A PROPERTY OF COMBINATIONS IN NORMAL FORM

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The notion of the normal form of a combination was introduced in [4]. In his thesis [2] Lercher has studied several properties of those combinations that are in normal form; he has shown that they are exactly the combinations that are strongly irreducible. In this note we prove that if $[Z/x]Y$ is in normal form, then both Z and Y are in normal form, provided for Z that Y contains x. In the proofs we use several ideas of [2].

We consider a system of combinations generated by three primitive combinators: S, K, I and also other atoms that are called indeterminates. We assume there are infinitely many indeterminates. The primitive combinators and indeterminates are combinations; if X and Y are combinations, then the ordered pair consisting of X and Y in that order, which we write (XY) or simply XY, is a combination. Letters X, Y, Z, U. V will denote combinations; letters x, y, z denote indeterminates. We shall omit parentheses with the understanding that the association is to the left. A combination is open if it is of one of the forms: S, K, I, SX, KX, SXY; it is closed if it is of the form $xX_1 \ldots Xk$, $k \geqq 0$, for some indeterminate x. A redex is a combination of one of the forms: $SXYZ$, KXY, IX; the contracta of those redexes are respectively: $XZ(YZ)$, X ,X. If U is a combination and V is obtained by replacing a part of U which is a redex by its contractum, we say that V is a contraction of U. Now $X \geqq Y$ means that there are combinations $X_1, \ldots, X_k$, $k \geqq 1$, such that X_1 is X, X_k is Y, and for each $i = 1, \ldots, k-1$ X_{i+1} is a contraction of X_i. The following fundamental theorem, that was proved by Rosser for a restricted system of combinatory logic, holds also in our system: If $X \geqq Y$ and $X \geqq Z$, there is a combination U such that $Y \geqq U$ and $Z \geqq U$. A combination that does not contain redexes is called irreducible. The notation $X = Y$ means that X and Y are the same combination. Note that this is not the usual meaning of $=$ in combinatory logic.

We introduce now the abstraction operator. Given a combination X and an indeterminate x, we define a combination $[x]X$ by the following rules:

(i) If X is x, then $[x]X = I$

(ii) If X does not contain x, then $[x]X = KX$.

(iii) If $X = Yx$ where Y does not contain x, then $[x]X = Y$

(iv) If $X = YZ$, and no other rule can be applied, then $[x]X = SUV$ where $U = [x]Y$, $V = [x]Z$.

We define also a substitution operator. If X and Z are combinations and x is an indeterminate, then $[Z/x]X$ is the combination defined by the following rules:

(i) If X is x, then $[Z/x]X = Z$

(ii) If X is atomic, but is not x, then $[Z/x]X = X$

(iii) If $X = UV$ then $[Z/x]X = U_1V_1$ where $U_1 = [Z/x]U$, $V_1 = [Z/x]V$.

Lemma 1.

a) $[x]X = [y][y/x]X$ if y does not occur in $[x]X$.

b) $[x]\,[Z/y]X = [Z/y]\,[x]X$ if x is distinct from y and Z does not contain x.

c) If $Y = [x]X$, then $YZ \geqq [Z/x]X$.

Proofs on these properties are given in [1] and [4].

The number of occurrences of S, K, and I in X is denoted $n(X)$. It is clear that if $X = [x]Y$ then $n(Y) \leqq n(X)$.

Now we define by induction the combinations that are in normal form:

(i) Every indeterminate is in normal form.

(ii) If $X = xX_1 \ldots X_k$, $k \geqq 1$, and $X_1, \ldots, X_k$ are combinations in normal form, then X is a combination in normal form.

(iii) If X is in normal form then $[x]X$ is in normal form.

Note that a combination in normal form is either open or closed.

Lemma 2. If U is in normal form, Z is closed and in normal form, then $[Z/x]U$ is in normal form.

This is a special case of Theorem 11 in [4], Chapter I.

Lemma 3. If U is in normal form and Ux is irreducible, then Ux is in normal form.

This is clear when U is closed or it is S or K. If U is SV then for z not occurring in V is SVz in normal form. By Lemma 2 is SXx in normal form.

Corollary. If U is in normal form, Z is closed and in normal form, and UZ is irreducible, then UZ is in normal form.

Theorem 1. If X is in normal form then every part of X is in normal form.

The proof is by induction on $n(X)$. The case $n(X) = 0$ is trivial. For $n(X) > 0$ we consider the following cases.

(i) X is S, K or I. This case is trivial.

(ii) X is KY. Hence Y is in normal form, and by the induction hypothesis every part of Y is in normal form.

(iii) X is SY. Hence SYx is in normal form for x not occurring in Y. Hence there is a combination U such that $SYx = [y]U(xy)$ and $Y = [y]U$ and $U(xy)$ is in normal form. Since $n(U(xy)) \leqq n(Y) < n(X)$ it follows from the induction hypothesis that Y is in normal form and every part of Y is in normal form.

(iv) X is SYZ. Hence there is a combination UV in normal form such that $X = [x]UV$, $Y = [x]U$, $Z = [x]V$. By the induction hypothesis we have that Y and Z are in normal form, and every part of Y and Z is in normal form. We have to show only that SY is in normal form. By the Corollary to Lemma 3 $U(yx)$ is in normal form, hence SYy and SY are in normal form.

(v) X is closed. For this case we use induction on the number of atoms of X. Suppose $X = yX_1 \ldots X_k$. If for all i, $n(X_i) < n(X)$ we use the induction hypothesis on $n(X)$. If for some i, is $n(X_i) = n(X)$ and X_i is closed we use the induction on the number of atoms of X; *if* X_i is open we use the same argument as in (*i*)—(*iv*).

Theorem 2. If $X = [Z/x]Y$ where Y is irreducible and contains at most one occurrence of x, every part of Y not containing x is in normal form, and $X \geqq U$ where U is in normal form, then Y is also in normal form.

The proof is by induction on $n(U)$. If $n(U) = 0$ we use induction on the structure of Y. Since Y is irreducible and $n(U) = 0$ it follows that Y is closed. We may assume that $Y = yY_1 \ldots Y_k$ where y is not x. Hence $U = yU_1 \ldots U_k$ and $[Z/x]\ Y_i \geqq U_i$ and by the induction hypothesis on the structure of Y each Y_i is in normal form, hence Y is in normal form.

If $n(N) > 0$ we consider the following cases:

(i) Y is S,K of I. This case is trivial.

(ii) Y is KY_1. Hence $X = KX_1$ and $X_1 \geqq U_1$. Since $n(U_1) < < n(U)$ it follows that Y_1 is in normal form, so Y is in normal form.

(iii) Y is SY_1Y_2, hence $X = SX_1X_2$, $U = SU_1U_2$ and $X_1 \geqq U_1$, $X_2 \geqq U_2$. It follows that Y_1 *and* Y_2 are in normal form. Furthermore there is a combination $V = V_1V_2$ such that V, V_1 and V_2 are in normal form, $U = [y]V$, $U_1 = [y]V_1$, $U_2 = [y]V_2$, and we may assume that y is distinct from x and does not occur in Z or Y. Since Y_1 and Y_2 are in normal form there are combinations M_1 and M_2 in normal form such that $Y_1 = [y]M_1$, $Y_2 = [y]M_2$. Let M be the combination M_1M_2. We want to show that M is in normal form and $Y = [y]M$. Since $X_1 = [Z/x]Y_1 = [y][Z/x]M_1$ it follows that $X_1y \geqq [Z/x]M_1$. Also $X_1y \geq U_1y \geq V_1$. By Rosser's lemma, since V_1 is irreducible, we have that $[Z/x]M_1 \geqq V_1$. By the same argument $[Z/x]M_2 \geqq V_2$. Since V contains y, it follows that M contains y; furthermore if $M_2 = y$ and M_1 does not contain y, then $V_2 = y$ and V_1 does not contain y, and this is impossible. Hence $Y = [y]M$. To show that M is in normal form we may assume that M contains exactly one occurrence of x. Hence every part of M not containing x is a part of M_1 or M_2, so is in normal form. The only possible redex in M is M itself; but if M is a redex then V is a redex and this is impossible. Hence M is irreducible. We have also that $[Z/x]M \geqq V$ and $n(V) < n(U)$, so by the induction hypothesis M is in normal form.

(iv) Y is SY_1. In this case we apply the argument of (iii) to the combination SY_1z where z is an indeterminate that does not occur in Y_1.

(v) Y is closed. We may assume that $Y = yY_1 \ldots Y_k$ where y is not x. We use induction on the number of atoms in Y. We have that $X = yX_1 \ldots X_k$ and $U = yU_1 \ldots U_k$ where $X_i \geqq U_i$ for $i = 1, \ldots, k$. If $n(U_i) < n(U)$ for all i, then by the induction hipothesis each Y_i is in normal form, hence Y is in normal form. If for some i is $n(U_i) = n(U)$ and Y_i is open we use the argument of (*i*) — (*iv*); if Y_i is closed we use the induction hypothesis on the number of atoms in Y.

Theorem 3. If $[Z/x]Y$ is in normal form, then Y is in normal form.

The proof is by induction on the number of occurrences of x in Y. If there is no occurrence then Y is in normal form. Suppose there is at least one occurrence of x in Y, and let Y_1 be a combination that contains exactly one occurrence of z, $[x/z]Y_1 = Y$. Suppose $U = [Z/x]Y_1$; then $[Z/z]U = [Z/x]Y$ is in normal form. It is clear that U is irreducible and contains exactly one occurrence of z; furthermore every part of U not containing z is in normal form. By Theorem 2 U is in normal form, and by the induction hypothesis Y_1 is in normal form; hence Y is in normal form.

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CRONICA

SOBRE LA ENSEÑANZA MATEMATICA

En este campo cabe señalar dos noticias de interés:

Con fecha 15 de diciembre de 1964 se ha constituido en la Argentina una Comisión nacional para la enseñanza de la matemática, "con el objeto de llevar a la práctica las recomendaciones de la Primera conferencia Interamericana sobre educación matemática (ver esta *Revista*, Vol. XIX, p. 363), que ha iniciado ya sus funciones.

Además, la Comisión internacional de enseñanza matemática ha hecho conocer su participación en el próximo Congreso de Moscú (agosto de 1966), decidiendo:

a) proponer a la Comisión organizadora del Congreso la inscripción en el programa de una conferencia general a cargo de un matemático ruso, sobre el tema: La enseñanza del análisis numérico en la Universidad.

b) Presentar tres informes sobre los siguientes temas:

Programa de la formación matemática universitaria del futuro físico: necesidad o no de cursos particulares.

Empleo del método axiomático en la enseñanza media.

Desarrollo de la actividad matemática de los alumnos. Papel de los problemas en ese desarrollo.

Los relatores serán respectivamente: C. Pisot (París); H. G. Steiner (Münster) y Z. Krygowska (Cracovia).