A PREDICATIVE EXTENSION OF ELEMENTARY LOGIC — PART I

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We can extend elementary logic by allowing quantification over class variables and introducing a comprehension axiom. In this paper we shall use Gentzen's methods to study the predicative extension obtained by restricting the comprehension axiom to propositional functions containing only quantification over individuals. We prove a theorem that permits to pass from a deduction in the extended logic to a deduction in elementary logic; from this theorem several non-derivability and relative consistency results can be obtained.

This simple predicative extension has been studied before, mainly in connection with the Bernays-Gödel formalization of set theory. We have not followed the customary procedure of identifying the set of individuals with a subset of the universe of classes. This procedure seems to be natural in dealing with set theory; for a more general investigation the identification is rather artificial. In this way we return to the original method of Bernays who seems to be the first in studying an extension of this kind.

In his paper [3] Maehara has considered a system very similar to that studied in this paper. He uses it as an auxiliary system to obtain a result on Hilbert's ϵ-symbol. Our investigation is more akin to that of [6]. Leaving aside the identification of individuals with classes the result proved in this paper seems to generalize the result obtained by Shoenfield using an extension of the first ϵ-theorem.

1. The system *LKP*. We introduce a system that extends Gentzen's system *LK*. This system is called *LKP*. It is constructed as a syntactial system; i. e. we assume that the formal entities are strings of symbols. Since these symbols are never exhibited it

is clear that the assumption is irrelevant; we might as well assume that the formal entities are n-tuples of some given primitive atoms.

The primitive symbols are classified in several groups:

1. A denumerable list of free individual variables. .

2. A denumerable list of bound individual variables.

3. Individual constants and functors; each functor whit a fixed number of arguments.

4. A denumerable list of free class variables.

5. A denumerable list of bound class variables.

6. Class constants and predicate constants; each predicate with a fixed number of arguments.

7. Special symbols: ϵ, $\neg$, $\supset$, $\vee$, $\wedge$, $\forall$, $\exists$, λ, $=$, (,), $\vdash$.

Let A and B be expressions (i. e. finite strings of primitive symbols) and let u be a primitive symbol. We denote with $[B/u]A$ the expression obtained by substitution of B for u in A.

We shall use the following notation: letters $x, y, z, ..$ for free individual variables; letters a, b, c, d ... for bound individual variables; letters $X, Y, Z, \ldots$ for free class variables; letters $H, J, ..$ for bound class variables.

Definition of individual terms. 1. Each free individual variable and individual constant is an individual term. 2. If $t_1, \ldots, t_k$ are individual terms, $k \geqq 1$, and f is a functor with k arguments, then $f(t_1, \ldots, t_k)$ is an individual term. We shall use letters $t, h, ..$ for individual terms.

Definition of class terms and formulas. 1. If $t, h, t_1, \ldots, t_k$ are individual terms, $k \geqq 1$, and F is a predicate constant with k arguments, then $F(t_1, \ldots, t_k)$, $t \epsilon h$, $t = h$ are formulas. 2. Every free class variable and class constant is a class term. 3. If t is an individual term and U is a class term then $t \epsilon U$ is a formula. 4. If A and B are formulas then $\neg A$, $(A \supset B)$, $(A \vee A)$, $(A \wedge B)$ are formulas. 5. If A is a formula which does not contain the bound variable b, then $(\forall b[b/x]A)$ and $(\exists b[b/x]A)$ are formulas. 6. If A is a formula which does not contain the bound variable H then $(\forall H[H/X]A)$ and $(\exists H[H/X]A)$ are formulas. 7. If A is a formula which does not contain the bound variable b, and in which there is no occurrence of bound class variables, then $\lambda b[b/x]A$ is a class term.

We shall use letters $A, B, C, ..$ for formulas, and letters $U, V, ..$ for class terms. Parentheses that are not necessary for the understanding of a formula will be omitted.

Let A be a formula; by $Grade_1(A)$ we understand the number of occurrences of $\forall$ and $\exists$ preceding a bound class variable; $Grade_2(A)$ is the number of occurrences of symbols $\neg$, $\supset$, $\vee$, $\wedge$, $\forall$, $\exists$, λ, where the quantifiers $\forall$ and $\exists$ are counted only if they precede a bound individual variable. Now $Grade\ (A) = Grade_1(A) + Grade_2(A)$. In the same way can define $Grade(U)$ for a class term U.

A prime formula is a formula by classes 1 or 3 of the definition, with U a class variable or a class constant. A λ - prime formula is a formula of the form $t \epsilon \lambda b\,[b/x]A$.

Lemma 1. If A is a formula, U a class term and t an individual term, then $[t/x]A$ is a formula and $[t/x]U$ is a class term.

Lemma 2. If A is a formula, U is a class term and V is a class term such that no bound variable of V occurs in A or U, then $[V/X]A$ is a formula and $[V/X]U$ is a class term.

Corollary. An expression of the form $\forall\ b[b/x]A$ where A does not contain the bound individual variable b is a formula if and only if A is a formula. The same property holds for $\exists\ b[b/x]A$, $\forall\ H[H/X]A$, $\exists\ H[H/X]A$ and $\lambda b[b/x]A$ with the restriction in the last case that A does not contain bound class variables.

Lemma 1 and Lemma 2 can be easily proved by induction on $Grade(A)$ and $Grade(U)$. The corollary follows inmediately.

A finite sequence of formulas of the form $A_1, \ldots, A_k$ $(k \geqq 0)$ is called an L-sequence. Letters $M, N, S, T \ldots$ are used for L-sequences. The formulas $A_1, \ldots, A_k$ are called the components of the L-sequence. The notation $M < N$ means that every component of M is also a component of N. An expression of the form $M \vdash N$ is called an L-formula. We shall use also the obvious notation $[B/u]M$.

Definition of thesis. Some L-formulas are called thesis according with the inductive definition given by the following rules:

Rule (Ax) For each prime formula A, $A \vdash A$ is a thesis.

Rule $(=_1)$ For every individual term t, $\vdash t = \mathrm{t}$ is a thesis.

Rule $(=_2)$ If $M \vdash N, [t/x]A$ and $M \vdash N, t = h$ are thesis and A is a prime formula then $M \vdash N,\ [h/x]A$ is a thesis.

Rule (U) If $M \vdash N$ is a thesis, $M < M_1$, and $N < N_1$ then $M_1 \vdash N_1$ is a thesis.

Rule $(\neg *)$ If $M, A \vdash N$ is a thesis then $M \vdash N, \neg A$ is a thesis

Rule $(* \neg)$ If $M \vdash N, A$ is a thesis then $M, \neg A \vdash N$ is a thesis.

Bule ($\supset$*) If $M, A \mid\!- N, B$ is a thesis then $M \mid\!- N, A \supset B$ is a thesis.

Rule (*$\supset$) If $M \mid\!- N, A$ and $M, B \mid\!- N$ are thesis then $M, A \supset B \mid\!- N$ is a thesis.

Rule ($\vee$*) If $M \mid\!- N, A$ is a thesis then $M \mid\!- N, A \vee B$ and $M \mid\!- N, B \vee A$ are thesis.

Rule (*$\vee$) If $M, A \mid\!- N$ and $M, B \mid\!- N$ are thesis then $M, A \vee B \mid\!- N$ is a thesis.

Rule ($\wedge$*) If $M \mid\!- N, A$ and $M \mid\!- N, B$ *are* thesis then $M \mid\!- N, A \wedge B$ is a thesis.

Rule (*$\wedge$) If $M, A \mid\!- N$ is a thesis then $M, A \wedge B \mid\!- N$ and $M, B \triangle A \mid\!- N$ are thesis.

Rule ($\forall$ *) If $M \mid\!- N, A$ is a thesis, A does not contain the bound variable b, and the free variable x does not occur in M or N, then $M \mid\!- N, \forall b[b/x]A$ is a thesis.

Rule (*$\forall$) If $M, [t/x]A \mid\!- N$ is a thesis and A does not contain the bound variable b, then $M, \forall b[b/x]A \mid\!- N$ is a thesis.

Rule ($\exists$*) If $M \mid\!- N, [t/x]A$ is a thesis and A does not contain the bound variable b, then $M \mid\!- N, \exists b[b/x]A$ is a thesis.

Rule (*$\exists$) If $M, A \mid\!- N$ is a thesis, A does not contain the bound variable b, and the free variable x does not occur in M or N, then $M, \exists b[b/x]A \mid\!- N$ is a thesis.

Rule ($\forall_1$*) If $M \mid\!- N, A$ is a thesis, A does not contain the bound variable H and the free variable X does not occur in M or N, then $M \mid\!- N, \forall H[H/X]A$ is a thesis.

Rule (*$\forall_1$) If $M, [U/X]A \mid\!- N$ is a thesis and A does not contain the bound variable H, then $M, \forall H[H/X]A \mid\!- N$ is a thesis

Rule ($\exists_1$*) If $M \mid\!- N, [U/X]A$ is a thesis and A does not contain the bound variable H, then $M \mid\!- N, \exists H[H/X]A$ is a thesis.

Rule (*$\exists_1$) *If* $M, A \mid\!- N$ is a thesis, A does not contain the bound variable H, and the free variable X does not occur in M or N, then $M, \exists H[H/X]A \mid\!- N$ is a thesis.

Rule (λ*) If $M \mid\!- N, [t/x]A$ is a thesis, A does not contain bound class variables, and the bound variable b does not occur in A, *then* $M \mid\!- N, t \epsilon \lambda b[b/x]A$ is a thesis.

Rule (*λ) If $M, [t/x]A \mid\!- N$ is a thesis, A does not contain bound class variables, and the bound variable b does not occur in A, then $M, t \epsilon \lambda b[b/x]A \mid\!- N$ is a thesis.

The variable x in rules ($\forall$*) and (*$\exists$), and the variable X in rules ($\forall_1$*) and (*$\exists_1$) are called the proper variable of that application of the rule. We note that the variable x in rule $(^*\forall)$ is used just to indicate a substitution and can be taken arbitrarily. The same is true for rules $(\exists^*)$, $(^*\forall_1)$, $(\exists_1{}^*)$, (λ^*) and $(^*\lambda)$.

We define the relation $M \vdash N$ is a thesis with order n by the following rules:

i) If $M \vdash N$ is case of rule (Ax) or of rule $(=_1)$ then it is a thesis with order 0.

ii) If $M_1 \vdash N_1$ is a thesis with order n, and $M \vdash N$ results of the application of some rule to $M_1 \vdash N_1$ then $M \vdash N$ is a thesis with order $n+1$.

iii) If $M_1 \vdash N_1$ and $M_2 \vdash N_2$ are thesis with order n_1 and n_2 respectively, and $M \vdash N$ results of the application of some rule to $M_1 \vdash N_1$ and $M_2 \vdash N_2$ then $M \vdash N$ is a thesis with order $Max(n_1, n_2) + 1$.

We can define also the relation $M \vdash N$ is a thesis with general order n by the same rules with the only exception of rule (U) in which case the general order does not increase.

We state now several lemmas that can be easily proved by induction on n; they hold also if we replace order by general order.

Lemma 3. If $M \vdash N$ is a thesis with order n, then $[t/x]M \vdash [t/x]N$ is a thesis with order n.

Lemma 4. If $M \vdash N$ is a thesis with order n, $M_1 \vdash N_1$ is an L-formula obtained by changing all occurrences of a bound variable b (H) in $M \vdash N$ by another bound variable c (J) then $M_1 \vdash N_1$ is a thesis with order n.

Lemma 5. If $M \vdash N$ is a thesis with order n then $[Y/X]M \vdash [Y/X]N$ is a thesis with order n.

It is easy to show that for every formula A, $A \vdash A$ is a thesis. In order to generalize rule $(=_2)$ for an arbitrary formula A we need the following theorem.

Theorem 1. Let $S \vdash T, t = h$ be a thesis. For every formula A and number n the following hold: If $M \vdash N$ is a thesis with order n, $N < N_1$, $[t/x]A$ and $M < M_1$, $[t/x]A$ then

$$(1)\quad S, M \vdash T, N_1, [h/x]A$$

$$(2)\quad S, M_1, [h/x]A \vdash T, N$$

are both thesis.

Note that from $S \vdash T, t = h$ and $\vdash t = t$ we get by rule

($=_2$) the thesis $S \mid\!- T, h = t$. Note also that if A is a prime formula (1) follows immediately by rule ($=_2$). Furthermore (1) is trivial if $[t/x]A$ is not *a* component of N and (*2*) is trivial if $[t/x]A$ is not a component of M.

The proof of the theorem is by induction. We assume the property holds for every formula A' and number m if one at least of the three following conditions is satisfied.

(i) $Grade_1(A') < Grade_1(A)$

(ii) $Grade_1(A') = Grade_1(A)$ and $Grade_2(A') < Grade_2(A)$.

(iii) $Grade_1(A') = Grade_1(A)$, $Grade_2(A') = Grade_2(A)$, $m < n$.

In every application of part (iii) of the induction hypothesis the formula A' will be the same formula A. For the proof of (*1*) we assume that $[t/x]A$ is a component of N, and for the proof of (*2*) that it is a component of M.

a) $M \mid\!- N$ is a case of rule (Ax). Hence $M \mid\!- N$ is the thesis $[t/x]A \mid\!- [t/x]A$ with A a prime formula. We have remarked above that (*1*) follows by rule ($=_2$); (*2*) also follows by rule ($=_2$) from the thesis $S \mid\!- T, h = t$ and $[h/x]A \mid\!- [h/x]A$.

b) $M \mid\!- N$ is a case of rule ($=_1$) is clear by the remark we have made above.

c) $M \mid\!- N$ is obtained by some rule from a premise or premises with smaller order and $[t/x]A$ is not the formula introduced by the rule. Hence that formula is a component of N_1 if it is a right rule, or a component of M_1 if it is a left rule. We show with one example how this case is handled. Suppose we have a derivation by rule ($\supset$*) in this way

$$\frac{M, B \mid\!- N', C}{M \mid\!- N', B \supset C}$$

Note that $M, B < M_1, B, [t/x]A$ and $N', C < N_1, C, [t/x]A$. Using part (iii) of the induction hypothesis we get

$$(1') \quad S, M, B \mid\!- T, N_1, C, [h/x]A$$

$$(2') \quad S, M_1, B, [h/x]A \mid\!- T, N', C$$

and now we get (*1*) *from* (*1'*) by rule (U), rule ($\supset$*) and again rule (U) since $B \supset C$ is a component of N_1; from (*2'*) we obtain (*2*) with rule (U) and rule ($\supset$*).

To handle any rule with restriction on a proper variable we use Lemma 3 or Lemma 5.

d) $M \vdash N$ is obtained by some rule and $[t/x]A$ is the formula introduced by the rule. Here we apply part (iii) of the induction hypothesis and afterward part (i) or part (ii) of the induction hypothesis. We show this in several examples.

The formula A is $\neg B$ and we have an application of rule $(\neg^*)$

$$\frac{M, [t/x]B \vdash N'}{M \vdash N', [t/x] \neg B}$$

Using part (iii) of the induction hypothesis as in *c*) we get

$$(1') \quad S, M, [t/x]B \vdash T, N_1, [h/x] \neg B$$

$$(2') \quad S, M_1, [t/x]B, [h/x] \neg B \vdash T, N'$$

Now (2) follows from (2') with rule (U) and rule $(\neg^*)$. Since $Grade_2(B) < Grade_2(\neg B)$ we apply part (ii) of the induction hypothesis to (1') to obtain

$$S, M, [h/x]B \vdash T, N_1, [h/x] \neg B$$

and from this (1) is obtained by rule (U) and rule $(\neg^*)$.

The formula A is $g \in \lambda b[b/y]B$ and rule (λ^*) is applied. We can assume that y is distinct from x and does not occur in t or h. Hence $[t/x]A$ is the formula $[t/x]g \in \lambda b[b/y][t/x]A$ and $[h/x]A$ is the formula $[h/x]g \in \lambda b[b/y][h/x]B$.

$$\frac{M \vdash N', [[t/x]g/y][t/x]B}{M \vdash N', [t/x]g \in \lambda b[b/y][t/x]B}$$

Using part (iii) of the induction hypothesis we get

$$(1') \quad S, M \vdash T, N_1, [[t/x]g/y][t/x]B, [h/x]g \in \lambda b[b/y][h/x]B$$

$$(2') \quad S, M_1, [h/x]g \in \lambda b[b/y][h/x]B \vdash T, N', [[t/x]g/y][t/x]B$$

Now (2) follows from (2') using rule (λ^*); to get (1) note that the formula $[[t/x]g/y][t/x]B$ is identical with the formula $[t/x][g/y]B$ and since $Grade_2([g/y]B) < Grade_2(A)$ we apply part (ii) of the induction hypothesis to obtain from (1')

$S, M \mid\!- T, N_1 \ [[h/x]g/y][h/x]B, \ \ [h/x]g \in \lambda b[b/y][h/x]B$
and from this (*1*) follows using rule (λ*).

The formula A is $\forall H[H/Y]B$ and rule (*$\forall_1$) is applied. Now $[t/x]A$ is the formula $\forall H[H/Y][t/x]B$ and $[h/x]A$ is the formula $\forall H[H/Y][h/x]B$.

$$\frac{M', [U/Y][t/x]B \mid\!- N}{M', \forall H[H/Y][t/x]B \mid\!- N}$$

Using part (iii) of the induction hypothesis we get

(*1'*) $S, M', [U/Y][t/x]B \mid\!- T, N_1, \forall H[H/Y][h/x]B$

(*2'*) $C, M_1, [U/Y][t/x]B, \forall H[H/Y][h/x]B \mid\!- T, N$

Now (*1*) follows from (*1'*) using rule (*$\forall_1$). To obtain (*2*) note that by changing the formula B we can assume that the variable x does not occur in U; this change consists just in replacing a free variable by another free variable. Hence the formula $[U/Y][t/x]B$ is also the formula $[t/x][U/Y]B$; *since* $Grade_1([U/Y]B) < Grade_1(A)$ we can apply part (i) of the induction hipothesis to obtain

$$S, M_1 \ [U/Y][h/x]B, \forall H[H/Y][h/x]B \mid\!- T, N$$

and from this (*2*) follows by rule (*$\forall_1$).

The other cases can be handled in the same way; if there is a restriction on a proper variable we use Lemma 3 or Lemma 5.

Corollary. If $M \mid\!- N, t = h$ and $M \mid\!- N, [t/x]A$ are thesis then $M \mid\!- N, [h/x]A$ is a thesis.

Theorem 2. If $M \mid\!- N$ is a thesis, and $[U/Y]M \mid\!- [U/Y]N$ is an L-formula, then $[U/Y]M \mid\!- [U/Y]N$ is also a thesis.

Proof by induction on the order of $M \mid\!- N$; the Corollary to Theorem 1 is used to handle the case in which rule ($=_2$) is applied.

Theorem 3. If $M \mid\!- N$ is a thesis with order n, and M_1 is obtained from M by eliminating components of the form $t = t$, then $M_1 \mid\!- N$ is a thesis with order n.

The proof by induction on n is completely trivial.

Theorem 4. For every formula A and numbers n and m the following hold: given L-sequences S, T, M, N, T_1, M_1 such that

(*1*) $S \mid\!- T$ is a thesis with order n

(*2*) $M \mid\!- N$ is a thesis with order m

(*3*) $T < T_1,A$ and $M < M_1,A$

then

(*4*) $S, M_1 \mid\!- T_1,N$ is also a thesis.

The proof is by induction. We assume the theorem is true for a formula *A'* and number n_1 and m_1 if one of the three following conditions is satisfied

(i) $Grade_1(A') < Grade_1(A)$

(ii) $Grade_1(A') = Grade_1(A)$ and $Grade_2(A') < Grade_2(A)$

(iii) $Grade_1(A') = Grade_1(A)$, $Grade_2(A') = Grade_2(A)$, $n_1 + m_1 < n + m$.

Note that the theorem is trivial if A is not a component of T or if it is not a component of M. The cases $n = 0$ or $m = 0$ follow easily using rule (U) or Theorem 3. In the following we shall assume $n + m > 0$.

a) (*1*) is obtained by a left rule. We show how this case is handled with an example. Suppose rule ($* \supset$) is applied.

$$\frac{S' \mid\!- T,B \qquad\qquad S',C \mid\!- T}{S',B \supset C \mid\!- T}$$

Using part (iii) of the induction hypothesis we get

$$S',M_1 \mid\!- T_1,B,N$$

$$S',C,M_1 \mid\!- T_1,N$$

Now we get (*4*) using rule ($* \supset$).

b) (*1*) is obtained by a right rule and the formula introduced by the rule is a component of T_1. We proceed as in case *a*) using rule (U) to eliminate the formula introduced by the rule.

c) (*2*) is obtained by a right rule or by a left rule such that the formula introduced by the rule is a component of M_1. We use here the same method as in cases *a*) or *b*).

d) (*1*) is obtained by a right rule such that the formula introduced by the rule is not a component of T_1, and (*2*) is obtained by a left rule such that the formula introduced by the rule is not a component of M_1. In both cases this formula must be the formula A; hence A is not a prime formula and the case in which (*1*) is obtained by rule $(=_2)$ cannot occur. We shall show in one example the procedure used to handle this case. Suppose A is the formula $\forall H[H/X]B$ with the following derivations

$$\frac{S \mid\!- T', B}{S \mid\!- T', \forall H[H/X]B} \qquad \frac{M', [U/X]B \mid\!- N}{M', \forall H[H/X]B \mid\!- N}$$

Using Lemma 5 we can assume that the variable X does not occur in M_1, T_1 or N. First we apply part (iii) of the induction hypothesis to obtain

$$S, M_1 \mid\!- T_1, B, N$$

$$S, M_1, [U/X]B \mid\!- T_1, N$$

By Theorem 2 we have also the thesis

$$S, M_1 \mid\!- T_1, [U/X]B, N$$

and since $Grade_1([U/X]B) < Grade_1(A)$ we can apply part (i) of the induction hypothesis to obtain (*4*).

Corollary. If $S \mid\!- T, A$ and $M, A \mid\!- N$ are thesis then $S, M \mid\!- T, N$ is also a thesis.

Theorem 5. If $M \mid\!- N$ is a thesis with general order n, which does not contain bound class variables, $N < N_1$, $t_1 = h_1, \ldots, t_k = h_k$ $(k \geqq 0)$ where for each $i = 1, \ldots, k$, t_i is not identical with h_i, and furthermore the symbol $=$ does not occur in M and occurs in N_1 only in prime formulas, then $M \mid\!- N_1$ is also a thesis with general order $m \leqq n$.

The proof by induction on n is easy. We consider only the case in which $M \mid\!- N$ is obtained by rule $(=_2)$. Suppose we have the derivation

$$\frac{M \mid\!- N', t = h \qquad M \mid\!- N', [t/x]A}{M \mid\!- N', [h/x]A}$$

If t and h are the same term the right premise and the conclusion are the same L-formula and we apply the induction hypothesis. If this is not the case it follows that

$$N', t = h < N_1, t = h, t_1 = h_1, \ldots, t_k = h_k$$

hence by the induction hypothesis applied to be left premise it follows that $M \mid\!- N_1$ is a thesis.

We say that a thesis is independent of some rules of the system *LKP* if we can show it is a thesis without using those rules.

Corollary. If $M \mid\!- N$ is a thesis with general order n, which does not contain bound class variables, and the symbol $=$ does not occur in it, then $M \mid\!- N$ is independent of rules $(=_1)$ and $(=_2)$.

Proof by induction on n using Theorem 5.

2. THE SYSTEM *LK**. It does not seem possible to prove the Herbrand-Gentzen Theorem in the system *LKP*. For this reason we shall study a system *LK**, which is a subsystem of *LKP*. The new system is obtained from *LKP* if we drop rules (λ^*) and $(^*\lambda)$ and furthermore we allow in rules (Ax) and $(=_2)$ that the formula A be a prime formula or a λ-prime formula. A thesis in the system *LK** will be called an elementary thesis. We define order and general order as before.

The system *LK** is essentially the system *LK* of first order logic and the λ-prime formulas behave as prime formulas since no rule allows the introduction of such a formula. Theorem 4 can be proved by any of the standard methods of proving the elimination theorem; our proof can be applied replacing parts (i) and (ii) of the induction hypothesis by a condition on the number of quantifiers and propositional symbols occurring in the formula A not counting those occurring in a λ-term. In this way we do not need the restriction on the occurrence of bound class variables in a λ-term.

A λ-axiom is a formula which is either of the form $t \in \lambda b/b[x]B \supset [t/x]B$ or of the form $[t/x]B \supset t \in \lambda b[b/x]B$. It is easy to show that $M \mid\!- N$ is a thesis if and only if for some T consisting of λ-axioms is $M,T \mid\!- N$ an elementary thesis.

A Q-free formula is a formula in which every quantifier occurs inside a λ-term. Now every Q-free formula is a formula in prenex normal form. If A is in prenex normal form then $\forall\, b[b/x]A$, $\exists\, b[b/x]A$, $\forall\, H[H/X]A$ and $\exists\, H[H/X]A$ are formulas in prenex normal form, provided they are formulas.

An application of rule (U) from $M \vdash N$ to $M_1 \vdash N_1$ is called restricted if $M_1 < M$, $N_1 < N$ and furthermore no component is repeated in M_1 or N_1.

We say that $M \vdash N$ is Q-derivable from $S \vdash T$ if it can be obtained starting with $S \vdash T$ and applying rule (U) or quantifier rules. If every application of rule (U) is restricted we say that $M \vdash N$ is QR-derivable from $S \vdash T$.

Lemma 6. If $M \vdash N$ is Q-derivable from $S \vdash T$ then there is $M_1 \vdash N_1$ which is QR-derivable from $S \vdash T$ and $M_1 < M, N_1 < N$.

Lemma 7. If $M \vdash N$ is Q-derivable from $S \vdash T$, the formulas B and C are Q-free, S' and M' are obtained by eliminating all components B in S and M respectively, T' and N' are obtained by eliminating all components C in T and N respectively, then $M' \vdash N'$ is Q-derivable from $S' \vdash T'$.

Now suppose $M \vdash N$ is QR-derivable from $S \vdash T$ and x is a proper variable in the derivation. Let y be a variable not occurring in $S \vdash T$; then $M \vdash N$ is QR-derivable from $[y/x]S \vdash [y/x]T$ and x is not a proper variable in the new derivation. The same procedure can be used for a class variable X.

Let A be a formula in prenex normal form; then for some Q-free formula B the formula A is of the form

$$Q_1 \ldots Q_n [R_1/T_1] \ldots [R_n/T_n] B$$

where R_i is a bound variable, T_i is a free variable of the same kind, and Q_i is a quantifier followed by R_i, $i = 1, \ldots, n$. Now let $U_1, \ldots, U_n$ be terms such that U_i is of the same kind as T_i, and if Q_i is a universal quantifier then U_i is a variable not occurring in $A, U_1, \ldots, U_{i-1}$, $i = 1, \ldots, n$. Then if

$$[U_1/T_1] \ldots [U_n/T_n] B$$

is a formula it is called a right reduced form of A. In this definition the substitution prefix must be understood in the sense of a simultaneous substitution of $U_1, \ldots, U_n$ for $T_1, \ldots, T_n$ in B. If Q_i is an universal quantifier we say that U_i is a proper variable in the i^{th} place of the right reduced form.

We define a left reduced form of A in a similar way, but we require that U_i be a variable when Q_i is an existential quantifier, and in this case is also called a proper variable in the i^{th} place of the left reduced form.

Let $B_1, \ldots, B_k$ be right reduced forms of a formula A, and suppose B_i is determined by $U^i{}_1, \ldots, U^i{}_n$, $i = 1, \ldots, n$. We say they are compatible right reduced forms if whenever a variable is proper in B_i and B_j then it is proper in the same place, say the s^{th} place, and for every $m < s$ the terms $U^i{}_m$ and $U^j{}_m$ are identical, $i = 1, \ldots, n$, $j = 1, \ldots, n$.

In the same way are defined compatible left reduced forms of a formula A.

Our next result is the so called Herbrand-Gentzen Theorem. We give a new proof of this Theorem which seems to be more convenient than the proof given by Gentzen.

Theorem 6. Let $M \mid\!- N$ be an elementary thesis with all the components in prenex normal form. Then there are elementary thesis $S \mid\!- T$ and $M_1 \mid\!- N_1$ such that:

i) The components of $S \mid\!- T$ are Q-free formulas, $M_1 \mid\!- N_1$ is QR-derivable from $S \mid\!- T$ and $M_1 < M$, $N_1 < N$.

ii) With each component of M_1 we can associate formulas $B_1, \ldots, B_k$ $(k > 0)$ in S that are compatible left reduced forms of that component. Under this association every formula in S corresponds to some formula in M_1. The same property holds for N_1 and T with right reduced forms.

iii) If under the correspondence of ii) a variable is proper in two forms in $S \mid\!- T$, then both corresponds to the same component of $M_1 \mid\!- N_1$, and the variable is proper in the QR-derivation from $S \mid\!- T$ to $M_1 \mid\!- N_1$.

Suppose $M \mid\!- N$ is an elementary thesis with order n. It is easy to show by induction on n that the elementary thesis $S \mid\!- T$ exists and $M \mid\!- N$ is Q-derivable from it. This is done using Lemmas 6 and 7. We show in one example a standard procedure which can be applied in any case. Suppose $M \mid\!- N$ is obtained by rule $(=_2)$ in this way

$$\frac{M \mid\!- N',\ t = h \qquad\qquad M \mid\!- N',\ [t/x]A}{M \mid\!- N',\ [h/x]A}$$

By the induction hypothesis $M \mid\!- N'$, $t = h$ is Q-derivable from $S_1 \mid\!- T_1$ and $M \mid\!- N'$, $[t/x]A$ is Q-derivable from $S_2 \mid\!- T_2$. From Lemma 7 follows that $M \mid\!- N'$ is Q-derivable from $S_1 \mid\!- T'_1$ where T'_1 is obtained by eliminating all *components* $t = h$ in T_1. We may assume no proper variable in this derivation occurs in M, N' or

$[h/x]A$. Also from Lemma 7 follows that $M \mid\!- N'$ is Q-derivable from $S_2 \mid\!- T'_2$ where T'_2 is obtained by eliminating all components $[t/x]A$ in T_2; we may assume no proper variable in this derivation occurs in S_1, T'_1, M, N' or $[h/x]A$. Now from the elementary thesis $S_1 \mid\!- T'_1,\ t = t$ and $S_2 \mid\!- T'_2,\ [t/x]A$ we get using rule $(=_2)$ the thesis $S_1, S_2 \mid\!- T'_1, T'_2, [h/x]A$ and from this by Q-derivation is obtained $M \mid\!- N', [h/x]A$.

Once it is proved the existence of $S \mid\!- T$ we apply Lemma 6 to obtain $M_1 \mid\!- N_1$ which is QR-derivable from $S \mid\!- T$. The correspondence of parts ii) and iii) of the theorem are related in an abvious way with the steps of the QR-derivation. It can be shown in detail by induction on the number of rules applied.

We shall write $A \equiv B$ as an abbreviation for the formula $(A \supset B) \wedge (B \supset A)$. It is well-known that for every formula A there is a formula B in prenex normal form, such that $\mid\!- A \equiv B$ is an elementary thesis. A formula is called elementary if it does not contain bound class variables or class terms. It is called quasi-elementary if it does not contain bound class variables, and every class term occurrin in it is a λ-term. It is easy to show that for every quasi-elementary formula A there is an elementary formula B such that $\mid\!- A \equiv B$ is a thesis. This formula B is obtained by replacing parts of the form $t \in b[b/x]C$ by $[t/x]C$. An $\forall$-formula ($\exists$-formula) is a formula of the form $\forall H[H/X]C$ ($\exists H[H/X]C$) where C is an elementary formula in prenex normal form.

An elementary form of an $\forall$-formula $\forall H[H/X]C$ is either a formula $\forall b[b/X]C$ or is a formula obtained from the formula $[U/X]A$, where U is a quasi-elementary term, by prefixing universal quantifiers for all variables in U, with the understanding that those quantifiers must not bind occurrences of the variables outside U. We define similarly the elementary form of an $\exists$-formula but using existential quantifiers instead of universal quantifiers.

Theorem 7. Let $M, S \mid\!- T, N$ be a thesis, where the components of M and N are elementary formulas in prenex normal form, the components of S are $\forall$-formulas, and the components of T are $\exists$-formulas. Then there is a thesis $M, S' \mid\!- T', N$ where the components of S' and T' are elementary forms of the formulas in S and T respectively.

Proof: there is an elementary thesis $M, S, M_1 \mid\!- T, N$ where the components of M_1 are λ-axioms, and we may assume they have been replaced by formulas in prenex normal form. From Theore m6 follows there is an elementary thesis $S_1 \mid\!- T_1$ from which the former thesis can be obtained by a Q-derivation. We obtain a thesis $S'_1 \mid\!- T'_1$ if we replace every free class variable and class constant by some individual constant. This substitution does not affect those formulas that are reduced forms of the formulas in M or N. We can repeat the Q-derivation but now the class quantification is replaced by the quantification of all individual variables in the corresponding terms. In place of M_1 we get formulas M'_1 which are again λ-axioms, hence can be eliminated. In place of S and T we get S' and T' which are elementary forms of the formulas in S and T respectively.

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