ON THE ENERGY MODERATION OF NEUTRONS IN A HEAVY MONATOMIC GAS (*)

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ABSTRACT

The subject of this investigation is the space and energy dependent neutron spectrum in a heavy monatomic gas due to a point source as described by the the Wilkins differential equation for the case of constant neutron scattering cross sections. Utilizing Laplace transformation methods the solution (Green's function) is given in closed analytical form.

1. INTRODUCTION

During the process of neutron moderation the mean logarithmic loss of energy due to collisions decreases as the energy of the neutrons tend towards the thermal energy of the moderating nuclei. In an interesting paper Koppel (1) has investigated the problem of energy moderation in a heavy monatomic gas and has calculated the spectrum due to a source of high energy neutrons with help of the Wilkins differential equation (2).

The physical problem discussed in the following can be envisioned by considering a concentrated source located at $x = 0$ which generates neutrons at energy E_0. The principal aim is to determine the energy spectrum (neutron flux) $\phi\,(E, x; E_0)$ of neutrons of energy E as a function of the distance x from the source. Koppel obtained a physically meaningful solution to his problema by applying a Laplace transformation technique which required however certain approximations of an asymptotic character. In the following

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an alternate method of attack is suggested which yields for the above mentioned source condition, which differs somewhat from the one formulated by Koppel, an exact analytical solution suitable for evaluation and discussion of the spectrum for all $x \geqslant 0$.

2. ANALYSIS

The Green's fuction of Wilkins differential equation $\varphi(E, x; E_0)$ is defined to satisfy the following equations

$$\left(\frac{1}{\xi \Sigma_s}\right) \frac{\partial \varphi}{\partial x} = E \frac{\partial^2 \varphi}{\partial E^2} + E \frac{\partial \varphi}{\partial E} + \left(1 - \frac{\Sigma_a}{\xi \Sigma_s}\right) \varphi$$

$$0 < E < \infty, \ x > 0 \tag{1}$$

$$\varphi(E, 0; E_0) = \delta(E - E_0) \tag{2}$$

$$\varphi(0, x; E_0) = 0, x \geqq 0. \tag{3}$$

Furthermore φ $(E, x; E_0)$ is assumed to be of exponential order in E, as $E \rightarrow \infty$. Equations (1), (2) describe the neutron thermalization in a spatially infinite moderator medium due to a neutron pulse of energy E_0 released at position $x = 0$. Equation (3) prescribes that the neutron flux φ should vanish at zero energy. The absorption cross section Σ_a, the free atom scattering cross sextion Σ_s, and the average logarithmic energy decrement per collision ξ are assumed to be constant and thus independent of the dimensionless energy variable E. Once the fundamental solution φ $(E, x; E_0)$ has been determined the space dependent neutron flux, due to an arbitrary integrable source distribution $g(E_0)$, is given by

$$\varphi(E, x) = \int_0^\infty \varphi(E, x; E_0)\, g(E_0)\, dE_0. \tag{4}$$

With

$$z = \xi \Sigma_s x; \quad \alpha = \frac{\Sigma_a}{\xi \Sigma_s}; \quad \phi(E, z; E_0) = e^{\alpha z} \varphi(E, z; E_0) \tag{5}$$

equations (1) to (3) transform into

$$\frac{\partial \phi}{\partial z} = E \frac{\partial^2 \phi}{\partial E^2} + E \frac{\partial \phi}{\partial E} + \phi ; \quad \phi < E < \infty , \; z > 0 \tag{6}$$

$$\phi (E, 0; E_0) = \delta (E - E_0) \tag{7}$$

$$\phi (0, z; E_0) = 0 \tag{8}$$

which do not contain the parameter α. In contrast to Koppel who takes the Laplace transform of similar equations with respect to z we take the transform with respect to E. The Laplace transform of $\Phi (E, z; E_0)$ with respect to the variable E is defined as usual by

$$\Theta (s, z; E_0) = \int_0^\infty e^{-sE} \phi (E, z; E_0) \, dE. \tag{9}$$

In view of the assumption about φ Φ is sufficiently well behaved as $E \rightarrow \infty$ to insure the existence of the integral be haved as $E \rightarrow \infty$ to insure the existence of the integral. The application of the integral transform to the differential equation (6) yields together with the boundary condition (8) a first order partial differential equation for Θ $(s, z; E_0)$

$$\frac{\partial \Theta}{\partial z} + s (s + 1) \frac{\partial \Theta}{\partial s} + 2 s \Theta = 0. \tag{10}$$

Its solution must satisfy the tansformed initial condition (7) *i.e.*

$$\Theta (s, 0; E_0) = e^{-sE_0}. \tag{11}$$

The general solution of (10) is

$$\Theta = \left(\frac{e^z}{s} \right)^2 F \left(e^z \frac{s+1}{s} \right). \tag{12}$$

The arbitrary function F is evaluated with help of the initial condition (11) and leads to the desired solution

$$\Theta (s, z; E_0) = \left(\frac{1}{1 + s (1 - e^{-z})} \right)^2 \exp \left\{ - \frac{s E_0 e^{-z}}{1 + s (1 - e^{-z})} \right\} \tag{13}$$

a result which is readily verified. We note in passing that this expression represents a properly behaved Laplace transform for all $z > 0$.

The expression for $\Theta\ (s, z; E_0)$ must now be inverted in order to recover the function $\Phi\ (E, z; E_0)$. For reasons of manipulation let

$$(1 - e^{-z}) = a; \quad E_0 e^{-z} = b \tag{14}$$

then

$$\Theta(s, z; E_0) = \left(\frac{1}{1+as}\right)^2 \exp\left\{-\frac{bs}{1+as}\right\} =$$
$$= e^{-\frac{b}{a}} \left(\frac{\exp\left\{\frac{b}{a(1+as)}\right\}}{1+as}\right) \cdot \left(\frac{1}{1+as}\right) \tag{15}$$

If I_ν denotes the modified Bessel function of the first kind order ν one obtains for the inverse Laplace transform of the two bracketed terms

$$L^{-1}\left(\frac{\exp\left\{\frac{b}{a(1+as)}\right\}}{1+as}\right) = \frac{e^{-E/a}}{a}\, I_0\left(\frac{2}{a}\sqrt{bE}\right)$$
$$L^{-1}\left(\frac{1}{1+as}\right) = \frac{1}{a}\, e^{-\frac{E}{a}}. \tag{16}$$

The application of the convolution theorem to equation (15) with these inversion results yields

$$\phi(E, z; E_0) = \frac{e^{-\frac{b+E}{a}}}{a^2} \int_0^E I_0\left(\frac{2}{a}\sqrt{bx'}\right) dx'. \tag{17}$$

But

$$\int_0^E I_0\left(\frac{2}{a}\sqrt{bx'}\right) dx' = \frac{a}{b}\sqrt{bE}\; I_1\left(\frac{2}{a}\sqrt{bE}\right). \tag{18}$$

The substitución of this into equation (17) followed by the replacement of a and b with (14) and the replacement of Φ with help of equation (5) results finally in the desired fundamental solution

$$\varphi(E, z; E_0) = \left(\frac{e^{-az}}{1-e^{-z}}\right)\left(\frac{E}{E_0 e^{-z}}\right)^{1/2} I_1\left(\frac{2\sqrt{EE_0 e^{-z}}}{1-e^{-z}}\right) \exp\left\{-\frac{E+E_0 e^{-z}}{1-e^{-z}}\right\} \qquad (19)$$

with z defined by (5). It is tedious but not difficult to verify that equation (19) does indeed satisfy the defining equations of the problem (1) — (3). The reader might compare our result with that given by Koppel *i.e.* his equation (41).

The solution to this problem is in a convenient form for discussion as well as numerical evalution. The principal properties of the spectrum are recidily arcertained from equation (19). For very small z, if one utilizes the asymptotic development of the modified Bessel functiton I_1, one obtains the representation

$$\varphi(E, z; E_0) \doteq \frac{1}{\sqrt{4\pi z}}\left(\frac{E}{E_0^3}\right)^{1/4} \exp\left\{-\frac{(\sqrt{E}-\sqrt{E_0})^2}{z}\right\}, \qquad \text{for } z \ll 1. \qquad (20)$$

That this representation exhibits the proper Dirac delta function behavior as $z \rightarrow 0$ can be seen from the fact that for $E \neq E_0$, $\lim\limits_{z\to 0} \varphi = 0$, while for $E = E_0$ the $\lim\limits_{z\to 0} \varphi$ does not exist. Furthermore a calculation shows that

$$\lim_{z\to 0} \int_0^\infty \varphi(E, z; E_0)\, dE = 1.$$

The representation (20) is closely related to Kelvin's well known "heat source function". For very large values of z and $a > 0$, φ tends of course to zero. In case of $a = 0$ (no absorption) one can expand (19) in powers of e^{-z}. There results

$$\varphi(E, z; E_0) = Ee^{-E} + Ee^{-E}[1-(E_0+E)]e^{-z} + 0(e^{-2z}), \qquad z \gg 1. \qquad (21)$$

From equations (20) and (21) one can form the following qualitative ideas about the spatial development of the thermal spectrum. According to equation (20) and the properties of the heat source function, the neutron pulse spreads in energy space in the neighborhood of the source in a maner similar to the spread of a temperature pulse in real space at small times. According to equation (21) the spectrum takes on a Maxwellian distribution as $z \to \infty$ *i.e.* far from the source as is expected. If one focuses attention on an individual energy group one notes from (21) that the approach to equilibrium is determined by the algebraic sign of the square bracketed term. φ is either a decreasing function of distance for a sufficiently high energy group regardless of the value E_0 or an increasing function of distance for a sufficiently low energy group provided E_0 is also sufficiently small. It is interesting to note that the energy level E_0 of the source influences the approach to equilibrium.

REFERENCES

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CRONICA

LA REUNION ANUAL DE LA UMA

Conforme a lo anunciado, en los días 11 a 13 de octabre de 1962 se realizó en Rosario la Reunión anual de comunicaciones científicas de la Unión Matemática Argentina. La Reunión tuvo lugar en los locales del Instituto de matemática de la Facultad de ciencias matemáticas de la Universidad Nacional del Litoral, cedidos gentilmente al efecto por las autoridades de la misma.

La Reunión fue muy concurrida, participando en ella, además de numerosos socios de la UMA de Buenos Aires y de Rosario, los representantes y delegados de las siguientes instituciones:

Instituto de matemática y estadística de la Facultad de Ingeniería y Agrimensura de Montevideo: Enrique M. Cabaña; Jorge Lewowicz; Alfredo Gandulfo.

Consejo Nacional de Investigaciones científicas y técnicas: Andrés Valeiras.