-1}_{\alpha_1,\ldots,\alpha_n} = e^j_{\alpha_1,\ldots,\alpha_n} I^i \text{ with } e^j_{\alpha_1,\ldots,\alpha_n} = 1 \text{ or } -1 \qquad [1]$$

We have $e^j_{1,\ldots,n} = (-1)^{n-1}$ and $e^j_{n+1,\ldots,2n} = (-1)^n$. We shall take when n is odd $J = J_{1,\ldots,n}$ and when n is even $J = J_{n+1,\ldots,2n}$ in order that for all n

$$J I_j J^{-1} = I^i \quad \text{and} \quad J I^i J^{-1} = I_j \qquad [2]$$

The inner automorphism $\Gamma \longrightarrow J \Gamma J^{-1}$ of G_n corresponds to the transposition of the matrices of the elements associated to the $\vec{\Omega}$, in the above representation of G_n.

Since $J N_j J^{-1} = \hat{N}_j$, we have $J P J^{-1} = Q$ and

$$J P^{k_1,\ldots,k_s}_{j_1,\ldots,j_r} J^{-1} = Q^{j_1,\ldots,j_r}_{k_1,\ldots,k_s} \qquad [3]$$

The equations [2] and [3] are not covariant for a change of the basic vectors $\vec{I}_j$. This results from the fact that J depends on the choice of the $\vec{I}_j$.

We shall denote $J_{n+1,..,2n}$ by C. It is easily seen that

$$C\, P^{j_1,\dots,j_r} = (-1)^{g_r} \quad \{ (n-r)! \}^{-1} \quad \varepsilon_{j_1,\dots,j_n} \; P^{j_{r+1},\dots,j_n} \qquad [4]$$

$$2\, g_r = (2\,n + r)\, (r-1)^k \qquad [5]$$

Hence

$$C \{ (r!)^{-1}\, \xi_{j_1,\dots,j_r}\, P^{j_1,\dots,j_r} \} = \Sigma \{ (n-r)! \}^{-1}\, \hat{\eta}^{k_1,\dots,k_{n-r}}\, P^{k_1,\dots,k_{n-r}} \quad [6]$$

with

$$(r!)\; \hat{\eta}^{j_{r+1},\dots,j_n} = (-1)^{g_r}\; \varepsilon^{j_1,\dots,j_n}\; \xi_{j_1,\dots,j_r} \qquad [7]$$

Equations [6] *and* [7] *show that the linear operator on the* ξ *corresponding to the element* C *transforms* ξ *into the* $\hat{\eta}$ *whose components are defined by* [7] *with* $r = 0, 1,..,n$. *The element* C *is therefore related to the duality of the antisymmetric tensors of the n-dimensional space.*

Alternative construction of G_n.

4. The tensor product of the linear spaces of the $\vec{V}$ and $\underset{\rightarrow}{U}$ gives the linear space of the mixed tensors $A_j{}^k$. We can associate to $A_j{}^k$ the linear operator A on the $\vec{V}$ defined as follows: $A\,\vec{V} = A_j{}^k V^j \vec{I}_k$. *Thus the space of the tensors* $A_j{}^k$ *becomes an associative algebra, the product of the tensors* $A_j{}^k$ *and* $B_j{}^k$ *taken in a given order being the tensor associated to the product of the operators* A *and* B *taken in the same order*: $(A_j{}^k)\,(B_j{}^k) = (C_j{}^k)$ with $C_j{}^k = A_h{}^k\, B_j{}^h$.

The tensor $A_j{}^k$ may also be associated to a linear operator $\tilde{A}$ on the $\underset{\rightarrow}{U}$ defined as follows: $\tilde{A}\,\underset{\rightarrow}{U} = U_k\, A_j{}^k\, \underset{\rightarrow}{I}{}^j$. *The space of the* $A_j{}^k$ *is now endowed with the structure of an associative algebra in which the product of* $A_j{}^k$ *and* $B_j{}^k$ *is the tensor of the product of the corresponding operators on the vectors* $\underset{\rightarrow}{U}$: $\{ A_j{}^k \}\; \{ B_j{}^k \} = \{ A_j{}^h B_h{}^k \}$. The present algebra of the $A_j{}^k$ is the reciprocal of that corresponding to the linear operators on the contravariant vectors $\vec{V}$.

The above considerations can obviously be applied to any pair of dual spaces of finite dimensionality. The linear spaces of the ξ and η of section 2 are dual ones. Since G_n is the algebra of the linear operators on the ξ, it is the algebra obtained from the tensor product of the ξ and η spaces that corresponds to the operators on the ξ. G_n is also the algebra of the linear operators on the η obtained from the tensor product of the ξ and η spaces. To any set of $(n+1)^2$ tensors $A^{k_1,..,k_s}_{j_1,..,j_r}$ $(r, s = 0,,1,..,n)$, antisymetrical with respect to the indices j and also with respect to the indices k, correspond two elements $\sum'_{r,s} A^{k_1,..,k_s}_{j_1,..,j_r} P^{j_1,..,j_r}_{k_1,..,k_s}$ and $\sum'_{r,s} A^{k_1..,k_s}_{j_1,..,j_r} Q^{j_1..j_r}_{k_1..k_s}$ of G_n, the former corresponding to a linear operator on the ξ and the latter to a linear operator on the η, (the sign Σ' indicates that the summation is taken only over the ordered sets with $j_1 < j_2 < \, < \ldots < j_r$ and $k_1 < k_2 < \ldots < k_s$).

The double rôle of G_n as operator algebra on the ξ and η is related to the tensor duality, in which $A^{k_1,..,k_s}_{j_1,..,j_r}$ corresponds to $B^{j_{r+1},..,j_n}_{k_{s+1},..k_n} = \, = \delta^{j_1,..,j_n}_{k_1,..,k_n} A^{k_1,..,k_s}_{j_1,..,r} (r!\, s!)^{-1}$, δ denoting the generalized Kronecker delta. This follows from the fact for any values of the j and k

$$Q^{k_1,..,k_s}_{j_1,..,j_r} = (-1)^{r\,(r-1)/2\,+\,s\,(s-1)/2} \{(n-r)!\,(n-s)!\}^{-1} \delta^{k_1,..,k_n}_{j_1,..,j_n} P^{j_{r+1},..,j_n}_{k_{s+1},..,k_n} \tag{1}$$

The passage from the P basis of G_n to its Q basis corresponds therefore to the replacement of the tensors A by their dual B.

The present definition of G_n as the algebra of the linear operators of the ξ — space leads immediately to the introduction of the $P^{(k)}_{(j)}$, which correspond to the dyadics associated to the basic vectors of the ξ — space. The element (1) corresponds to the operator unity. Hence we have

$$(1) = \sum_r (r!)^{-1} P^{j_1,..,j_r}_{j_1,..j_r} \tag{2}$$

We can now define the I_j and I^j in terms of the $P^{(k)}_{(k)}$

$$I_j = \sum_r (r!)^{-1} P^{j_1,..,j_r}_{j,j_1,..,j_r} \quad , \quad I^j = \sum_r (r!)^{-1} P^{j,j_1,..j_r}_{j_1,..,j_r} \tag{3}$$

Since $C^{(g)}_{2p}$ is the G_p of the totally isotropic p-dimensional ω-space introduced in section 1, we may define $C^{(g)}_{2p}$ as the algebra of the linear operators on the vectors of the direct sum of the linear spaces of the covariant antisymmetric tensors of all orders of the $\vec{\omega}$-space. Those 2^p-dimensional vectors are spinors of the 2 p-dimensional metric $\vec{V}$-space. *Thus we have a direct simple definition of the 2 p-dimensional spinors and also a definition of the Clifford algebra $C^{(g)}_{2p}$ by means of those spinors.*

It is important to note that in the above definition of G_n we may replace the ξ - space by the direct sum of the linear spaces of the antisymmetric covariant relative tensors of weight α and the η-space by the direct sum of the linear spaces of the contravariant antisymmetric relative tensors of weight - α. *The choice* $\alpha = 0$ *is actually not the most interesting one from the point of view of the tensor duality*: *by taking* $\alpha = - 1/2$ *we get in the linear space of the* ξ *the invariant bilinear forms*

$$(\xi, \xi')_f = \sum_{r=0}^{n} \{ r! \, (n - r)! \}^{-1} \, \varepsilon^{j_1,..,j_n} \, \xi_{j_1,..,j_r} \, \xi'_{j_{r+1},..,j_n} \, f_r \qquad [4]$$

$\varepsilon^{j_1,..,j_n}$ *denoting the well known Ricci tensor density and the* f_r *arbitrary numerical coedficients.* The existence of those forms follows from the duality

$$\eta^{j_{r+1},..,j_n} = (r!)^{-1} \, \varepsilon^{j_1,..,j_n} \, \xi_{j_1,\ldots,j_r} \qquad [5]$$

The choice of the weight $\alpha = - 1/2$ *is however not possible in the real domain, because a real relative tensor of weight* $- 1/2$ *in a cartesian system has imaginary components in another system with opposite orientation.*

The equation $(\xi, \xi')_f = O$ is invariant with respect to the changes of cartesian coordinates in the n-dimensional affine space. It does not depend on the choice of the weight α given to the $\xi_{j_1,..,j_r}$. Let us interpret the components ξ_ρ of ξ as homogeneous projective coordinates in a space of dimensionality $2^n - 1$, and the ξ_ρ' in a similar way. The equation $(\xi, \xi')_f = 0$ associates a hyperplane of the projective ξ — space to a point of the ξ — space and conversely.

The fundamental polarity in the projective ξ*-space*

5. We shall now treat the components of ξ as homogeneous projective coordinates of a point in a $2^n - 1$ dimensional space and the components of η as homogeneous hyperplane coordinates in the same space. We shall write the equations of the tensor duality as follows

$$\hat{\eta}^{j_{r+1},..,j_n} = (-1)^{g_r} \, \varepsilon^{j_1,..,j_n} \, \xi_{j_1,..,j_r} \, (r!)^{-1} \qquad [1]$$

$$2 \, g_r = (2 \, n + r) \, (r - 1) \qquad [2]$$

The set of equations [1] for $r = 0, 1, .., n$ defines a correlation $\hat{\eta}^{\rho} = C^{\rho,\sigma} \xi_{\sigma}$ in the ξ-space. *Since* $C^{\sigma,\rho} = (-1)^{n(n+1)/2} C^{\rho,\sigma}$ *the correlation is involuctoric. There is a fundamental polarity in the projective ξ-space associated to the duality of the antisymmetric tensors.* The bilinear equation of the polarity is $(\xi, \xi') = 0$

$$(\xi, \xi')_{k_1,..,k_n} = \sum_r \{r!\,(n-r)!\}^{-1} \delta^{j_1,..,j_n}_{k_1,..,k_n} \xi_{j_1,..,j_r} \xi'_{j_{r+1},..,j_n} (-1)^{a_r} \quad [3]$$

$$(\xi', \xi) = (-1)^{n(n+1)/2} (\xi, \xi') \quad [4]$$

When $n(n+1)/2$ is even, we have a polarity with respect to the hyperquadric $(\xi, \xi) = 0$. When $n(n+1)/2$ is odd, the form (ξ, ξ') is antisymmetric, so that the polarity is taken with respect to a linear complex.

The ξ-polarity corresponds to the well known polarity of the spinor theory. *The above discussion shows that the spinor polarity of a 2 p-dimensional metric space is essentially the polarity of the projective ξ-space of a p-dimensional affine space, arising from the duality of the p-dimensional antisymmetric tensors.* It is known from the spinor theory that the spinor polarity is invariant for the linear transformations of the spinors induced by the 2 p-dimensional rotations of the metric space. Since the ξ are the spinors of the 2 n-dimensional space of the $\vec{\Omega} = \vec{V} + \underset{\rightarrow}{U}$ endowed with the maximally indefinite metric $<V, U>$, we see that the ξ-polarity must be invariant for the linear transformations of the ξ induced by the $2n$-dimensional rotations of the $<V, U>$ metric, which constitute a group larger than the n-dimensional linear group.

The ξ-polarity is obviously related to the transformation of the linear space of the ξ defined by the element C of G_n introduced in section 3. We have $\hat{\eta} = C_{op}\, \xi$, C_{op} denoting the linear operator corresponding to the element C. It follows from the equations [3] and [1] that

$$(\xi, \xi')_{k_1..,k_n} = \varepsilon_{k_1,..,k_n} \sum_r (r!)^{-1} (C_{op}\, \xi)^{j_1,..,j_r} \xi'_{j_1,..,j_r} \quad [5]$$

The above coefficients $C^{\rho,\sigma}$ are simply matrix elements of C_{op}.

The ξ as functions of n two-valued variables.

6. Let us consider n variables N'_j which take only the values 0 and 1. The linear space of the real functions $F(N')$ is of dimen-

sionality 2^n. It is possible to establish a one-one linear correspondence between the ξ and the $F(N')$ by associating to ξ the function $\xi(N') = \sum'_{j,r} \xi_{j_1,\dots,j_r} F_{j_1,\dots,j_r}(N')$, the summation being taken only over the ordered sets of values $j_1 < j_2 < \dots < j_r$ and $F_{j_1,\dots,j_r}(N') = \delta(N'_{j_1} - 1) \dots \delta(N'_{j_r} - 1)\, \delta(N'_{j_{r+1}}) \dots \delta(N'_{j_n})$, with $\delta(u) = \delta_{u,0}$.

We may write $P^{k_1,\dots,k_s}_{j_1,\dots,j_r} = P(N', N'')$ with $F_{j_1,\dots,j_r}(N')\, F_{k_1,\dots,k_s}(N'') = 1$, $j_1 < \dots < j_r$ and $k_1 < \dots < k_s$. There is a one-one linear correspondence between the elements Γ of G_n and the functions

$$\Phi(N', N'') : \Gamma \longrightarrow \Phi_\Gamma(N', N''),$$

with

$$\Gamma = \sum_{N', N''} \Phi_\Gamma(N', N'')\, P(N', N'').$$

The values of the function Φ_Γ are the matrix elements of Γ in the representation of G_n discussed in section 3.

The two-valued variable N_j' is closely related to the element N_j, since $N_j\, P^{j_1,\dots,j_r} = N_j'\, P^{j_1,\dots,j_r}$ and $P_{j_1,\dots,j_r}\, N_j = N_j'\, P^{j_1,\dots,j_r}$, with $N_j' = 1$ when j coincides with one of the indices $j_1, \dots, j_r$ and $N_j' = 0$ when this does not happen. *There is an idempotent element* $N_\Omega = \langle V, U \rangle^{-1}\, U\, V$ *associated to any non-isotropic direction of the space of the* $\vec{\Omega}$, *since* N_Ω *is not changed when* $\vec{\Omega}$ *is replaced by* $a\,\vec{\Omega}$, *a being a non-null number.* N_j is the N_Ω of the direction of $\vec{I}_j + \vec{I}^j$. The correspondence between the N_Ω and the non-isotropic directions of the $\vec{\Omega}$ — space is not one-one, for the same N_Ω is associated to all the $a\,\vec{V} + b\,\vec{U}$ with $ab \neq 0$. There is another idempotent element $\hat{N}_\Omega = \langle V, U \rangle^{-1}\, V\, U = (1) - N_\Omega$ associated to the $a\,\vec{V} + b\,U$ with $ab \neq 0$.

We can establish a one-one linear correspondence between the η and the functions of the n two-valued variables $\hat{N}_j' = 1 - N_j'$, which are related to the elements $\hat{N}_j$ of G_n. There is also a one-one linear correspondence between the elements Γ of G_n and the functions $\Psi(\hat{N}', \hat{N}'')$ of the two-valued variables $\hat{N}_j'$ and $\hat{N}_j''$.

It may be more convenient to use the variables $2\,N_j' - 1 = N_j' - \hat{N}_j'$, that take the values 1 and -1, instead of the N_j'

or $\hat{N}_j'$ with the values 0 and 1. Since $2 N_j - (1) = (I_j + I^j)(I_j - I^j)$, the $2 N_j' - 1$ correspond to indices of inner orientation of two dimensional manifolds parallel to the pairs of vectors $\vec{I}_j + \underset{\rightarrow}{I^j}$ and $\vec{I}_j - \underset{\rightarrow}{I^j}$ of the $\vec{\Omega}$ — space. The n-dimensional geometrical interpretation of those indices is not quite obvious.

The close relation between the theory of the antisymmetric tensors of an affine space and that of the functions of two-valued variables is very interesting, especially in the four dimensional case, because of its physical importance. The existence of such a relation indicates that there are fundamental discrete properties of space and space-time, as yet not well understood, that may play an important part in the theory of the elementary particles.

The identification of the ξ with the functions of n two-valued variables N_j' corresponds to the description of the states of a quantized field of fermions by functions of an enumerable infinity of two-valued variables analogous to the above N_j'. The states of the quantized fermion field are vectors of a linear representation-space of the Jordan-Wigner algebra of the emission and absorption operators, which is isomorphic to a G_∞. The field variables analogous to our N_j' give the numbers of fermions in the different particle-states. They have the values 0 and 1, because in each particle state there may be at most one fermion, as a consequence of the Pauli principle.

The possibility of using either the $\hat{N}_j'$ or the N_j' in our formalism corresponds to the well known symmetry of the formalism of the quantized fermion field with respect to the particles and the "holes". The exchange of the N_j' and the $\hat{N}_j'$ corresponds to that of the particles and the "holes", because we have $\hat{N}_j' = 1 - N_j'$, so that the value 1 of N_j' corresponds to the value 0 of $\hat{N}_j'$ and conversely.

ξ may be regarded as a function of the sub-sets of the set of the n basic vectors $\vec{I}_j$, whose value for the sub-set $(\vec{I}_{j_1}, \ldots, \vec{I}_{j_r})$ with $j_1 < j_2 < \ldots < \ldots < j_r$ is $\xi_{j_1, \ldots, j_r}$. The antisymmetric covariant tensors of the order r are thus associated to the sub-sets with r elements of the set of the $\vec{I}_j$. In particular the scalars correspond to the functions of the empty sub-set.

The 2p-dimensional spinors.

7. The spinors of a n dimensional metric space are the vectors of a linear space in which $C_n^{(g)}$ has an irreducible representation. The identity of $C_{2p}^{(g)}$ and the algebra generated by the W_a and W_a^+ allows to obtain in a simple way the spinors of $C_{2p}^{(g)}$, as we shall show. Let us introduce the idempotent elements $\hat{N}_a = W_a W_a^+$. The $\hat{N}_a$ are all commutable, so that $P = \hat{N}_1 \dots \hat{N}_p$ is also idempotent: $P^2 = P$. Since $W_a \hat{N}_a = \hat{N}_a W_a^+ = 0$ and both W_a and W_a^+ are commutable with $\hat{N}_b$ when $b \neq a$, we have

$$W_a P = P W_a^+ = 0 \qquad [1]$$

Let us introduce the elements

$$P_{a_1,..,a_r,\, b_1,..,b_s} = W_{a_1}^+ \dots W_{a_r}^+ \; P \; W_{b_s} \dots W_{b_1} \qquad [2]$$

which are antisymmetrical with respect to the r indices a and with respect to the to the s indices b too. The $P_{(a)\,;\,(b)}$ will be taken with $a_1 < a_2 < \dots < a_r$ and $b_1 < b_2 \dots < b_s$. We have

$$P_{(a)\,;\,(b)} = P_{(a)\,;\,(\,)} P_{(\,)\,;\,(b)} \qquad P_{(\,)\,;\,(b)} P_{(a)\,;\,(\,)} = \delta_{(a)\,;\,(b)} P \qquad [3]$$

Hence

$$P_{(a)\,;\,(b)} P_{\underline{(a)}\,;\,\underline{(b)}} = \delta_{(b)\,;\,\underline{(a)}} P_{(a)\,;\,\underline{(b)}} \qquad [4]$$

It follows from equation [4] that the elements $P_{(a)\,;\,(b)}$ are all linearly independent and, since there are 2^{2p} of them, any element of $C_{2p}^{(g)}$ must be a linear combination of the $P_{(a)\,;\,(b)}$, the number of linearly independent elements of a C_{2p} being also 2^{2p}. Moreover $C_{2p}^{(g)}$ must be equivalent to the algebra of the matrices with 2^p lines and 2^p columns, because of the multiplication rule [4]. $C_{2p}^{(g)}$ *is the algebra of the linear operators of the vector space* $C_{2p}^{(g)} P$ *(constituted by the right products of its elements by* P*) since the left multiplication of any element of* $C_{2p}^{(g)} P$ *by an element of* $C_{2p}^{(g)}$ is *again an element of that vector space.*

The general form of the elements of $C_{2p}^{(g)} P$ is $A\, P + A^a w_a^+ P + {} + \Sigma' A^{a_1, a_2} w_{a_1}^+ w_{a_2}^+ P + \dots + \Sigma' A^{a_1,\dots,a_r} w_{a_1}^+ \dots w_{a_r}^+ P + \dots + {} + A^{1,\dots,p} w_1^+ \dots w_p^+ P$, the numerical coefficients $A^{a_1,\dots,a_r}$ being the components of a p-dimensional antisymmetrical contravariant tensor of order r. Σ' denotes that the summation is taken only for sets (a) such that $a_1 < a_2 < a_3 < \dots$. *The structure of the elements* *of $C_{2p}^{(g)} P$ shows that the 2^p components of a spinor of $C_{2p}^{(g)}$ constitute a set of antisymmetric tensors of the orders 0, 1, 2,.., p of the p-dimensional $\vec{\omega}$ — space.*

The existence of relations between the spinors of a $2p$-dimensional metric flat space and the p-dimensional antisymmetric tensors is well known from the Cartan approach to the theory of spinors (3). The above discussion is essentially the same given by Schönberg (4) in the theory of G_n. Cartan defines firstly the spinor in connection with the flat manifolds of maximal dimensionality lying on the null hypercone $g_{jk} x^j x^k = 0$ of a space of odd dimensionality $n = 2p + 1$ and later introduces the Clifford algebra. We start from the Clifford algebra and obtain the spinor from a left-ideal.

Let us consider now the right ideal $P\, C_{2p}^{(g)}$ constituted by the products $P\, \Gamma$, the Γ being the elements of $C_{2p}^{(g)}$. The elements of $C_{2p}^{(g)}$ may be viewed as linear operators on the vectors of the 2^p-dimensional linear space $P\, C_{2p}^{(g)}$ since to any Γ corresponds the linear transformation $P\, C_{2p}^{(g)} \rightarrow P\, C_{2p}^{(g)}\, \Gamma$. Thus we get a linear representation of the reciprocal algebra of $C_{2p}^{(g)}$, because the operator corresponding to the product $\Gamma_1 \Gamma_2$ is the product of the operators corresponding to Γ_1 and Γ_2 taken in the reversed order. The general form of the elements of the right ideal $P\, C_{2p}^{(g)}$ is $P\, B + P\, W_b\, B^b + \Sigma'\, P\, W_{b_2} W_{b_1} B^{b_1, b_2} + \ldots + \Sigma'\, P\, W_{b_r} \ldots W_{b_2} W_{b_1} B^{b_1, \ldots, b_r} + \ldots + P\, W_n \ldots W_1\, B^{1, \ldots, n}$ the $B^{b_1, \ldots, b_r}$ being numerical coefficients antisymmetrical with respect to the indices b, which are components of antisymmetrical tensors of the $\vec{\omega}^{+}$ — space. The numbers B, B^b, $B^{b_1, b_2}, \ldots, B^{1, \ldots, p}$ are the components of a covariant spinor of $C_{2p}^{(g)}$.

The n elements $N_a = W_a^+ W_a$ are idempotent and commutable as the $\hat{N}_a$. By means of them we build $Q = N_1 \ldots N_n$, which satisfies the conditions $W_a^+ Q = 0$, $Q\, W_a = 0$ and $Q^2 = Q$. Let us introduce the $Q_{(b)\,;\,(a)} = W_{b_1} \ldots W_{b_s} Q\, W_{a_r}^+ \ldots W_{a_1}^+$, analogous to the $P_{(a)\,;\,(b)}$, taking them with $b_1 < b_2 < \ldots < b_s$ and $a_1 < a_2 < \ldots < a_r$. We have

$$Q_{(b)\,;\,(a)} = Q_{(b)\,;\,(\,)}\, Q_{(\,)\,;\,(a)} \qquad Q_{(\,)\,;\,(a)}\, Q_{(b)\,;\,(\,)} = \delta_{(b),\,(a)}\, Q \tag{5}$$

$$Q_{(b)\,;\,(a)}\, Q_{\underline{(b)}\,;\,\underline{(a)}} = \delta_{(a),\,\underline{(b)}}\, Q_{(b)\,;\,\underline{(a)}} \tag{6}$$

It follows fron [6] that qhe $Q_{(b)\,;\,(a)}$ are a set of 2^{2p} linearly independent elements.

The left ideal $C_{2p}^{(g)}Q$ can be taken as a vector space whose linear operators are the elements of $C_{(g)}^{2p}$. It is easily seen that the left ideal $C_{2p}^{(g)}Q$ is related to the antisymmetric tensors of the $\vec{\omega}^{+}$—space in the same way as $C_{2p}^{(g)}P$ to those of the $\vec{\omega}$ — space. The properties of $C_{2p}^{(g)}$ must obviously be completely symmetrical with respect to the $\vec{\omega}$ and $\vec{\omega}^{+}$ vector spaces. The W_a^{+} play in the Q formalism the role played by the W_a in the P formalism and conversely.

The P and Q formalisms are covariant for a change of basic vectors in the $\vec{\omega}$ — space and the associate change in the $\vec{\omega}^{+}$ — space, for

$$P = W_1 \dots W_n W_n^{+} \dots W_1^{+} \quad , \quad Q = W_1^{+} \dots W_n^{+} W_n \dots W_1 \quad [7]$$

The w-algebra of a finite set.

8. Let us consider a set S_p constitEted by p objects E_a. We shall associate to the E_a the symbols w_a and w_a^{+} of the generators of a p-dimensional Jordan-Wigner algebra, which will be regarded as an algebra of the set S_p and called the w-algebra of the set. *It follows from the considerations at the end of section 6 that the w-algebra is equivalent to that of the linear operators on the numerically-valued functions of the sub-sets of S_p.* The formalism of the spinors of a $C_{2p}^{(g)}$ developed in section 7 can be applied to the numerically-valued functions of the sub-sets of S_p.

We have regarded the Jordan-Wigner algebra of a quantized fermion field as an algebra of a separable Hilbert space, but it is also possible to view it as an algebra of the numerically-valued functions of the finite sub-sets of an enumerable infinite set of objects, namely the functions of a complete orthonormal set for the states of a fermion of the field. Thus the Jordan-Wigner algebra of the field appears as the infinite-dimensional analogue of the above w-algebras of finite sets.

The w-algebra over the complex numbers of a set with a single object is isomorphic to the Pauli-algebra of the spin operators, i.e. to the algebra of the linear operators of a two-dimensional complex vector space. This results from the fact that a S_1 possesses only two sub-sets: the empty sub-set and S_1 itself. Since the Pauli algebra is a fundamental geometric algebra of the three dimensional euclidean space and the four-dimensional minkowskian space-time:

the geometry of the flat space-time is fundamentally related to the w-algebra of a single object.

The w-algebra of a S_2 taken over the complex numbers is iso morphic to a C_4 over the complex numbers, hence to the Dirac algebra of the γ-matrices of the leptons and baryons. The above results indicate that the two simplest sets S_1 and S_2 are likely to play an important part in the theory of the structure of the world.

The Pauli algebra as an algebra of the space-time is that of the half-spinors, which are related to a distinguished screw-orientation of space. The w-algebra of S_1 is therefore associated to a minkowskian space-time with a preferred screw-orientation of space, the actual situation for the neutrino. The Dirac algebra includes the two screw-orientations of space. This suggests that the above S_1 and S_2 could be the sets of one and two screw-orientations, respectively.

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