A TEST FOR MARKOV TIMES (*)

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1. *Introducción*

In this paper it is proved a necessary and sufficient condition for a non-negative Borel function of the path in an arbitrary stochastic process to be a Markov time. It is also included a similar condition for a Borel set to belong to the sigmafield corresponding to a given Markov time.

These tests are mentioned by H.P. McKean and H. Tanaka in sections 2 and 12 of [1] as a private communication of the author.

Their application yields, in most cases, simpler proofs than those obtained by using the current definitions.

2. *Definitions and Notations*

We will need only a few. Most of them are in section 2 of [1].

According to J. L. Doob [2] a stochastic process is a family of random variables

$$[x(t, \omega),\ t \in T,\ \omega \in \Omega]$$

The time range T satisfies $T \subset [0, \infty]$.

We will consider the sample space Ω consisting of all arbitrary onevalued functions ω (paths) from the time range T into a locally compact Hausdorff space E (state space). We will call B_E the

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sigmafield of all Borel subsets of E. In the canonical representation of the process (see E. G. Dynkin [4] $x(t, \omega) =$ "$\omega(t)$" for all $t \in T$, $\omega \in \Omega$.

The proofs below do not change if the paths ω are assumed to be continuous as in section 2 of [1] . ; i. e., if Ω consists only of continuous fuctions from T into E.

We define en Ω the sigmafield B generated by all subsets of the type

$$(2.1) \qquad [\omega : x(s, \omega) \in A]$$

where $A \in B_E$ and $s \in T$.

For every $t \in T$, all subsets of type (2.1) with $s \leq t$ generate a subsigma field $\underline{\underline{B}}_t$.

A *Markov time* $m(\omega)$ is a non-negative Borel function of the path whose range is T such that, for every $t \in T$.

$$(2.2) \qquad [\omega : m(\omega) < t] \in \underline{\underline{B}}_t.$$

Finally we introduce, for a given Markov time $m(\omega)$, the subsigmafield $\underline{\underline{B}}_{m}+$, which consist of all events $B \in \underline{\underline{B}}$ such that, for every $t \in T$,

$$(2.3) \qquad B \cap [\omega : m(\omega) < t] \in \underline{\underline{B}}_t.$$

3. *Lemmas.*

We will first present two simple lemmas that contain an optative definition of the family of sub-sigmafields $\underline{\underline{B}}_t$ which mill make the proof of the theorems 4.1 and 5.1 neater.

Lemma 3.1: If for a Borel set $B \in \underline{\underline{B}}$, two paths ω_1 and ω_2 and a number $t \in T$, the following conditions hold:

$$(3.1) \qquad B \in \underline{\underline{B}}_t$$

$$(3.2) \qquad x(s, \omega_1) = x(s, \omega_2) \text{ for all } s \in T, s \leq t$$

$$(3.3) \qquad \omega_1 \in B,$$

then also

$$(3.4) \qquad \omega_2 \in B.$$

Lemma 3.2: Conversely, if conditions (3.2) and (3.3) imply (3.4) for every pair of paths ω_1, ω_2 and fixed $B \in \underline{\underline{B}}_t$ and $t \in T$, then $B \in \underline{\underline{B}}_t$.

Proof of lemma 3.1:

The property in lemma 3.1 is obviously true for generators of $\underline{\underline{B}}_t$, which are subsets of the type (2.1) with $s \leq t$. As this property is preserved by countable unions and intersections as well as by taking complements, it is also true for all Borel sets $B \in \underline{\underline{B}}_t$ as asserted.

Proof of lemma 3.2:

Let us call $\underline{\underline{B}}_t{}^*$ the set of all Borel subsets in $\underline{\underline{B}}$ for which (3.2) and (3.3) imply (3.4) for every pair ω_1, ω_2 and fixed $t \in T$.

It is clear that $\underline{\underline{B}}_t$ contains all generators of type (2.1) belonging to $\underline{\underline{B}}_t$. Furthermore if a generator of type (2.1) does not belong to $\underline{\underline{B}}_t$ it can not belong to $B_t{}^*$ either. In fact, this generator must be of the type:

$$B = [\omega : x\ (s, \omega) \in A],\ A \in B_E,\ s > t.$$

We can certainly choose ω_1, ω_2 satisfying (3.2) and (3.3) with $x(s, \omega_2)$ not belonging to A, unless $A \equiv E$ in which case $B = \Omega \in \underline{\underline{B}}_t$.

As $\underline{\underline{B}}_t{}^*$ is closed under countable unions and intersections and under complementations it must be identical to $\underline{\underline{B}}_t$. Hence lemma 2 follows.

4. *Test for Markov Times.*

THEOREM 4.1: Let $m(\omega)$ be a non-negative Borel function of the sample path ω. Then the following statements (4.1.a) and (4.1.b) are equivalent:

4.1.a) $[\omega : m(\omega) < t] \in \underline{\underline{B}}_t$ for every $t \in T$ i. e., m is a Markov time).

(4.1.b) If for two sample path ω_1, ω_2 and a nmber $t \epsilon T$ the following conditions hold:

(4.2) $m(\omega_1) < t$

4.3) $x(s, \omega_1) = x(s, \omega)$ for all $s \leq t$, $s \epsilon T$

then:

4.4) $m(\omega_1) = m(\omega_2)$.

Proof that (4.1.*a*) *implies* (4.1.*b*)

Let $m(\omega)$ satisfy (4.1.*a*) and, with respect to some fixed ω_1, ω_2, t, also satisfy (4.2), (4.3). We must prove (4.4).

In fact, suppose $m(\omega_2) > m(\omega_1)$ and call $t' = \min\ [a, m(\omega_2)]$.

Let $B' = [\omega : m(\omega) < t']$.

Clearly:

4.5) $B' \epsilon \underline{\underline{B}}_t'$

(4.6) $\omega_1 \epsilon B$

(4.7) ω_2 does not belong to B'.

But (4.5), (4.3) and (4.6) should imply, by lemma 3.1, that $\omega_2 \epsilon B'$, in contradiction to (4.7).

Similarly, if we assume $m(\omega_2) < m(\omega_1)$ and call $t' = m(\omega_1)$, we would get $\omega_2 \epsilon B'$, ω_1 does not belong to and again a contradiction to lemma 3.1. Hence, $m(\omega_1) = m(\omega_2)$ as it had to be proved.

Proof that (4.1.*b*) *implies* (4.1.*a*)

Consider the set

$$B = [\omega : m(\omega) < t].$$

To prove that $B \epsilon \underline{\underline{B}}_t$ it is enough, by lemma 3.2, to show that (3.2) and (3.3) imply (3.4).

From condition (3.3):

4.8) $m(\omega_1) < t$.

(4.8) and (3.2) are the conditions (4.2) and (4.3) in the hypothesis of (4.1.*b*). Therefore $m(\omega_1) = m(\omega_2)$.

Then clearly $\omega_2 \in B$, which is (3.4), and the proof is complete.

5. *Test for the Subsigmafields* $\underline{\underline{B}}_{m+}$

THEOREM 5.1: Let $m(\omega)$ be a Markov time and B a Borel subset of Ω. Then the following statements (5.1.*a*) and (5.1.*b*) are equivalent.

5.1.*a*) $B \in \underline{\underline{B}}_{m+}$.

(5.1.*b*) If for two sample paths ω_1, ω_2 and a number $t \in T$ the following conditions hold:

5.2) $m(\omega_1) < t$

5.3) $x(s, \omega_1) = x(s, \omega_2)$ for all $s \leq t$, $s \in T$.

(5.4) $\omega_1 \in B$,

then also

(5.5) $\omega_2 \in B$.

Proof that (5.1.*a*) implies (5.1.*b*)

Let B satisfy (5.1.*a*) and, for fixed ω_1, ω_2, t, also satisfy (5.2), 5.3), (5.4). We must prove (5.5).

By (5.1.*a*):

(5.6) $B \cap [\omega : m(\omega) < t] \in \underline{\underline{B}}_t$.

By (5.2) and (5.4):

(5.7) $\omega_1 \in B \cap [\omega : m(\omega) < t]$.

Conditions (5.6), (5.3) and (5.7) are the same as in the hypothesis of lemma 3.1. Therefore $\omega_2 \in B \cap [\omega : m(\omega) < t]$ and (5.5) follows.

Proof that (5.1.*b*) *implies* (5.1.*a*)

To prove that $B \cap [\omega : m(\omega) < t] \in \underline{\underline{B}}_t$ it is enough, by lemma 3.2, to show that (3.2) and (3.3) imply (4.4) imply (3.4).

From (3.3) applied to $B \cap [\omega : m(\omega) < t]$:

5.8) $m(\omega_1) < t$

(5.9) $\omega_1 \in B$.

(5.8), (3.2) and (5.9) are conditions (5.2), (5.3), (5.4) in (5.1.*b*). Therefore:

(5.10) $\omega_2 \in B$.

As $m(\omega)$ is a Markov time, and as (5.8) and (3.2) are conditions (4.1) and (4.2) in theorem 4.1, by that theorem it follows:

(5.11) $m(\omega_1) = m(\omega_2)$.

From (5.10), (5.8) and (5.11) we clearly obtain $\omega_2 \in B \cap [\omega : m(\omega) < t]$, which is (3.4), and the proof is complete.

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