ON THE EXTENSION OF CURRENTS

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1. Let A be an open set in number space R^n and F a closed subset of A. Let T be a closed o-continuous current on $A-F$ and $\lambda > 1$ a real number such that the following condition $C\lambda$ is satisfied on A: $C\lambda$ — For each relatively compact open set G such that $G^- \subset A$ there exists a constant $k(G) \geqslant 0$ such that $\| T \|_{Br} \leqslant$ $\leqslant k(G)\, r^\lambda$ for every open ball B_r of radius r contained in G.

P. Lelong has proved in [2] (theor. 4) that, if F is a submanifold of class C^s $(s \geqslant 2)$ of A of dimension $d < \lambda - 1$, then the simple extension of T on A exists and is closed. This is used in [2] to prove that the integration current on a complex analytic set is closed. Here we give a consequence of the first result and show to apply it to the integration on a semianalytic (orientable) set. The properties of currents we use implicitly here are to be found in [1] y [2].

2. For each open set A in R^n, let $\mathcal{D}(A)$ be the space of C^∞ — differential forms on A with compact support. The norm $\| \phi \|$ of a form $\phi \in \mathcal{D}(A)$ is the maximum on A of the absolute values of the coefficients of ϕ. If T is a linear form on $\mathcal{D}(A)$ and G is a relatively compact open set in R^n, the norm $\| T \|_G$ of T on G is defined by

$$\| T \|_G = \sup \, (| T(\phi) | : \phi \in \mathcal{D}(G \cap A) \text{ and } \| \phi \| \leqslant 1).$$

A o-continuous current T on A is a linear form on $\mathcal{D}(A)$ such that $\| T \|_G$ is finite for each relatively compact open set G with $G^- \subset A$. All the currents occurring here are o-continuous. For every open subset U of A, $T|U$ denotes the restriction of T on U.

If A is open in R^n, F closed in A and T is a current on $A-F$ such that $\| T \|_G$ is finite for each relatively compact open set G

with $G^- \subset A$, then it is proved in [2] that there exists one and only one extension T' of T on A (i.e., $T'|A - F = T$) that verifies $\| T' \|_G = \| T \|_G$ for such G; T' is called the simple extension of T on A.

2.1. Definition: *Let T be a current on the open set A in R^n and F a subset (not necessarily closed) of A. The norm of T on F is zero* ($\| T \|_F = 0$) *if, for each compact set K in A and each* $\varepsilon > 0$, *there exists a relatively compact open set G in A such that* $K \cap F \subset G \subset G^- \subset A$ and $\| T \|_G \leqslant \varepsilon$.

$\| T \|_F = 0$ if and only if $\| T|U \|_{U \cap F} = 0$ for each member U of an open covering of A. If (F_i) is a denumerable family of subsets of A such that $\| T \|_{F_i} = 0$ for all i, then $\| T \|_{\cup F_i} = 0$.

2.2. Proposition: *Let A be an open set in R^n, F a closed subset of A, T a current on $A - F$ and T' an extension of T on A. Then the following statements are equivalent*:

(i) T' *is the simple extension of T on A.*

(ii) $\| T \|_G = \| T' \|_G$ *for any relatively compact open G with* $G^- \subset A$.

(iii) $\| T' \|_F = 0$.

Proof: The equivalence between (i) and (ii) has been proved in [2]. Let us suppose (ii) true but not (iii). Then there exists a compact set $K \subset A$ and $\varepsilon > 0$ such that for each relatively compact open set G, $K \cap F \subset G \subset G^- \subset A$ implies $\| T' \|_G > \epsilon$. Choose one such G and, by condition (ii), a form $\phi \in \mathcal{D}(G - F)$ with support K_1 such that $T'(\phi) > \varepsilon$ and $\| \phi \| \leqslant 1$. By the same argument choose a $\phi_1 \in \mathcal{D}(G - (F \cap K_1))$ such that $T'(\phi_1) > \epsilon$ and $\| \phi_1 \| \leqslant 1$. After a finite number of steps a form $\psi = \phi + \phi_1 + \dots$ is obtained such that $\psi \in \mathcal{D}(G)$, $\| \phi \| \leqslant 1$ and $T'(\psi) > \| T' \|_G$, which is absurd.

Conversely, if (iii) holds but not (ii), let G be a relatively compact open set in A such that $\| T \|_G \neq \| T' \|_G$; then $\| T' \|_G - \| T \|_G = 2d > 0$ and there exists $\phi \in \mathcal{D}(G)$ such that $\| \phi \| \leqslant 1$ and $T'(\phi) > \| T \|_G + d$; let K be the support of ϕ. By (iii) there exists an open set U which verifies $K \cap F \subset U \subset U^- \subset A$ and $\| T' \|_U < d$. Let $\phi_1 \in \mathcal{D}(U)$ and $\phi_2 \in \mathcal{D}(G - F)$ be forms such that $\phi = \phi_1 + \phi_2$, $\| \phi_1 \| \leqq 1$ and $\| \phi_2 \| \leqq 1$. Then $| T(\phi_2) | \leqq \| T \|_G$ and consequently $T'(\phi_1) = T'(\phi) - T'(\phi_2) \geqq T'(\phi) - \| T \|_G > d$, which contradicts the choice of U.

2.3. Corollary: *Let A be an open set in R^n and* $F_0 \subset F_1$

closed subsets of A. Let T_0 be a current on $A - F_0$ such that $\| T_0 \|_{F1-F0} = 0$ and let $T_1 = T_0|A - F_1$. Then if one of T_0 or T_1 has a simple extension on A, so does the other, and both simple extensions are equal.

2.4. Lemma: *Let A be an open set in R^n and T a current on A which verifies condition C_λ $(\lambda > 1)$ on A. Then $\| T \|_M = 0$ for each submanifold M of A of class C^s $(s \geqq 1)$ and dimension $d < \lambda$.*

Proof: Choose $P \epsilon M$ and a coordinate map $\chi : U \rightarrow R^n$ of class C^s of a neighborhood U of P such that $\chi(U \cap M) = R^d \subset R^n$. It suffices to prove $\| T|U \|_{M \cap U} = 0$. As χ is a $\mathcal{C}^s$ — isomorphism, this is equivalent to $\| T' \|_{R^d} = 0$, where $T' = \chi(T/U)$. As $T|U$ satisfies $C\lambda$ on U, so does T' on R^n and, because of an usual reasoning in measure theory (see [2], theor. 3), we have $\| T' \|_{R^d} = 0$.

2.5. Proposition: Let *A be an open set in R^n and F a closed subset of A. Let T be a closed current on A—F. Let us suppose the following two conditions are verified*:

(a). — F is contained in a denumerable disjoint union of submanifolds V_i of A, of class C^s $(s \geqq 2)$, of dimensions $d_i \leqq d$ $(0 \leqq d < n)$, and such that $(V_i^- - V_i) \cap A \subset \cup (V_j : j > i)$ and $\cup (V_j : j \geqq i)$ is closed for all i.

(b). — T verifies condition $C\lambda$ on A with $\lambda > d + 1$.

Then the simple extension of T on A exists and is closed.

Proof: By (b) $\| T \|_G$ is finite for each relatively compact open G with $G^- \subset A$; then the simple extension $\tilde{T}$ of T on A exists ([2], Prop. 1). By (b) and 2.4 $\| T \|_{V-F} = 0$ for all i, therefore $\|T\|_{\cup V_i - F} = 0$ and the simple extension $\tilde{T}'$ of $T' = T|A - (\cup V_i)$ on A exists and is equal to $\tilde{T}$ by 2.3. Consequently it suffices to prove that $\tilde{T}'$ is closed. For each i let T'_i be the restriction of $\tilde{T}'$ on $A - (\cup V_j : j \geqq i)$, where $(\cup V_j : j \geqq i)$ is closed because of (a). Let us prove by induction that each T'_i is closed. Since T is closed, so is $T'_1 = T'$. Let us suppose T'_i is closed. By 2.3 T'_{i+1} is the simple extension of T'_i on $A - (\cup V_j : j \geqq i + 1) = (A - (\cup V_j : j \geqq i)) \cup V_i$ and, according to the theorem of Lelong quoted in § 1, it is closed. This implies that $\tilde{T}'$ is closed, as was to be shown.

By simplicity we have worked in an open set in R^n. All remains valid in a differentiable manifold of class C^s $(s \geqq 2)$.

3. AN APPLICATION.

3.1. Let X be a connected real analytic manifold. For each $x \in X$, let $S(x)$ be the smallest family of germs of sets in x such that: (1) $a \in S(x)$ and $b \in S(x)$ imply $a \cup b \in S(x)$ and $a - b \in S(x)$; (2) $S(x)$ contains all the germs defined in x by sets of the form $(f(y) > 0)$, where f is a real analytic function defined in a neighborhood of x. Following S. Lojasiewicz ([6]), a subset M of X is semianalytic if, for each $x \in X$, the germ of M in x belongs to $S(x)$.

Let M be a semianalytic set in X; a point $x \in M$ is p-regular if there exists an open neighborhood U of X such that $U \cap M$ is an analytic submanifold of X of dimension p. The set of regular points (*i.e.*, p-regular points for some p) of M is dense in M. The dimension *dim* (M) of M is $\leqq p$ if there are not q-regular points in M with $q > p$. We set *dim* $(M) = p$ if *dim* $(M) \leqq p$ but not *dim* $(M) \leqq p - 1$. If *dim* $(M) = p$, we denote by M_p the set of p-regular points of M and define $\partial(M) = M - M_p$; $\partial(M)$ is a semianalytic set of dimension $\leqq p - 1$ and we can call it the singular part of M; if M is closed, so is $\partial(M)$. Now it is possible to decompose $\partial(M)$ in its $(p-1)$ — regular part and its singular part, and so on.

If M is closed semianalytic of dimension p, we can then write $M = (\cup V_i : i = 1, \ldots, p)$ where the V_i are disjoint analytic submanifolds of X of dimension i and $V_i^- - V_i \subset \cup (V_j : j < i)$ for each $i = 0, \ldots, p$ (*cf.* 2.5). The family of the connected components of M_p (*dim* $(M) = p$) is locally finite (¹).

3.2. Let M be a closed semianalytic set of dimension p of an open set A of R^n such that M_p is oriented. The restriction of each form $\phi \in \mathcal{D}(A - \partial M)$ to M_p is a form $\phi^* \in \mathcal{D}(M_p)$ with compact support. Then the current

$$I^0_M(\phi) = \int_{M_p} \phi$$

on $A - \partial M$ is well defined, and is a o-continuous current of di-

(¹) All this properties have been proved by Lojasiewicz (unpublished). They were enunciated in a course given in 1964 at the University of Buenos Aires. Summaries of results will be given in [4] and [5]. Some of the facts are treated in [3] and [6]..

mension p. In a forthcoming paper it will be proved that I^0_M satisfies condition C_λ of § 1 with $\lambda = p$. Then the simple extension I_M of I^0_M on A exists; we call it the integration current on M. (cf. [2]). If we recall the decomposition of $\partial(M)$ into submanifolds (3.1) and Proposition 2.5, we see that I_M is closed when dim $\partial(M) \leqslant \dim(M) - 2$. Trivial examples (consider the interval [0,1] in R) show that I_M is not closed in general if this restriction is not imposed. Moreover, the extension of this definition and properties to a semianatytic set M in a real analytic manifold X is inmediate. In the case X is a complex analytic manifold and M a complex analytic set in X of (complex) dimension p, the current I_M defined in this way by considering the canonical orientation of M_p coincides with the one defined by Lelong in [2]. It is always closed because $\dim_R \partial(M) \leqslant \dim_R M - 2 = 2p - 2$.

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