ON THE ROTATION - VIBRATION INTERACTION IN DEFORMED NUCLEI

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Introduction

A description based on rotational, vibrational and single particle degrees of freedom has proved to be adequate for many nuclei. The vibrational modes can be described either with collective variables or as linear combinations of elementary excitations. Both methods yield the same result in the adiabatic limit.

The collective and intrinsic degrees of freedom can be mixed through residual terms of the Hamiltonian, such as the Coriolis force and the rotation - vibration interaction. This leads to a mixing of the states belonging to the rotational band of the ground state with those belonging to bands of the vibrational state.

The aim of this paper is to show that, in the adiabatic limit, this mixing is the same whether we choose to consider the vibrations as collective modes or as superposition of intrinsic excitations.

1) *The nuclear Hamiltonian*

We assume a single particle Hamiltonian with axial symmetry (the Hartree field) plus residual two body interactions:

$$H = Hsp + Hp + Hq \tag{1}$$

Here Hp represents the short range part of the interaction which preserves the axial symmetry and is supposed to be diagonalised with Hsp to give a spectra of elementary excitations.

The term Hq is the quadrupole force, whose strength is given by a parameter κ and which tends to deform the field.

It is treated either as a perturbation (adiabatic method) or in the quasi - boson approximation. The elementary excitations will be denoted by lower case letters. We shall then write:

$$| i > = \Gamma_i^+ | 0 > \tag{2}$$

The vacuum of excitations is defined by

$$\Gamma_i | 0 > = 0$$

We call Γ_{-i}^+ the creation operator for an excitation having the same quantum numbers as $| i >$ except the z - projection of angular momentum whose sign is changed ($Ki = -K_{-i}$)

Our elementary excitations could be either twe quasi —particle states obtained by solving $Hsp + Hp$ in the Bogoliubov— Valatin formalism, or particle — hole states. Our derivation, however, is independent of the particular choice.

We note that if we wish to use the quasi-particle language for β - vibrations care should be taken to eliminate the spurious state arising in the solution of the superconducting equations.

2) *Rotational motion* (ref. (1))

We suppose the nucleus is an ellipsoid and work in the principal axis system defined by

$$\left.\begin{aligned} Q_1 = Q_{-1} = 0 \\ Q_2 = Q_{-2} \end{aligned}\right\} \tag{4}$$

where the Q_i's are the quadrupole moments. The nuclear shape is determined by two parameters

$$\left.\begin{aligned} Q = Q_0 \\ S = \sqrt{2}\, Q_2 \end{aligned}\right\} \tag{5}$$

If any of these is different from 0 the system can rotate. The rotational angular momentum is the diference between the total $\vec{(I)}$ and intrinsic $(\vec{J})$ angular momentae. The Hamiltonian is then

$$H_{rot} - \sum_i \frac{1}{2\mathcal{J}_i} (I_i - J_i)^2 \tag{6}$$

$i = 1, 2, 3$ label the intrinsic axis.
From now on we suppose the equilibrium shape to have basically axial symmetry and write:

$$\left.\begin{aligned} H_{rct} &= H^0{}_{rot} + H' \\ H^0{}_{rot} &= \frac{1}{2\mathcal{J}_0}(I_1{}^2 + I_2{}^2) \\ H' = -\left(\frac{I_1 J_1}{\mathcal{J}_1} + \frac{I_2 J_2}{\mathcal{J}_2}\right) &- \frac{1}{2}\left(\frac{\delta\mathcal{J}_1}{\mathcal{J}_0{}^2} I_1{}^2 + \frac{\delta\mathcal{J}_2}{\mathcal{J}_2{}^2} I_2{}^2\right) + \\ &+ \frac{1}{2\mathcal{J}_3}(I_3 - J_3)^2 + H'' \end{aligned}\right\} \quad (7)$$

whre $\mathcal{J}_0 = \mathcal{J}_1 - \delta\mathcal{J}_1 = \mathcal{J}_2 - \delta\mathcal{J}_2$ Both $\delta\mathcal{J}_1$, and $\delta\mathcal{J}_2$ must be small corrections according to the hypothesis of axial symmetry.

The four terms in H' are the Coriolis interaction (H_c) conpling the intrinsic and rotational motions, the rotation - vibration interaction H_{RV}, a term to be assimilated in the Hamiltonian for γ vibrations (see below) and H'' which is incorporated to the intrinsic Hamiltonian.

In first order the nuclear wave function is then

$$\psi_{IMK\tau} = \sqrt{\frac{2I+1}{16\pi^2}}\left[|\tau> D^I{}_{MK} + (-)^{I+K} T\,|\,\tau> D^I{}_{M-K}\right] \quad (8)$$

for $I_3 = K \neq 0$

$$\psi_{IM0\tau} = \sqrt{\frac{2I+1}{8\pi^2}}\,|\tau> D^I{}_{M0} \quad (9)$$

for $I_3 = K = 0$

Here M is projection of I on the z-axis of the laboratory system, τ denotes a set quantum numbers characterizing the vibrational and intrinsic motions and T is the time reversal operator. The D's are the usual rotation matrices.

The form of (8) is such as to take into account the reflexion symmetry of the nucleus. (ref. (2))

3) *Vibrational motion* (refs. (1),)3))

The vibrations can be studied either by introducing a convenient set of collective variables or by treating them as a superposition of elementary excitations. We shall briefly outline both methods.

A) *Collective description*

We choose as collective variables the parameters Q and S. Deviations from the equilibrium values ($Q = Q_0, S = 0$) give rise to vibrations which may conserve the axial symmetry (β vibrations) or not (γ vibrations).

For small oscillations the corresponding Hamiltonians are

$$H_\gamma = \frac{1}{2} C_\gamma S^2 - \frac{1}{2B_\gamma} \frac{1}{S} \frac{\partial}{\partial S} S \frac{\partial}{\partial S} + \frac{1}{2\mathcal{J}_3} (I_3 - J_3)^2 \qquad (10)$$

$$H_\beta = \frac{1}{2} C_\beta (Q - Q_0)^2 - \frac{1}{2B_\beta} \frac{\partial^2}{\partial Q^2} \qquad (11)$$

H_γ leads to the radial equation for a two dimensional harmonic oscillator and H_β is a linear oscillator Hamiltonian.

According to this description the first β and γ states are due to one phonon excitations of the oscillators with energies

$$\omega_\gamma = \sqrt{\frac{C_\gamma}{B_\gamma}} \qquad \omega_\beta = \sqrt{\frac{C_\beta}{B_\beta}} \qquad (12)$$

The complete wave functions for these states are

$$\psi^c_{IM2\gamma} = \sqrt{\frac{2I+1}{16\pi^2}} \varphi^{n_\gamma - 1} |0\rangle [D^I_{M2} + (-)^I D^I_{M-2}] \qquad (13)$$

and

$$\psi^c_{IM0\beta} = \sqrt{\frac{2I+1}{8\pi^2}} \varphi^{n_\beta - 1} |0\rangle D^I_{M0} \qquad (14)$$

where we have called $\varphi^{n}\beta(or\,\varphi^{n}\gamma)$ the vibrational state with $n_\gamma\,(or\,n_\beta)$ phonons. The superscript c stands for collective.

The matrix elements of the collective variables between the ground and first excited states are

$$<\varphi^{n}\gamma^{=1}|S|\varphi^{n}\gamma^{=0}>=(B_\gamma\,\omega_\gamma)^{-1/2};\ <\varphi^{n}\beta^{=1}|Q|\varphi^{n}\beta^{=0}>= \\ =(2\,B_\beta\,\omega_\beta)^{-1/2} \qquad (15)$$

To evaluate the parameters $B_\beta, B_\gamma, C_\beta, C_\gamma$ and $\mathcal{I}_3$ we follow the approach of ref. (3). The existence of deviations from the equilibrium values of Q and S implies a contribution of the form $-\kappa\,(S\,S_{op}+Q\,Q_{po})$ to the Hartree field.

The perturbed ground state is given by

$$|\,0>'=|\,0>+\kappa R\sum_i \frac{r_i}{\epsilon_i}\,|\,i>\equiv \Psi^{c}_{IMOO} \qquad (16)$$

R stands for S or Q, $r_i=<0\,|\,R\,|\,i>$ and $|\,i>$ is a state with $K=Ki=2$ if $R=S$ or $K=0$ if $R=Q$.
C and B are given in formulae (29) and (33) of ref. (3).

$$C=\frac{1}{2\sum_i r_i^{2}/\epsilon_i}-\kappa \qquad (17)$$

$$B=2\,(\sum_i r_i^{2}/\epsilon_i^{3})^{-2}\sum_i r_i^{2}/\epsilon_i^{3} \qquad (18)$$

$\mathcal{I}_3$ is a function of S and can be evaluated by using the cranking model.

$$\mathcal{I}_3=2\sum_l \frac{|<l\,|\,J_3\,|\,0>'\,|^2}{\epsilon_l}=2\,\kappa^2\,S^2\sum_{lk}\frac{|<l\,|\,J_3\,|\,k>|^2}{\epsilon_k^{2}\,\epsilon_l}\,s_k^{2}= \\ =8\,k^2\,S^2\sum\frac{s_k^{2}}{\epsilon_k^{3}}=4\,B\,S^{r} \qquad (19)$$

where we have used the fact that $\kappa=(2\sum_k s_k^{2}/\epsilon_k)^{-1}$ in the adiabatic limit. Explicit expressions of δI, and δI_2 can be given as well:

$$\left.\begin{aligned}\mathcal{I}_i=2\sum_l\frac{|<l\,|\,J_i\,|\,0>'\,|^2}{\epsilon_l}&=2\sum_l\frac{|<l\,|\,J_i\,|\,0>|^2}{\epsilon_l}+\\ &+4\,\kappa\,R\sum\frac{<l\,|\,J_i\,|\,0><l\,|\,J_i\,|\,k>r_k}{\epsilon_k\,\epsilon_l}\\ &=I_0+\delta I_i\ \ i=1,2\end{aligned}\right\} \qquad (20)$$

If we introduce the operators $J^{\pm} = J_1 \pm i J_2$, with the usual phase conventions to make their matrix elements real, we readily get

$$\delta \mathcal{T}_1 = -\delta \mathcal{T}_2 = 2 \kappa S \sum_{k,l} \frac{< 0 \mid J^- \mid l > < l \mid J^- \mid k >}{\epsilon_k \epsilon_l} s_k \quad ; \quad K_k = 2 \tag{21}$$

for γ vibrations, and

$$\delta \mathcal{T}_1 = \delta \mathcal{T}_2 = 2 \kappa Q \sum_{k,l} \frac{< 0 \mid J^- \mid l > < k \mid J^- \mid l >}{\epsilon_k \epsilon_l} q_k \quad ; \quad K_k = 0 \tag{22}$$

for β vibrations.

B) *The Quasi-Boson aproxmation* (*QBA*)
(see for instance ref. (4))

In the *Q B A* the assumption is made that the operators

$$X_j = \frac{1}{\sqrt{2}} (\Gamma_j{}^+ + \Gamma_{-j}) \qquad P_j = \frac{i}{\sqrt{2}} (\Gamma_{-j}{}^+ - \Gamma_j) \tag{23}$$

obey the conmutation relation

$$[X_k, P_j] = i \delta_{kj} \tag{24}$$

The validity of (24) depends of course on our particular model. If the Γ_i's are two quasi-particle creation operators the neglected terms are proportional to the number of quasi-particles and therefore small for low energy states.

Within this approximation we write

$$R_{op} = \sqrt{2} \sum_i r_i \, X^i \tag{25}$$

Furthemore, for $H^0 = Hsp + Hp$ we have

$$[X_i, H^0] = i P_i{}^+ \epsilon_i \qquad [P_i, H^0] = -i X_i{}^+ \epsilon_i \tag{26}$$

which means that H could be written, apart from ferms conmuting with X_i and P_i as

$$\mathrm{H} = \sum_i \frac{\epsilon_i}{2} (P_i P_i{}^+ + X_i X_i{}^+) - \kappa \mid \Sigma r_i \, X^i \mid^2 \tag{27}$$

which is a system of coupled oscillators.

We define new variables

$$\left.\begin{aligned} Y_j &= \sum_i \mu_{ji} X_i \\ \Pi_j &= \sum_i \lambda_{ji} P_i \end{aligned}\right\} \sum_j \mu_{ji}\, \lambda_{jk} = \delta_{ik} \tag{28}$$

such that

$$H = \sum_j \frac{1}{2}\left(\frac{\Pi_j^+ \Pi_j}{B_j} + C_j Y_j Y_j^+\right) \tag{29}$$

and require that

$$[H, Y_j] = -\frac{2\,\Pi_j^+}{B_j} \quad \text{and} \quad [H, \Pi_j] = C_j Y_j^+ \tag{30}$$

to get after some algebra

$$\mu_{kl} = \frac{\lambda_{kl}}{\epsilon_l B_k} \quad \text{and} \quad \lambda_{jl} = \frac{\Lambda_j\, \epsilon_e\, r_e}{\epsilon_l^2 - \omega_j^2} \tag{31}$$

where

$$\Lambda_j = 2\kappa \sum \lambda_{ji}\, r_i = \left[\sum \frac{\epsilon_i\, r_i^2}{(\epsilon^2{}_i - \omega^2{}_j)\, B_j}\right]^{-1/2} \tag{32}$$

and

$$\omega_j^2 = \frac{C_j}{B_j} \tag{33}$$

These frequencies are obtained as roots of the equation

$$\frac{1}{2\kappa} = \sum \frac{\epsilon_l r_e}{\varepsilon_l^2 - \omega_j^2} \tag{34}$$

The first root ω, reduces to (12) calculated with the values of C and B given in (17) and (18) in the limit $\omega_1^2 << \epsilon_l^2$. We shall therefore identify the first excited state of (29) with the 1 phonon vibrational state which is doubly degenerate for γ vibrations. Both modes should be separated to be able to write down the correctly

symmetrized wave function. The degenerate wave function are obtained by applying Y_1, or Y_1^+, to the ground state. We notice that the *QB* method introduces correlations in the ground state through the operator $\exp\left(-\frac{1}{2}\Sigma_1 | Y'_j |^2\right)$ —

The dimensionless variable Y^1 is defined as $Y'_j = (\omega_j B_j)^{1/2} Y_j$. The correlated ground state will be called

$$\psi_{IM_{00}}{}^{QB} = \sqrt{\frac{2I+1}{8\pi^2}}\, O' > D^I{}_{MO} \tag{35}$$

Introducing the notation

$$|n_\gamma = 1> = \sqrt{2}\, Y_1\, exp\left(-\frac{1}{2}\Sigma | Y_i |^2\right)|0>$$

$$|n_\gamma{}^+ = 1> = \sqrt{2}\, Y_1{}^+\, exp\left(-\frac{1}{2}\Sigma | Y_i |^2\right)|0> \tag{36}$$

The complete wave function for the γ state, analogous to (13), is

$$\psi_{IM_{2\gamma}}{}^{Q.B.} = \sqrt{\frac{2I+1}{16\pi^2}}\,(D_{M_2}{}^I |n_\gamma = 1> + (-)^I D_{M-2}{}^I |n_\gamma{}^+ = 1>) \tag{37}$$

For the β — vibrations we have

$$\psi_{IM_{0\beta}}{}^{QB} = \sqrt{\frac{2I+1}{8\pi^2}}\, D_{M_0}{}^I |n_\beta = 1> \tag{38}$$

The notation being obvious.

4) *The rotation — vibration interaction*

We are now in a position to study the mixing of the ground state with the vibrational states produced by the terms Hc and H_{RV} of (7).

It is seen that Hc will be effective only if we use the *QBA* wave functions since it does not depend on the collective variables.

The reverse is true for H_{RV} since it does not contain intrinsic variables and could therefore only act between states of the form (13) and (14).

Applying ordinary perturbation theory we can write

$$(\psi^c_{IM2\gamma})' = \psi^c_{IM2\gamma} + \frac{<\psi^c_{IM2\gamma}|H_{RV}|\psi^c_{IM00}>}{\varepsilon_\gamma}\psi^c_{IM00} \tag{39}$$

and a similar expression for β

The form of H_{RV} is, according to (21) and (22),

$$-\frac{1}{\mathcal{J}_0} Q A_Q \,(I_+ I_-) \quad \text{for } \beta \text{ vibrations} \tag{40}$$

$$-\frac{1}{2\mathcal{J}_0} S A_S \,(I^2_+ + I^2_-) \quad \text{for } \gamma \text{ vibrations} \tag{41}$$

The coefficient of Ψ^c_{IMDO} in (39) are then (using the adiabatic value of κ)

$$z^c_{\beta(I)} = -\frac{1}{2\mathcal{J}_0}\omega_\beta^{-3/2}\left(\sum_k q^2_k/\varepsilon^2_k\right)^{-1/2}\left(\sum_k \frac{<0|J^-|l><k|J^-|l>}{\varepsilon_k\,\varepsilon_l} q_k\right)$$
$$(I)\,(I+1) \tag{42}$$

$$z^c_{\gamma(I)} = -\frac{1}{2\mathcal{J}_0}\left(\sum_k s^2_k/\varepsilon^2_k\right)^{-1/2}\left(\sum_k \frac{<0|J^-|l><l|J^-|k>}{\varepsilon_k\,\varepsilon_l} s_k\right)$$
$$(I)^{1/2}\,(I+1)^{1/2}\,(I+2)^{1/2}\,(I-1)^{1/2} \tag{43}$$

For the *QBA wave* functions we have: (there is no mixing, in first order):

$$(\psi^{QB}_{IM2\gamma})^1 = \psi^{QB}_{IM2\gamma} + \sum_l \frac{<\psi^{QB}_{IM2\gamma}|H_c|Il><Il|H_c|\psi_{IM00}>}{\varepsilon_\gamma(\varepsilon_\gamma - \varepsilon_l)}$$
$$\psi^{QB}_{IM00} \tag{44}$$

and a similar expression for β.

By $|Il>$ we denote elementary excitations with $K = \pm 1$ having the form

$$|Il> = \sqrt{\frac{2I+1}{8\pi^2}} (D^I_{M1} |l> + (-)^{I+1} |l> D^I_{M-1}) \qquad (45)$$

We can then write

$$z^{QB}_{\beta(I)} = \frac{1}{2\mathcal{J}_0} \sum_l \frac{<n_\beta = 1|J_-|l><l|J_+|0'>}{\varepsilon_\beta (\varepsilon_\beta - \varepsilon_l)} I(I+1) \qquad (46)$$

and

$$z^{QB}_{\gamma(I)} = \frac{1}{2\mathcal{J}_0} \sum_l \frac{<n_\gamma|J_+|l><l|J_+|0'>}{\varepsilon_\gamma (\varepsilon_\gamma - \varepsilon_l)} (I+2)^{1/2} (I+1)^{1/2} (I)^{1/2} (I-1)^{1/2} \qquad (47)$$

To evaluate then sum in (46) we consider the operator

$$\hat{A}_{\beta(I)} = \frac{1}{2\mathcal{J}_0} \sum_l \frac{|J_-|l><l|J_+|}{\varepsilon_\beta (\varepsilon_\beta - \varepsilon_l)} (I)(I+1) \qquad (48)$$

then, obviously

$$z^{QB}_{\beta(I)} = <n_\beta = 1| \hat{A}_{\beta(I)} |0^1> \qquad (49)$$

If we now write $\hat{A}_{\beta(I)}$ in second quantization

$$\hat{A}_{\beta(I)} \simeq \sum <j|\hat{A}_{\beta(I)}|0> \sqrt{2} X_j \qquad (50)$$

W can express X in terms of Y_i's and put $|n_\beta = 1> = \sqrt{2} Y_1 |0'> —$. It is then easy to take advantage of the harmonic oscillator matrix elements to find

$$e^{QB}_{\beta(I)} = \frac{1}{2\mathcal{J}_0} I(I+1) \sum_{l,i} \frac{<i|J_-|l><l|J_+|0>}{\varepsilon_\beta (\varepsilon_\beta - \varepsilon_l)} \lambda_{\beta i} \omega_\beta^{-1/2} \qquad (51)$$

$$= \frac{1}{2\mathcal{I}_0} I(I+1)\Big(\sum_i \frac{q_i^2 \varepsilon_i}{\varepsilon_i^2 - \omega_\beta^2}\Big)^{-1/2} \sum \frac{\langle i|J_-|l\rangle \langle l|J_+|0\rangle q_i \varepsilon_i}{\varepsilon_\beta^{3/2}(\varepsilon_\beta - \varepsilon_l)(\varepsilon_i^2 - \varepsilon_\beta^2)},$$

$$K_i = 0$$

and in the same way

$$\mathcal{L}^{QB}{}_{\gamma(I)} = \frac{1}{2^{3/2}\mathcal{I}_0}[I(I+1)(I-1)(I+2)]^{1/2}\Big(\frac{s_i^2 \epsilon_i}{(\varepsilon_i^2 - \omega_\gamma^2)}\Big)^{-1/2}$$

$$\sum_{il} \frac{\langle i|J_\pm|l\rangle \langle l|J_\pm|0\rangle}{\varepsilon_\gamma^{3/2}(\varepsilon_\gamma - \varepsilon_l)(\varepsilon_i^2 - \varepsilon_\gamma^2)} (q_2)_i \varepsilon_i \qquad (52)$$

Remembering that $(q_2)_i = \sqrt{2}\ s_i$ we see that in the limit of low energies for the vibrational states

$$\begin{aligned} Z^{Q\,B}_{\gamma(I)} &\longrightarrow Z^{\;\;C}_{\gamma(I)} \\ Z^{Q\,B}_{\beta(I)} &\longrightarrow Z^{\;\;C}_{\beta(I)} \end{aligned} \qquad (53)$$

It is seen that the two different approachs lead to the same result. This is not surprising since H_{RV} an Hc have a common origin in the inertia forces due to the introduction of the rotating system.

In fact, in such a system the lowest order modification to the Hamiltonian due to individual motions is of the form I.J, the simplest possible scalar we could construct with the available variables.

On the other hand a sligth modification in the shape of the nucleus should also lead to a change in the rotational energy due to a variation in the moments of inertia.

The preceeding discussion shows that (53) could be qualitatively predicted and also clarifies the fact that when dealing with vibrations Hc and H_{RV} should be considered essentially the same term and counted only once.

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