# ON THE SPACES $L^1$ WHICH ARE ISOMORPHIC TO A $B^*$

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1. *Introduction.* The problem which we consider in this paper is a particular case of the following one: characterize the Banach spaces which are isomorphic (isometrically isomorphic) to the dual of another Banach space, (cf. [$D$]). This particular case is when the given space is of type $L$ in the sense of Kakutani ([$K_1$]) and it was explicitly proposed by Dieudonné ([$Di$]).. However the stem of Dieudonné's problem is much older and goes back to Gelfand, (cf. [$Ge$] and also [$KM$]). The best result in the isomorphic case, as far as we know, was given by Phelps, (cf. [$Ph$] and also [$P$]).

The purpose of this work is to give an account of the subject and to underline some problems. The paper is almost self-contained and precise references are given when a theorem is not proved. For the nomenclature and general references we mention [$DS$]. Finally, we want to note that many points of this paper were clarified thanks to a discussion with A. Benedek.

2. *Preliminary results.* Along the paper we consider only real Banach spaces and real valued Banach function spaces.

a) If $L^1$ is isomorphic (isometric) to the dual space $G^*$ of a Banach space G, there exists a closed subspace of $L^\infty$, $B$, such that $L^1$ is isomorphic (isometric) to $B^*$. Every $b^* \in B^*$ is the image of an element $f$ of $L^1$ verifying $b^*(b) = \int_X fb \, d\mu$ for every $b \in B$.

In the case of isometry $L^1$ may be called $B$-reflexive (cf. [$S_1$] and [$S_2$]) and $B$, $^*L^1$, ([$G$]). We shall sometimes use the word "dual" for this situation.

b) Let $L^1$ $(x, \Sigma, \mu)$ and $B$ be as in a) and $S$ a $\sigma$-finite set of $X$ with $\mu(S) = \infty$. Suppose $S = \Sigma_1^\infty A_k$, $0 < \mu(A_k) < \infty$. Then the space $L^1(X, \Sigma, \bar{\mu})$ with $\bar{\mu}$ defined by: $\bar{\mu}(E \cap A_k) = 2^{-k} \mu(A_k)^{-1} \mu(E \cap A_k)$, $\bar{\mu}(E) = \mu(E)$ if $E \cap A_k = \phi$ for every $k$, is isometric and lattice isomorphic to $L^1(\mu)$ so it is isomorphic (isometric) to the dual of a space $B$. This particularly means that the problem for a $\sigma$-finite space may be reduced to the same problem for a finite space.

c) Let $(X, \Sigma, \mu)$ be a measure space. Let $\|.\|$ be the $L^1$-norm and $[.]$ be a locally equivalent norm, that is, for every set of finite measure there exists $K$ and $k$, both greater than zero, such that: $k\,\|f\| \leqslant [f] \leqslant K\,\|f\|$ for every $f$ with support on this set.

LEMMA. *i)* $[.]$ *is an $L^1$-norm if and only if it is determined by a measure $\nu$ equivalent with $\mu$.*

*ii)* $[.]$ *verifies $[f+g] = [f] + [g]$ for $f, g \geqslant 0$, $f, g \in L^1$ with disjoint support if and only if $[\chi_E(x)] = \nu(E)$ is a measure equivalent with $\mu$ and $\int |f|\, d\nu = [f]$.*

*iii)* $[.]$ *is an $L^1$-norm if and only if for every $f, g, \in L^1$ with disjoint support $[f+g] = [f] + [g]$.*

*Proof.* i) Follows inmediately from the Radon-Nikodym theorem that $\mu$ and $\nu$ have the same null sets on every finite set of $\Sigma$. ii) implies iii). Let us prove ii). We define $\nu(E) = \infty$ if $\chi_E \,\bar{\in}\, L^1(X, \Sigma, \mu)$, $E \in \Sigma$. From the local equivalence of the norms and $\nu(\sum_1^{\infty} E_i) = \sum_1^{N} [\chi_{E_i}] + [\sum_{N+1}^{\infty} \chi_{E_i}]$, it follows that the last term tends to zero if and only if $\mu(\sum_{N+1}^{\infty} E_i)$ tends to zero, which proves the $\sigma$-additivity of the measure $\nu$. Besides, $\nu << \mu$ and $\mu << \nu$, and since $[f - \sum_1^{N} \chi_{E_i}] \to 0$ if and only if $\|f - \sum_1^{N} \chi_{E_i}\| \to 0$, we get, $[f] = \int f (d\nu/d\mu)\, d\mu$ for $f \geqslant 0$, which proves the lemma. (Notice that in the measure spaces $(X, \Sigma, \mu)$ which we consider, $\Sigma$ is a $\sigma$-field, and $X$ is the union of a disjoint family (perhaps uncountable) of finite measure sets).

3. *Examples.* The next three examples correspond to the case of $L^1$-spaces which are $B$-reflexive.

I) Following $[K_1]$ we say that a Banach space is of type $L$ if it is a Riesz space ($x \geqslant y$, $y \geqslant z \to x \geqslant z$; $x \geqslant y$, $y \geqslant x \to x = y$; there exist $x \cup y$, $x \cap y$; $x \geqslant y \to x + z \geqslant y + z$; $x \geqslant 0$, $\lambda \geqslant 0 \to \lambda x \geqslant 0$) such that $x_n \geqslant y_n$, $x_n \to x$, $y_n \to y$, $\to x \geqslant y$;

$x \geqslant 0,\ y \geqslant 0 \rightarrow \|x+y\| = \|x\| + \|y\|$; $x \frown y = 0 \rightarrow \|x+y\| = \|x-y\|$. Let $C^*(X)$ be the dual space of the space $C(X)$ of the continuous functions on the Hausdorff compact space $X$. The Riesz representation theorem says that $C^*$ is isometric to the family of regular countable additive functions on the Borel sets of $X$ ($\sigma$-field generated by the closed sets), with the norm $\|\mu\| = \sup \{\sum_1^N |\mu|(E_j)\,;\ \sum_1^N E_j = X\}$ = total variation of $\mu$ on $X$. This isometry satisfies $y^*(f) = \int f(x)\, d\mu$, for any $f \in C(X)$ and preserves order. It is well-known that $C^*$ is an $L$-space. Next we prove this fact and also that for $\mu_1,\ \mu_2 \geqslant 0$,

a) $\mu_1 (<) \mu_2$ in the sense of $[K_1]$ (1) is equivalent to $\mu_1 << \mu_2$, that is, $\mu_1$ is absolutely continuous with respect to $\mu_2$.

b) $\mu_1 \frown \mu_2 = 0$ is equivalent to $\mu_1 \perp \mu_2$, that is, $\mu_1$ and $\mu_2$ are mutually singular.

Defining $\mu \geqslant 0$ if $\mu = \mu^+$ the Riesz representation theorem asserts that $\mu \geqslant 0$ if and only if $y^*(f) \geqslant 0$ for every $f \geqslant 0$. All the properties for being a space of type $L$ are easily verified; the only one which needs a little more of attention is the existence of the sup and inf. Since the existence of one of them is reduced to the existence of the other we shall only consider the inf. We define $\mu_1 \frown \mu_2 = (1/2)\ [\mu_1 + \mu_2) - \nu(\mu_1 - \mu_2)]$ where $\nu(\mu)$ is a regular $\sigma$-additive measure called the total variation of $\mu$ and defined by $\nu(\mu)(E) = \sup \{\sum_1^N |\mu(E_j)|\,;\ \sum_1^N E_j = E\}$. Since for every $f \geqslant 0$, $\int f\, d(\mu_2 - \mu_1) \leqslant \int f\, d\nu(\mu_2 - \mu_1)$, we get $\int f d(\mu_1 + \mu_2)/2 - \int f\, d\nu\,((\mu_1 - \mu_2)/2) \leqslant \int f\, d\mu_1$ and therefore $\mu_1 \frown \mu_2 \leqslant \mu_1$. Analogously $\mu_1 \frown \mu_2 \leqslant \mu_2$. On the other hand, there exist two sets $E_1$, $E_1'$, $E_1 \smile E_1' = X$, such that $\mu_1 - \mu_2 = (\mu_1 - \mu_2)^+$ on $E_1$ and $\mu_2 - \mu_1 = (\mu_1 - \mu_2)^-$ on $E_1'$. On $E_1$, $\mu_1 \frown \mu_2 = \mu_2$ and on $E_1'$, $\mu_1 \frown \mu_2 = \mu_1$. From this easily follows that any $\mu$, $\mu \leqslant \mu_i$, $i = 1, 2$, verifies $\mu \leqslant \mu_1 \frown \mu_2$, and also that $\mu_1 \frown \mu_2 = 0$ if and only if $\mu_1$ is singular with respect to $\mu_2$, this is, that each one is concentrated on a zero set of the other. Besides, $\|\mu_1 + \mu_2\| = \|\mu_1 - \mu_2\| = \nu\,(\mu_1 + \mu_2)(X) = \nu\,(\mu_1 - \mu_2)\,(X)$. Let us prove a). Suppose $\mu_1 << \mu_2$ and $\mu_2 \frown \nu = 0$. Then, $\nu$ is singular with respect to $\mu_2$. Obviously, $\nu$ is singular with res-

(1) $\mu_1 (<) \mu_2$ means $\mu_1 \frown \nu = 0$ for every $\nu \geqslant 0$ such that $\mu_2 \frown \nu = 0$.

pect to $\mu_1$, and therefore $\mu_1 \frown \nu = 0$. Suppose $\mu_1$, is not absolutely continuous wit respect to $\mu_2$. There exists $A$ such that $\mu_2(A) = 0$ and $\mu_1(A) > 0$. The restriction of $\mu_1$ to $A$ is singular with respect to $\mu_2$ and is not singular with respect to $\mu_1$.

*The Kakutani's representation theorem,* ([$K_1$]), asserts that any space of type $L$ is isometric and lattice isomorphic to a concrete $L^1(X, \Sigma, \mu)$ space. The set $X$ is the union of disjoint sets $X_s$, where each one is a part of a measure space $(X_s, \Sigma_s, \mu_s)$ such that $\mu_s(X_s) = 1$, and where $\Sigma_s$ is a $\sigma$ - algebra of subsets of $X_s$. $\Sigma$ is the $\sigma$ - algebra generated by $\smile_s \Sigma_s$ and $\mu$ is the measure on $\Sigma$ such that $\mu = \mu_\alpha$ on $\Sigma_\alpha$. Each $\Sigma_\alpha$ can be chosen as the family of Baire sets ($\sigma$ - algebra generated by the compact $G_\delta$ 's) of a totally disconnected Hausdorff compact space. (In this case the family of Baire sets coincide with the $\sigma$ - algebra generated by the clopen sets). Each $(L^1(X_s, \Sigma_s, \mu_s))^+$ is isometric and lattice isomorphic to a set $\{x; x \epsilon L, x \geqslant 0, x(<) x_s\}$ and $x_s \frown x_t = 0$ if $s \neq t$. Besides $x_s$ can be chosen to be the homologous of the function which is equal one on $X_s$ and zero on $X - X_s$, and $x\ (<)\ x_s$ means in $L^1$ to be integrable and zero on $X - X_s$; $x_s \frown x_t = 0$ means that each function is zero a.e. where the other is different from zero.

From all this it follows that a $C^*(X)$ is isometric and lattice isomorphic to an $L^1(\overline{X}, \overline{\Sigma}, \overline{\mu})$. However, the last result can be obtained in a more direct way. Since $C^*(X)$ is of type $L$, it can be decomposed into a direct sum of principals ideals $[\mu_\alpha]$, where $[\mu_\alpha] = \{\mu; \mu \epsilon C^*, (<) \mu_\alpha\}$ and $\mu_\alpha \frown \mu_\beta = 0$ for $\alpha \neq \beta$, and any $\nu \epsilon C^*$ can be uniquely expressed in the form $\nu = \sum_{n=i}^{\infty} \nu_{\alpha_{ni}}$, $\nu_{\alpha_n} \epsilon [\mu_{\alpha_n}]$, (theorem 2, [$K_1$]). From a) and b) we know that $[\mu_\alpha]$ is the set of measures which are absolutely continuous with respect to $\mu_\alpha$, and that $\mu_\alpha \frown \mu_\beta = 0$ is equivalent to $\mu_\alpha \perp \mu_\beta = 0$. Then, using Radon-Nikodym theorem it easily follows that each $[\mu_\alpha]$ is isometric and lattice isomorphic to a space $L^1(X, \Sigma, \mu_\alpha)$ where $\Sigma$ is the $\sigma$ - algebra of Borel sets on the given space $X$. Hence, a concrete space $L^1$, isometric and lattice isomorphic to $C^*(X)$, is obtained as the direct sum of a family $\{L^1(X, \Sigma, \mu_\alpha)\}$, i.e. of spaces $L^1$ defined on the same $(X, \Sigma)$. The only extremal points of the unit sphere of a space $L^1$ are the characteristic functions of atoms multiplied by adequate constants, and of a space $C^*(X)$, the signed measures $\delta$, concentrated on a point $x \epsilon X$ such that $\|\delta\| = 1$. Among those measures, the positive ones are in correspondence with the characteristics function of atoms of the isomorphic image $L^1$ of $C^*$, in other words, there are $|X|$ copies of $(X, \Sigma)$ with $\mu_\alpha$ a punctual measure of mass 1, and they contain the only atoms of the space $L^1$. The space $B$ (§ 1, a)) is now the class of functions $b$ of $L^\infty$ that coincide on each $(X, \Sigma, \mu_\alpha)$ with a fixed function of $C(X)$.

II) We denote by $1^1(X)$ the space of integrable functions of the measure space $(X, \Sigma, \mu)$ where $\Sigma = P(X)$ and $\mu(x) = 1$ for every $x \in X$. It is well-known that $1^1(X)$ is the dual space of $c_o(X) =$ the set of functions $f$ such that for every $\epsilon > 0$ there exist a finite set $E = E(f) < X$ which verifies $|f(x)| < \epsilon$ if $x \in E$.

A characterization of *the purely atomic spaces*, i.e. spaces which are isometric and lattice isomorphic to an $1^1(X)$, is given in [*Ph*].

III) It is well-known ([*KM*]) that an infinite dimensional Banach space cannot be the dual of another Banach space but if its unit sphere contains more than a finite number of extremal points. This means that if $L^1$ has a finite number of atoms and is a dual space, then it is purely atomic and finite - dimensional; besides, it is isometric and lattice isomorphic to the dual space of the continuous functions on a finite set of points.

4. *Case of separable $L^1$ - spaces.* In this section we prove a theorem due to R. R. Phelps and the proof is the same given in [$P_h$] except where me make use of a theorem of Dorothy Maharam.

D. Maharam's theorem. *The measure algebra of a measure space $(X, \Sigma, \mu)$, (all measurable sets mod. null sets), with $\mu(X) = 1$, is isomorphic to the measure algebra of a space $(Y, \Sigma', \mu')$, where this space is the direct sum of, at most, a denumerable set of typical homogeneous spaces $(Y_n, \Sigma'_n, \mu'_n)$, $n \geq 1$, and a purely atomic space $(Y_0, \Sigma'_0, \mu'_0)$. Each $(Y_n, \Sigma'_n)$ is an infinite product of intervals $[0, 1]$, $\prod_{0 \leq \alpha < \gamma_n} [0, 1]_\alpha$, where $\gamma_n$ is the least ordinal number corresponding to its cardinal class and $\gamma_n \neq \gamma_m$ if $m \neq n$; $\mu'_\alpha$ is the Lebesgue product measure except by a constant $c_n$, $0 \leq c_n \leq 1$, where $\sum_1^\infty c_n = 1 - \mu'_0(Y_0)$. The non atomic part of $(Y, \Sigma', \mu')$ is uniquely determined by the sequence $\{(\gamma_n, c_n)\}$, $n \geq 1$, ([M]).*

We want to recall that the measure algebra of the interval

[0, 1] with Lebesgue measure is isomorphic to the measure algebra of $\Pi_{\leq n<\omega} [0, 1]_n$ with the Lebesgue product measure. This isomorphism will be used without further comments in the following theorem.

PHELPS' THEOREM. *Every infinite dimensional separable $L^1$-space which is isomorphic to a $B^*$ is isomorphic to $1^1$ $(N)$.*

*Proof.* From the assumed separability and b), § 2, we can suppose that our space is an $L^1(X, \Sigma, \mu)$ with $\mu(X) = 1$. From Maharam's theorem and the separability of $L^1$ we know that $(X, \Sigma, \mu)$ is isomorphic, as a measure algebra, to the direct sum of a finite or denumerable set of atoms and eventually, of a [0, 1] interval. A theorem due to Gelfand, ([*A*], [*Ge*]) asserts that any function of strong bounded variation $(\sum_i \| \chi(a_i) - \chi(a_{i-1}) \| \leqslant K < \infty)$ from [0,1] to a separable Banach space which is isomorphic to some dual space, admits a strong derivative a.e., that is $[x(\alpha) - x(\beta)] / (\alpha - \beta)$ converges in the norm when $\alpha \rightarrow \beta$ for a.e. $\beta$. If the isomorphic image $\Sigma'$ of $\Sigma$ contained [0,1], then the function: $[0,1] \ni x \rightarrow \chi_{[0,x]}(t) \in L^1(Y, \Sigma', \mu')$ mould be of strong bounded variation but it is not differentiable in any point. Then $(Y', \Sigma', \mu')$ is purely atomic; besides, $L^1(Y', \Sigma', \mu')$ must be isometric and lattice isomorphic to $1^1$ $(N)$ because it is infinite dimensional. It is obvious that the space $L^1(X, \Sigma, \mu)$ in the proof can be chosen isometric and lattice isomorphic to the original $L^1$-space and therefore we have:

COROLLARY. *Every separable $L^1$ — space which is isomorphic to a $B^*$ is isometric and lattice isomorphic to $l^1$ $(N)$.*

*Remark.* If the given $L^1$—space is a complex one there exists an isometric isomorphism onto the complex $l^1$ which preserves conjugation.

5. *Case of non separable $L^1$—spaces.* In this situation the problem remains open in all its generality. However, something can be said in the case of an isometric isomorphism.

*Theorem* 1. *Let $L^1(X, \Sigma, \mu)$ be the dual space of $B, B < L^\infty$. If $L^1$ has countable many atoms, then $L^1$ is isometric and lattice isomorphic to $l^1$ $(N)$,* (cf. [*DS*], 458, and the example III above).

*Proof.* We may suppose that every atom of $\Sigma$ is of mass one. The unit sphere of $L^1$, $S_{L^1}$, has as the only extremal points the family of characteristic functions of atoms and their opposites. We call $E$ the family of positive extremal points, and prove next, that *$E$ is compact in the B-topology.*

Since $B < L^\infty$, from [*DS*], V. 8.11 and V. 8.6, we know that every extremal point of $S_{B^*}$ is an extremal point of $S_{(L^\infty)^*}$, and from V. 8.7, that the multiplicative linear functionals in $S_{(L^\infty)^*}$ form a set $Q$, compact in the $L^\infty$ - topology. $B$ induces on $S_{(L^\infty)^*}$ a weak topology for which $Q$ is still a compact set. Since $E = Q \frown$ canonic image of $S_{L^1}$, and since $S_{L^1}$ is $B$-compact, we get the desired result. (Observe that $Q$ is not necessarily a Hausdorff space in the $B$-topology, and therefore, the compactness of $E$ does not follow from an immediate argument. Notice that the natural quotient space associated to $Q$ and the $B$-topology is compact and Hausdorff, and in an obvius sense $E = Q' \frown$ canonic image of $S_{L1}$).

In our case, $E$ is a denumerable set, therefore $B^*$ is a separable space. Now, the theorem follows from the corollary of the preceding section.

Corollary. *An $L^1$ - space is isometric and lattice isomorphic to a purely atomic $L^1$ - space if, and only if, its restriction to every set of finite measure is a dual space.*

Remark. The theorem holds even for the complex case, if we take as the countable compact set, the set of all characteristic functions of atoms.

6. In the following, $\chi\,[A, x]$, or $\chi_A(x)$, or $\chi[A]$, will represent the characteristic function of the set $A$. A motivation of the following theorem will be found in the last paragraph of the next section, which is independent of this one.

Theorem 1. *Let $L^1$ $(X, \Sigma, \mu)$ be a $\sigma$ finite space, and $B$ a closed subspace of $L^\infty$ $(X, \Sigma, \mu)$. Let $A$ be a non-null subset of $X$ without atoms, and $u(x) \epsilon L^1$, a finite valued function different from zero on every point of $X$-$A$. If for every function $f \epsilon L^1$, the $B^*$-norm and the $L^1$-norm of $u.\chi\,[X\text{-}A] + f.\chi\,[A]$ coincide, then $B^*$ contains a functional which is not representable by a function of $L^1$.*

*Proof.* a) Without loss of generality we can restrict ourselves to the case of a finite measure space, i.e. $\mu\,(X) < \infty$ (cf. § 2, *b*)). Maharam's theorem enables us to replace $A$ by a direct sum of at most a denumerable collection of product spaces. Let $(P_\gamma, J) =$

$$= (\prod_{1 \leq \alpha < \gamma} [0,1]_\alpha,\ c \prod_{1 \leq \alpha < \gamma} m_\alpha),$$

be one of the terms of this sum, where $c > 0$, and $m_\alpha = m =$ Lebesgue measure on the unit interval. After multiplying $\mu$ by an adequate constant we can make $c = 1$.

If $I_{rs}$ is an interval with rational end points $r$ and $s$, then, by hpyothesis, the functions $(h_{rs})^{\pm} = u\,\chi\,[X—A] + \chi\,[A—P_\alpha] \pm \chi\,[I_{rs} \times \prod_{2 \leq \alpha < \gamma} [0,1]_\alpha] + \chi\,[\,(\,[0,1] — I_{rs}) \times \prod_{2 \leq \alpha < \gamma} [0,1]_\alpha]$ have the same $B^*$-norm and $L^1$ norm. Then, there exists a sequence $\{\check{g}^n{}_{rs}\} < B$ converging in measure to $(h_{rs})^+ + (1 — u)\,\chi\,[X — A] = 1$, such that $\| \check{g}^n{}_{rs} \| = 1$, $n = 1, 2, \ldots$, and another, with the same characteristics, converging in measure to $(h_{rs})^- + (1 — u)\,\chi\,[X — A]$. This implies the existence of a sequence $\{g^n{}_{rs}\}$ of functions of $B$, all of norm 1, which converges in measure to $\chi\,[I_{rs} \times \prod_{2 \leq \alpha < \gamma} [0,1]_\alpha]$ when $n \to \infty$. Moreover, we may suppose that the convergence is almost uniform. The function $u(x)$ of the hypothesis is used only to ensure the existence of such a sequence.

Since $\{g^n{}_{rs}\}$, $n = 1, 2, \ldots$, $r, s$ rational, is a denumerable family and, since every $g$ is determined by a denumerable number of sets $(\{x;\ g(x) > t,\ t\ \textit{rational}\})$, we may suppose that for every $n$, $r$, $s$, $g^n{}_{rs}(x) = g^n{}_{rs}(y)$ whenever the $\gamma$—tuples $x$ and $y$ coincide on a denumerable set of indices $(i_1 = 1, i_2, i_3, \ldots)$. Reordering the factors of $P_\alpha$ we may also have for $n \geqslant 2$, $i_n = n$. Besides, after replacing $\prod_{2 \leq n < \omega} [0,1]_n$ by $[0,1]$, we have: $P_\gamma =$

$$= [0,1] \times [0,1] \times \prod_{2 < \alpha < \gamma} [0,1] = \{(x, y, z),\ x \in [0,1],\ y \in [0,1],$$

$z \in \prod_{2 < \alpha < \gamma} [0,1]_\alpha\}$ and therefore, the family $\{g^n{}_{rs}\}$ verifies for all $n, r, s, x, y, z_1, z_2 : g^n{}_{rs}(x, y, z_1) = g^n{}_{rs}(x, y, z_2)$.

b) Next, we prove by an inductive process the existence of a subsequence of the g's with some special properties which permit the construction of a linear functional not representable by a function of $L^1(X, \Sigma, \mu)$. We denote by $\pi$ the measure $\prod_{2 < \alpha < \gamma} m_\alpha$ and by $R$ the product $\prod_{0 < \alpha < \gamma} [0,1]$

Inductive hypothesis.

i) There are $n$ functions of $B$, $h_i$, of $L^\infty$-norm equal to 1.

ii) There are $n$ intervals $I_i$ with rationals end points such that $I_1 = [0,1] > I_2 > I_3 \ldots > I_n$; $m(I_i) \leqslant 2^{-(i-1)}$, $1 \leqslant i \leqslant n$.

iii) For every $i$, and $j \geqslant i$, $i = 1, 2, \ldots, n$:

$$\frac{1}{m(I_j)} \iiint_{I_j \times [0,1] \times R} h_i(x,y,z)\; dxdyd\pi > 1/2 .$$

iv) Let $J_i$ be the set $\{ t \in X;\ | h_i(t) - \chi[I_i \times [0,1] \times R;\ t] | > 2^{-i} \}$. For $i = 1, 2, \ldots, n$, it is verified $\mu(J_i) < 2^{-i}\, m(I_i)$.

v) Let $Z_i$ be the set $\{ x \in [0,1];\ H_i(x) = \int_R d\pi \int_0^1 h_i(x, y, z)\, dy > 1/2 \}$. Then, $m(I_n \frown Z_i) > (n-1)\, m(I_n)/n$ for $1 \leqslant i \leqslant n$.

We find a function $h_{n+1}$ and an interval $I = I_{n+1}$ satisfying i) to v).

Let $Z'_i$ and $Z''_i$ be the set of points of differentiability of the function $H_i(x)$ *in* $Z_i$, and the set of points of density of $Z'_i$, respectively.

Obviously $Z_i > Z'_i > Z''_i$; since $H_i(x) \in L^1(0,1)$, and $m(Z'_i - Z''_i) = 0$, $i = 1, 2, \ldots, n$, we have $m(Z'_i) = m(Z''_i) = m(Z_i)$.

The condition v) implies, as it is easy to see, that $m(I_n \frown Z''_i) > (n-1)\, m(I_n)/n$, and $m(\bigcap_{i=1}^{n} (I_n \frown Z''_i)) > 0$. Therefore, there is an interior point $x_o$ of $I_n$ belonging to $\bigcap_{i=1}^{n} (In \frown Z''_i)$ and an $\epsilon > 0$ such that every interval $I$ with center $x_o$ and length less than $\epsilon$ verifies $m(Z'_i \frown I) > n\, m(I)/(n+1)$, v $1 \leqslant i \leqslant n$. *Hence* $m(Z_i \frown I) > n\, m(I)/(n+1)$, $1 \leqslant i \leqslant n$. Also, we may suppose $\epsilon \leqslant 2^{-n}$ and $I < I_n$. Since $x_o \in \bigcap_{i=1}^{n} Z'_i$, $\epsilon$ can be chosen as small as to verify $(1/m(I)) \int_I H_i(x)\, dx > 1/2$ for every $I$ with $m(I) < \epsilon$, and for every $i$, $1 \leqslant i \leqslant n$. Of course, $I$ may be chosen with rational end points, suppose they are $r$ and $s$. From the family $\{ g^k_{rs} \}$, we select a function $h$ with the following properties:

$$\|h\|_\infty = 1;\ \mu\{t;\ | h(t) - \chi[I \times [0,1] \times R; t] | < 2^{-n-1}\} < 2^{-n-1}\, m(\mathrm{I});$$
$$m(I \frown \{x \in [0,1];\ \int_R d\pi \int_0^1 h(x,y,z)\, dy > 1/2\}) > n\, m(I)/(n+1);$$
$$(1/m(I)) \iiint_{I \times [0,1] \times R} h(x,y,z)\; dxdyd\pi > ½ .$$

This is possible, because the g's converge almost uniformly to

$\chi[I \times [0,1] \times R; t]$ when$k \to \infty$. Now, we call $h_{n+1}$, to $h$ and $I_{n+1}$ to $I$.

c) Since for each fixed $i$, there is a sequence of $j$'s such that $(1/m(I_j)) \iiint_{I_j \times [0,1] \times R} h_i(x,y,z)\, dxdyd\pi$ converges to a number $\geqslant 1/2$, by a diagonal process it is possible to find a sequence $\{j_k\}$ of natural numbers such that

$$(*) \quad \varnothing(h_i) = \lim_{k \to \infty} \frac{1}{m(I_{j_k})} \iiint_{I_{j_k} \times [0,1] \times R} h_i(x,y,z)\, dxdyd\pi$$

exist for every $i$ and it is no less than 1/2. In the subspace $N < B$, algebraically spanned by $\{h_i\}$, (*) defines a bounded non zero functional $|\varnothing(h)| \leqslant \|h\|_\infty$. We extended now $\varnothing$ by continuity to the closure of $N$, and to the whole of $B$ by the Hahn-Banach theorem. If $\varnothing$ could be represented as $\varnothing(h) = \int_X \phi . h\, d\mu$ with $\phi \in L^1$, then conditions i), ii), iv) and $\varnothing(h_i) = \int_X \phi\, h_i d\mu = \int_{X-Ji} \phi\, h_i d\mu + \int_{Ji} \phi\; h_i\, d\mu$, would imply the convergence to zero of $\varnothing(h_i)$ when $i \to \infty$. This contradicts the fact that $\varnothing(h_i) \geqslant 1/2$.

7. *General considerations and problems.* a) Suppose there is an isometry of $L^1(X, \Sigma, \mu)$ onto $C^*(Y)$, $Y$ a compact space. We can assume without loss of generality that any atom of $\Sigma$ is of measure one. In this case, the isometry sends the extremal points of $C^*(Y)$ onto the characteristic functions of the atoms or their opposites. Since $C^*(Y)$ is isometric and lattice isomorphic to an $L^1(Z, \varnothing\, \nu)$ (cfr. § 3, I), and since the characteristic functions of the atoms or their opposites correspond to all the extremal points of $C^*(Y)$, we can, after eventually changing the isometry, suppose that $L^1(X, \Sigma, \mu)$ is isometric to $L^1(Z, \varnothing, \nu)$ ,and in such a way that characteristic functions of atoms correspond to characteristic functions of atoms. Since, $C(Y)$ is isometric and lattice isomorphic to an algebra with unit of functions of $L^\infty(Z, \varnothing, \nu)$ which are zero outside of the set of atoms of $\varnothing$, it follows that $L^1(X, \Sigma, \mu)$ is isometric to the dual of an algebra with unit of functions of $L^\infty(X, \Sigma, \mu)$. Conversely, if $L^1$ is isometric to the dual of an algebra with unit $B$, $B < L^\infty$; by [*DS*], V.8.11 and IV.6.20, this algebra is isomorphic (as an algebra, hence, lattice isomorphic and

isometric) to the family of continuous functions of a compact space $Y$. It follows that there is an isometry of $L^1$ onto $C^*(Y)$.

Besides, an $L^1$-space may be isometric to the dual of *several* algebras with unit. In fact, let $X$ and $Y$ be non homeomorphic compact spaces such that there is an isometry of $C^*(X)$ onto $C^*(Y)$. (Take for example, $X = [0,1] \cup [2,3]$, $Y = [0,1] \cup (\{2\})$. By the Banach-Stone theorem, $C(X)$ cannot be isometric to $C(Y)$.

b) Suppose $(X, \Sigma, \mu)$ is a $\sigma$-finite measure space the $L^1$ of which is isometric to a $B^*$, $B < L^\infty$. If every function of $L^1$ takes its norm on a function of the unit sphere of $B$, then $L^1$ *is finite-dimensional.* In fact, we may suppose that $\mu(X) = 1$. The function equal to 1 in a set $E$ and $-1$ in the set $X - E$ takes its norm only on itself, and therefore belongs to $B$. Since $B$ is a closed subspace and contains all the characteristic functions, it must coincide with $L^\infty$, which proves the proposition. If $L^1(X, \Sigma, \mu)$ is isometric to a $B^*$ and not $\sigma$-finite, does the same conclusion hold? This is equivalent to prove that an $L^1$-space is reflexive if (and only if) all its functions take their norm as linear functionals. If instead of asking that all the functions take their norm, we only ask it for a dense subset of $L^1$, the result is that it always holds. In fact, every $B^*$ contains a dense subset of elements which take their norm on elements of the unit sphere of $B$, ([*BP*]).

c) If the real $L^1$-space is isomorphic to a $B^*$ then the complex $L^1$-space is isomorphic to a $B^*$. We do not know if the result holds when "isomorphic" is replaced by "isometric" and whether or not the converses are true.

d) We do not know if a $\sigma$-finite $L^1$-space which is isomorphic to a $B^*$ is necesarily purely atomic. Another question, a particular case of the preceding one, may be stated as follows. Suppose $\mu(X)$ is finite and $A$ is a subset of $X$ of positive measure without atoms. Is it true that $L^1(X, \Sigma, \mu)$ cannot be isomorphic to a $B^*$ in such a way that $L^1(A, A \cap \Sigma, \mu)$ be isometric to a subspace of $B^*$? If instead of asking the isometry of $L^1(A)$, we require it for the subspace of the form $\lambda(\chi_{X-A} + \chi_A f)$, $f \in L^1$, $-\infty < \lambda < \infty$, then, as it is proved in § 6, $L^1$ *is not isomorphic to any* $B^*$.

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