CONJUGATE FUNCTION THEOREMS FOR DIRICHLET ALGEBRAS

by ALLEN DEVINATZ (1)

I. INTRODUCTION

1. During the past few years it has become apparent that a number of theorems about certain classes of functions, analytic in the unit disk, could be derived from a very few properties common to all of these classes. As is often the case, the abstraction of the problem has in many instances resulted in simplified proofs of the classical theorems.

The abstract approach had its beginning in papers by Arens and Singer [1, 2] and reached a certain decisive stage in a fundamental paper by Helson and Lowdenslager [9]. It was then soon recognized by Bochner [5] and subsequently Wermer [16] that the methods of the latter authors would work in an axiomatically defined context. In a recent paper [11] K. Hoffman has given an excellent account of the present development of the subject, and we shall make further reference to his paper later on. We also refer the reader to the report by Wermer [17].

The main object of this exposition is to show that a certain conjugate function theorem, originally proved by Helson and Szegö [10] for the circle group, is valid in a very general context. We need these results for [6a]. In part, this paper is expository in nature. We believe, however, that the results of III, § 5 are new, some even in the classical case of the circle group. In the next few paragraphs of this introduction we shall introduce the subject for the classical case of functions analytic in the unit disk. This will then serve as a guide for the abstract situation.

(1) Supported in part by N S F Grant G - 24834. This article was originally written for the second volume of the American Mathematical Association's "Studies in Analysis", but certain circumstances made publication in this journal preferable.

2. Let f be an analytic function with domain the unit disk $D=\{z:|z|<1\}$ which can be extended continuously to the boundary $S=\{z:|z|=1\}$ of D. Designate the class of such functions by A. A large collection of functions in A is obtained by considering polynomials in z, restricted to D, as well as those functions which may be uniformly approximated on $D \cup S$ by such polynomials. Indeed, it is not difficult to show that every element in A may be obtained in this way and that the boundary values of elements in A have Fourier coefficients which vanish on the negative integers. Any function, defined on S, whose Fourier coefficients are defined and vanish on the negative integers will be said to have an analytic Fourier series. With this terminology, the boundary functions of the elements of A have analytic Fourier series.

To obtain a wider class than A we consider functions f analytic in D and such that

$$\sup_{z \in D} |f(z)| < M < \infty .$$

Such functions are said to belong to the Hardy class H^∞. More generally an analytic function with domain D is said to belong to the Hardy class H^p, $p > 0$, if and only if

$$\| f \|_p^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} | f(re^{i\theta}) |^p d\theta < M < \infty.$$

We note that $\| f \|_p$ is a norm for $p \geqq 1$ and $\rho(f,g) = \| f - g \|_p^p$ is a metric for $0 < p < 1$.

Analogous to the situation for elements of A, there exists a Lebesgue measurable function $f(e^{i\theta})$, defined on S, such that for almost all θ,

$$\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta}),$$

and moreover

$$\lim_{r \to 1} \int_0^{2\pi} | f(re^{i\theta}) - f(e^{i\theta}) |^p d\theta = 0.$$

For proofs see [19 I].

It is clear that A is contained in every space H^p and it is not difficult to see that the closure of A in the metric of H^p is all of H^p. If $p \geqq 1$, every boundary function of an element of H^p has an analytic Fourier series, and conversely every element of the Lebesgue space $L_p(S,d\theta)$ which has an analytic Fourier series is the boundary function of a unique element of H^p. This establishes an isometry between the elements of H^p and the elements of $L^p(S, d\theta)$ which have analytic Fourier series. Hence, H^p may be considered as a subspace of $L^p(S, d\theta)$. This particular way of looking at H^p is useful in certain higher dimensional generalizations of these spaces.

3. If f is an analytic function defined on D and we write $f = u + iv$, u, v, real, then u and v are harmonic and v is called a conjugate to u. If v has the additional property that $v(0) = 0$, then we call v *the* conjugate function to u and designate it by Cu. If $f \in H^p$, then u and Cu have radial limits a.e. which belong to $L^p(S, d\theta)$. In particular, if u is the real part of an element of A, then both u and Cu have continuous extensions to S. If we identify the elements of A with their boundary functions, then C is a linear operator defined on ReA, the real linear space consisting of real continuous functions on S which are real parts of elements of A. By linearity, C can be extended to the complex linear space generated by ReA. Now, the continuous functions are dense in $L^p(S, d\theta)$ for every $p > 0$ and since by Fejer's theorem every continuous function on S is a uniform limit of trigonometric polynomials, it follows that C is defined on a dense linear subspace of L^p.

A famous theorem of M. Riesz [19 I, p. 253] states that if $1 < p < \infty$ then C is a bounded linear operator in the norm of L^p; i.e., there exists a $B_p > 0$ so that

$$\| Cf \|_p \leqq B_p \| f \|_p$$

for every f in the domain of C. This means that C can be uniquely extended as a continuous linear operator from L^p into itself.

For $p = 1$ or $p = \infty$ this theorem is no longer true. For $p = 1$ its place is taken by the statement that C acts as a bounded linear operator from L^1 into L^p for every $0 < p < 1$. For $p = \infty$, C acts as a bounded operator from L^∞ into L^p for every $0 < p < \infty$. More generally it turns out that for $|u| \leqq 1$, $\exp \lambda |Cu| \in L^1$ for $0 \leqq \lambda < \pi/2$. We shall give precise statements of these facts in Section III.

We shall now describe the result, which as we have mentioned before, is the main object of this exposition. If $0 \leqq w \in L^1(S, d\theta)$, then Helson and Szegö [10] have determined necessary and sufficient conditions on w in orden that the operator C, defined on the complex linear space generated by ReA, may be bounded in $L^2(S, wd\theta)$. Their result is that the operator C is bounded in this space if and only if

$$w = e^{u + \tilde{v}},$$

where u is a bounded real function and v is a real function with $\| v \|_\infty < \pi/2$. It also turns out that C is bounded in this space if and only if the projection operator from the trigonometric polynomials into H^2 is bounded in the space $L^2(S, wd\theta)$.

4. A somewhat different but cognate subject, whose results are needed in the proof of the Helson-Szegö theorem, is the problem of factorization of non-negative functions in $L^1(S, d\theta)$ into the square of the absolute value of a function in H^2. There is a well known and classical theorem due to L. Fejer and F. Riesz concerning the factorization of non-negative trigonometric polynomials. Suppose we have

$$p(e^{i\theta}) = \sum_{k=-n}^{n} p_k e^{ik\theta} \geqq 0, \ p_n \neq 0.$$

The theorem of Fejer-Riesz states that there exists a unique polynomial

$$q(z) = \sum_{k=0}^{n} q_n z^n, \ q_0 > 0,$$

such that $q(z) \neq 0$ for $|z| < 1$ and

$$p(e^{i\theta}) = | q(e^{i\theta}) |^2.$$

The classical proof of this factorization proceeds by quite algebraic methods and goes roughly as follows: Write $(pe^{i\theta}) = e^{-in\theta} h(e^{i\theta})$ where $h(z) = \Sigma_0^{2n} h_k z^k$. From the reality of p we get $p_k = \overline{p_{-k}}$ for $-n \leqq k \leqq n$ and one deduces inmediately that the roots of $h(z)$ are conjugate with respect to the unit circle S; i. e., if z is a root so is $1/\bar{z}$. Since $p \geqq 0$ any root on the unit circle must be of even order. One then takes q as that polynomial with $q_0 > 0$ whose roots are the roots of $h(z)$ which lie outside the unit circle and the roots which lie on the unit circle, but with half the multiplicity.

There is a generalization of this result, also classical, due to G. Szegö [15]. Suppose $0 \leq w \in L^1(S, d\theta)$; then w has a Fourier expansion

$$w \sim \sum_{n=-\infty}^{\infty} \hat{w}(n) e^{in\theta} .$$

Roughly speaking, w may be considered as an infinite positive trigonometric "polynomial". Szegö's result is that w has a factorization

$$w = | g |^2, \; g \sim \sum_{n=0}^{\infty} \hat{g}(n) e^{in\theta} ,$$

if and only if

$$\int_0^{2\pi} \log w d\theta > - \infty .$$

There is another statement, also due to Szegö [14], which is closely related to the previous statement:

If $0 \leqq w \in L^1(S, d\theta)$ then

$$\exp 1/2\pi \int_0^{2\pi} \log w d\theta = 1/2\pi \int_0^{2\pi} | 1 + p |^2 w d\theta,$$

where the infimum is taken over all trigonometric polynomials in H^2 with $p(0) = 0$.

An immediate consequence of these two statements is that if $f \in H^1$, then

$$\log | \hat{f}(0) | \leqq 1/2\pi \int_0^{2\pi} \log | f | d \theta.$$

It will be inmediately recognized that this last inequality can also be obtained from the classical Jensen formula.

Any non-zero element of H^1 for which equality persists in the previous formula is called an *outer factor.* If $f \in H^2$, let H_f be the smallest closed linear subspace of H^2 containing f which is invariant under multiplication by elements of A. A. Beurling [3] has shown that H_f is all of H^2 if and only if f is an outer factor. Moreover, he has shown that any $f \in H^2$ can be factored into $f = gh$, where h is an outer factor in H^2, $g \in H^\infty$ and $|g(e^{i\theta})| = 1$, a.e. Functions of the latter type are called *inner factors.* The factorization of any element in H^2 into an inner and outer factor is unique up to a multiplicative constant of unit modulus. Beurling has also shown that if H is any closed linear subspace of H^2, invariant under multiplication by elements of A, then $H = gH^2$ where g is an inner factor.

All of the results mentioned in this section can be generalized to matrix valued functions [9, 13, 18] or even to functions whose values are operators on a Hilbert space [6, 8].

The classical results of this paragraph were extended to higher dimentions by Helson and Löwdenslager. As we have mentioned before, the extension to an abstract situation forms the major part of the paper of Hoffman. We shall give the precise statements of these theorems in the abstract context after we have introduced the proper notions.

II. *Dirichlet Algebras*

1. If we identify A with the continuous functions on the unit circle which have analytic Fourier series, then A is an algebra over the complex numbers. We will put a norm on the elements of A by means of the formula

$$\| f \|_\infty = \sup_{0 \leq \theta \leq 2\pi} | f(e^{i\theta}) | .$$

With this norm A becomes a complex Banach algebra which satisfies the additional properties:

(a) $1 \in A$.

(b) A separates points of S; i.e., if $\theta \neq \phi$ there exists an $f \in A$ such that $f(\theta) \neq f(\phi)$.

(c) ReA is dense in $C_R(S)$, the real Banach space of real continuous functions on S with the uniform norm as given above. .

Suppose X is any compact Hausdorff space and A a complex Banach algebra of continuous functions on X with the uniform norm and which satisfies the properties (a), (b) and (c) with X replacing S. Such algebras were studied by Gleason [7] and were called by him Dirichlet algebras. Hoffman [11] replaces condition (c) by a weaker density condition and he calls his algebras logmodular. We will use Dirichlet algebras in this exposition since they are easier to work with. However, all of the results are valid for Hoffman's logmodular algebras.

2. The following are some additional examples of Dirichlet algebras.

a. Let T be torus given by the collection of all ordered pairs (θ, ϕ), $0 \leq \theta, \phi \leq 2\pi$ with the proper identification of the boundaries and suppose $\hat{T}$ is the set of all ordered pairs of integers (n, m). Let P be a subset of $\hat{T}$ containing the origin with the further conditions

(i) (n, m), $(p, q) \in P$ implies $(n+p, m+q) \in P$,

(ii) $(n, m) \in \hat{T}$ implies $(n, m) \in P$ or $(-n, -m) \in P$, but not both unless $n = m = 0$.

Such a set P is called a half-plane by Helson and Lowdenslager and was used previously by Bochner [4] for proving generalized conjugate function theorems. The algebra A is the set of all continuous functions on T whose Fourier transforms vanish for $(n, m) \bar{\in} P$.

A half-plane can be obtained by taking any irrational number α and setting $P = \{(n, m) : n + \alpha m \geqq 0\}$.
Another example is $P = \{(n, m) : n > 0\} \cup \{(0, m) : m \geqq 0\}$.

b. A generalization of the above example is obtained by letting T be any compact abelian topological group which has a linearly orderer dual group $\hat{T}$. P will then consist of all $\hat{t} \in \hat{T}$ such that $\hat{t} \geq 0$ and A will be the algebra of continuous functions on T whose Fourier transform vanishes for $\hat{t} < 0$.

A special case of this obtained by letting $\hat{T}$ be any subgroup of the additive group of real numbers with the discrete topology and P the non-negative numbers in $\hat{T}$. In this case T is not only linearly ordered but Archimdean ordered as well. The importance of this case lies in the fact that every discrete abelian group with an Archimedean order relation is order isomorphic to a subgroup of the additive reals with the discrete topology. For example, if $\hat{T}$ is the group of ordered pairs of integers as given in the first example and

$$P=\{(n,m): n+\alpha m \geqq 0\}$$

with α irrational, then P induces an Archimedean order on T.

c. If K is any compact set in the complex plane and A is the set of functions which are uniform limits of polynomials restricted to K, then A is a Dirichlet algebra.

A more trivial example is obtained by taking A to be $C(K)$, the space of all continuous functions on the compact set K.

More examples may be found in Hoffman's paper. .

3. Let X be any compact Hausdorff space and A a Dirichlet algebra of continuous complex valued functions defined on X. Let M be the space of continuous non-zero homomorphism from A into the complex plane with the weak* topology; i. e., a neighborhood basis of an $m_0 \epsilon M$ is the collection of all sets of the form

$$\{m: \mid m(f_k) - m_0(f_k) \mid < \epsilon, 1 \leqq k \leqq \mathrm{n}\} .$$

This is the so-called Gelfand topology on M and with this topology M is a compact Hausdorff space and each function defined by $f(m) = m(f)$ for $f \epsilon A$ is continuous on M. Indeed, since the functions f separate points of M there is only one compact Hausdorff topology on M such that all the functions f are continuous. From condition (b) for a Dirichlet algebra it follows that X may be embedded homeomorphically into M.

If $m \epsilon M$, since $1 \epsilon A$, it follows that

$$\| m \| = \sup_{\| f \|_\infty = 1} \mid m(f) \mid = 1.$$

Using the Hahn-Banach theorem, we may extended m to a continuous linear functional on $C(X)$, the space of all complex continous functions on X with the uniform norm. This extension may be made so that the norm of the extension remains one. By a well-known theorem of F. Riesz, it is easy to show that there exists a non-negative founded regular Borel measure $d\mu_m$ on X of total measure one, such that

$$\hat{f}(m) = \int_X f\, d\mu_m .$$

From condition (c) for a Dirichlet algebra, it follows immediately that $d\mu_m$ is unique.

Note that if $m \in X$, then $d\mu_m$ is the atomic measure which takes the value 1 on any Borel set containing m and takes the value zero otherwise. If X is the unit circle S, and A is the Dirichlet algebra of the introduction, then $M = D \cup S$, where by equality we mean that M is homeomorphic to $D \cup S$. If we take $m = 0$, the origin of coordinates, then $d\mu_m = d\theta/2\pi$. To see this we note that if $n \geq 1$, then

$$\int_S z^n d\mu_m = 0,$$

since this integral is just z^m evaluated at zero. Hence, taking complex conjugates and remembering that on $S, \overline{z^n} = z^{-n}$, we find that the preceeding integral is zero for all $n \neq 0$. The uniqueness theorem for Fourier-Stieltje's integrals then gives our assertion.

4. From this point on we shall fix $0 \in M$ *and* $d\mu$ shall be the corresponding representing measure, which we have previously designated $d\mu_0$. The context will always indicate whether the symbol '0' is this homomorphism or the number zero.

With respect to the measure $d\mu$, defined on the Borel field of X, we can construct the Lebesgue spaces L^p, $0 < p \leq \infty$, in the usual way. They are the collection of equivalence classes of Borel measurable complex valued functions with the norm given by

$$\| f \|_p = \left[\int_X | f(x) |^p d\mu \right]^{1/p}, \quad 0 < p < \infty ;$$

$$\| f \|_\infty = \text{ess. sup } | f(x) | , \quad p = \infty .$$

Two measurable functions are equivalent if and only if they coincide almost everywhere with respect to $d\mu$. There is usually no ambiguity in designating an equivalence class in L^p by any of its elements and we shall do this.

The spaces L^p for $1 \leqq p \leqq \infty$ are Banach spaces, and for $0 < p < 1$ they are complete metric spaces under the metric $\rho(f, g) = \| f - g \|_p^p$.

For $0 < p < \infty$, H^p shall be the closure of A in L^p, and H^∞ shall be the essentially bounded elements in H^1. For $p \geqq 1$, H^p may be charaterized as the collection of elements f in L^p such that $\int gf d\mu = 0$ for every $g \in A$ for which $\hat{g}(0) = 0$ (See [11, p. 298]).

5. We shall now list a number of results which we shall need later on. Proofs may be found in [11] or else can be easily constructed from the ingredients which appear there. The first theorem is a generalization of Jensen's inequality which we mentioned in the introduction. For $f \in L^1$ we write $\hat{f}(0) = \int f \, d\mu$.

Theorem 1. *If $f \in H^1$, then*

$$\log |\hat{f}(0)| \leqq \int_X \log |f| \, d\mu.$$

This theorem says, in particular, that if $\hat{f}(0) \neq 0$, then $f \neq 0$ almost everywhere. However, even if $\hat{f}(0) = 0$ it may still be true that integral of $\log |f|$ may be finite. For example, if f is a function on the circle group with an analytic Fourier series and for some $n > 0$, $\hat{f}(n)$ is the first Fourier coefficient different from zero, then $f_1(e^{i\theta}) = e^{-in\theta} f(e^{i\theta})$ has the property that $\hat{f}_1(0) = \hat{f}(n) \neq 0$ and $\log |f_1| = \log |f|$. This shows moreover that a function on the circle group, having an analytic Fourier series cannot vanish on a set of positive measure, unless it is identically zero. This property does not persist in higher dimensions and *a fortiori* for general Dirichlet algebras. This is the cause of some of the difficulties in trying to extend some theorems from the circle group to the more general context.

Functions which behave rather well with respect to the difficulty just mentioned, and indeed behave rather well in most contexts are the so called outer factors. If $f \in H^1$, it is called an *outer factor* if and only if $\hat{f}(0) > 0$ and

$$\log \hat{f}(0) = \int_X \log |f| \, d\mu.$$

Some authors define f to be an outer factor if and only if $\log \hat{f}(0)$ is replaced by $\log |\hat{f}(0)|$ in the above equality. However, it is clear that an outer factor in the first sense differs from an outer factor in the second sense only by a multiplicative constant of absolute value one. In the circle group case we see from Jensen's theorem that an outer factor can have no zeros inside the unit disk (considering now H^1 as analytic functions defined in the unit disk), although this is not a sufficient condition for an element of H^1 to be an outer factor. In the general Dirichlet algebra case, we may consider $M - X$ as taking the place of the open unit disk. However, in the general situation we do not know whether or not $\hat{f}(m)$ may vanish on $M - X$ if f is an outer factor in H^1.

The next two theorems are generalizations of the theorems of G. Szegö mentioned in the introduction.

Theorem 2. *Let w be a non-negative function on X summable with respect to $d\mu$. A necessary and sufficient condition for w to have a unique representation*

$$w = |f|^2,$$

with f an outer factor in H^2, is that

$$\int_X \log w \, d\mu > -\infty.$$

In the next theorem, $A_0 = \{g : g \in A \text{ and } \hat{g}(0) = 0\}$. It is clear that A_0 is a closed subalgebra of A and indeed it is an ideal in A.

Theorem 3. *Let w be as in theorem 2; then*

$$\exp \int_X \log w \, d\mu = \inf \int_X |1 + g|^2 w \, d\mu,$$

where the infimum on the right is taken as g ranges over elements of A_0. The left side is to be considered zero if $\int \log w \, d\mu = -\infty$.

Theorem 4. *If $f \in H^1$, $\int \log |f| \, d\mu > -\infty$, then we may write $f = gh$, where h is an outer factor in H^1 and $g \in H^\infty$, with $|g| \equiv 1$. The decomposition is unique. Further, $h = k^2$, k an outer factor in H^2.*

From the unicity in theorem 2 it is immediate that the k which appears in theorem 4 is unique. Further, it is an immediate corollary of theorems 1 and 4 that if h_1 and h_2 are outer factors in H^1 and $|h_1| = |h_2|$, then $h_1 = h_2$.

It is seen from theorem 4, that whenever an f in H^1 has the property that $\log |f|$ is in L^1, then f can be factored into a well-behaved part, an outer factor, and another part of unit modulus. An element g in H^∞ is said to be an *inner factor* if and only if $|g| \equiv 1$. It is clear that an element can be an inner and outer factor if and only if it is the constant 1. In those cases where an element in H^1 can be factored into an inner and outer factor, it is usually the inner factor which plays the dominant role in determining the properties of the element.

Theorem 5. *If f and $1/f$ belong to H^1, then f differs from an outer factor by a multiplicative constant of unit modulus. Conversely, if f is an outer factor and $1/f \in L^1$, then $1/f \in H^1$.*

A linear subspace $S \subseteq H^p$ is said to be *invariant* if and only if $gS \subseteq S$ for every $g \in A$. If $f \in H^p$ then S_f shall be the smallest closed invariant subspace containing f. The following result generalizes a theorem due to Beurling [3].

Theorem 6. *If $f \in H^2$*, then $S_f = H^2$ *if and only if f is, up to a multiplicative constant of unit modulus, an ouer factor.*

III. *Conjugate Functions*

1. If $u \in ReA$, then there exists a $v \in ReA$ such that $u + iv \in A$. Since the constants are in A we can choose v so that $\int v \, d\mu = 0$. Such a function v is unique almost everywhere. Indeed, if $u + iw \in A$ and $\int w \, d\mu = 0$, then $v - w \in A \cap ReA$; but the real functions in A differ from constants a.e., and this together with the normalization of v and w shows that they coincide a.e.

If $u \in ReA$, and $v \in ReA$ is normalized as in the preceding paragraph so that $u + iv \in A$, then designate by Cu the equivalence class of measurable functions on X which differ from v only a.e. Call

Cu the *conjugate* of *u* (As later facts will reveal, we should more precisely call it conjugate of the equivalence class of *u*) and *C* the conjugation operator. It is clear that *C* is a linear transformation acting on the real linear space *ReA* and by linearity it can be extended to the complex linear space $ReA + i\,ReA$.

2. We shall now show that *C* can be considered as a continuous linear operator acting in L^p. We begin with a theorem for $p = 1$. The proof follows the one given in [19 I, p. 254] for the circle group.

Theorem 7. *For every* $0 < p < 1$, *there exists a* $B_p > 0$ *so that*

$$\| Cf \|_p \leq B_p \| f \|_1 .$$

for every $f \in ReA + i\,ReA$.

Proof. Suppose $f \in A$ and $Re\hat{f}(m) > 0$ for every $m \in M$. Let $f = Re^{i\phi}$ $-\pi/2 < \phi < \pi/2$, and for p real, $\hat{f^p} = R^p e^{ip\phi}$. By the Lévy-Wiener theorem for Banach algebras, f^p is the representation of some $g \in A$. But $X \subseteq M$ and hence $\hat{g}(x) = g(x) = [\hat{f}(x)]^p = [f(x)]^p$, which implies $f^p \in A$, where, of course, we define $f^p = R^p e^{ip\phi}$. Since $\hat{f^p}(m) = \hat{g}(m) = [\hat{f}(m)]^p$ *we have*

$$\hat{f}(0)^p = \int_X f^p d\mu . \tag{3.1}$$

Suppose now that $u \in ReA$, $u(x) > 0$ for every $x \in X$. Let $v \in Cu$ so that $f = u + iv \in A$. We claim that $Re\hat{f}(m) > 0$ for every $m \in M$. Indeed, suppose $m \in M$ and $d\mu_m$ the representing measure. The measure $d\mu_m \neq 0$, for otherwise $\hat{g}(m) = 0$ for every $g \in A$ which is impossible since m is a non-zero homomorphism. Hence $Re\hat{f}(m) = \int u\, d\mu_m > 0$ and since $\hat{f}(0) = \int u\, d\mu$ we have, using (3.1) and the fact that the integral of u is real positive,

$$\int_X R^p \cos p\phi\, d\mu = \hat{f}(0)^p = [\int_X u\, d\mu]^p. \tag{3.2}$$

Since $|v| \leq |f| = R$ and $\cos \phi \geqq \cos \frac{1}{2} p\pi$ for $0 < p < 1$, we get from (3.2) for $0 < p < 1$,

$$(3.3) \qquad \| Cu \|_p = [\int_X | v |^p d\mu]^{1/p} \leqq B_p \int_X u d\mu = B_p \| u \|_1,$$

where $B_p^p = 1/\cos \frac{1}{2} p\pi$.

For any $u \in ReA$ write $u = u^+ - u^-$, where u^+, $u^- \in C_R(X)$, $u^+ \geqq 0$, $u^- \geqq 0$ and $u^+ u^- = 0$. Suppose $\{ v_n \}$, $\{ w_n \}$ are sequences in ReA such that $v_n > u^+$, $w_n < u^-$ and $v_n \to u^+$, $w_n \to u^-$ in the topology of $C_R(X)$. *Set* $u_n = v_n - w_n$; clearly $u_n - u > 0$ and belongs to ReA. Applying (3.3) we see that $Cu_n \to Cu$ in L^p. On the other hand $v_n > 0$ and there exists a sequence of positive numbers $\epsilon_n \to 0$ so that $w_n + \epsilon_n > 0$. Hence, since $C(w_n + \epsilon_n) = Cw_n$ we get

$$\| Cu_n \|_p^p \leq [\| Cv_n \|_p^p + \| Cw_n \|_p^p] \leqq B_p^p [\| v_n \|_1^p + \| w_n + \epsilon_n \|_1^p].$$

Allowing $n \to \infty$ we get $\| Cu \|_p \leqq 2^{1/p} B_p \| u \|_1$, which is our desired conclusion for elements of ReA.

In now $f \in ReA + i\, ReA$ we use the triangle inequality for the metric of L^p and we get the theorem with the B_p of (3.3) replaced by $4^{1/p} B_p$.

By very much the same techniques, the following theorem can also be proved. We forgo a proof here. The reader will be able to construct a along the lines given above by consulting [19 I, p. 256].

Theorem 8. *There exist two absolute constants A and B such that for every* $f \in ReA + i\, ReA$

$$\int_X | Cf | \, d\mu \leqq A \int_X | f | \log^+ | f | \, d\mu + B.$$

3. We now turn to the case $1 < p < \infty$ which is the so called M. Riesz conjugate function theorem. Again we could give a proof modeled along the lines of the proof of theorem 7. Instead we shall give a proof due to S. Bochner [5] which makes use of the Riesz-Thorin interpolation theorem [19 II].

Theorem 9. *If* $1 < p < \infty$, *there exists a* $B_p > 0$ *such that for every* $f \in ReA + i\, ReA$,

$$\| Cf \|_p \leqq B_p \| f \|_p.$$

Proof. Let $0 \neq u \in ReA$ and $v \in Cu$ such that $f = u + iv \in A$. If we set $\hat{u}(0) = \int u \, d\mu$ and $f = \hat{u}(0) + g$, it follows from the fact that $\int v \, d\mu = 0$ that $g \in A_0$ and hence for k a positive integer,

$$\int (u + iv)^{2k} d\mu = \int (\hat{u}(0) + g)^{2k} \, d\mu = \hat{u}(0)^{2k},$$

since any power of g is in A_0. Forther,

$$|\hat{u}(0)| = |\int f \, d\mu| \leqq \| f \|_p.$$

If we take the real part of the binomial expansion in the left hand integral of the next to the last equation we get

$$\int v^{2k} \, d\mu = 2 (-1)^k \hat{u}(0)^{2k} + \binom{2k}{2} \int v^{2(k-1)} u^2 \, d\mu$$

$$- \binom{2k}{4} \int v^{2(k-2)} u^4 \, d\mu + \ldots (-1)^{k-1} \binom{2k}{2} \int v^2 u^{2(k-1)} \, d\mu .$$

Applying Hölder's inequality to the terms on the right we get

$$\| v \|_{2k}^{2k} \leq 2 | \hat{u}(0) |^{2k} + \binom{2k}{2} \| v \|_{2k}^{2(k-1)} \| u \|_{2k}^{2} + \cdots$$

$$+ \binom{2k}{2} \| v \|_{2k}^{2} \| u \|_{2}^{2(k-1)}.$$

Divide both sides by $\| u \|_{2k}^{2k}$, use the fact that $| \hat{u}(0) | / \| f \|_{2k} \leqq 1$, and setting $X_0 = \| v \|_{2k} / \| u \|_{2k}$ we get

$$X_0^{2k} - \binom{2k}{2} X_0^{2(k-1)} - \binom{2k}{4} X_0^{2(k-2)} - \ldots - 2 \leq 0.$$

Hence, X_0 is less than or equal to the largest real root of the polynomial equation

$$X^{2k} - \binom{2k}{2} X^{2(k-1)} - \binom{2k}{4} X^{2(1-2)} - \ldots - 2 = 0.$$

If we call this largest root K_{2k}, we have $\| Cu \|_{2k} \leqq K_{2k} \| u \|_{2k}$.

By using the linearity of C and the Minkowski inequality this inequality can be extended to elements of $ReA + iReA$ with K_{2k} replaced by $2K_{2k}$. Now take $2k \leqq p \leqq 2k + 2$ and use the Riesz-Thorin interpolation theorem [19 II, p. 95] which will establish a bound for C in L^p. Hence, for $2 \leqq p < \infty$, there exists a $B_p > 0$ such that for $f \in ReA + iReA$,

$$\| Cf \|_p \leqq B_p \| f \|_p.$$

If $1 < p < 2$, then we have just shown that C is a bounded operator on the dual space L^q and hence the adjoint is a bounded operator on L^p. It turns out that $C^* = -C$ (Notice this means that iC is a self-adjoint operator on L^2). Indeed, if $u, v \in ReA$, $f = u + iCu$, $g = v + iCv$, then

$$\int fg d\mu = \hat{f}(0)\hat{g}(0) = \int u d\mu \,.\, \int v d\mu,$$

and since the integrals on the right are real we have

$$\int vCu d\mu = - \int uCv d\mu\,.$$

If now u and v are in $ReA + iReA$, then using the linearity of C we establish this last formula for this situation. This concludes the proof of the theorem.

4. We now turn to the generalization of the conjugate function theorem when $p = \infty$. The proof follows that in [19 I, p. 254].

Theorem 10. (a) *If f is real and $| f | \leqq 1$, then*

$$\int_X \exp.\lambda\, | Cf |\, d\mu \leqq B_\lambda$$

for $0 \leqq \lambda < \pi/2$.

(b) *If f is real and continuous, then* $\exp \lambda\, | Cf |$ *is integrable for all* $\lambda \geqq 0$.

Proof. (a) Suppose $u \in ReA$, $| u | \leqq 1$, and $v \in Cu$ such that $g = u + iv \in A$. By expanding $\exp(\pm i \lambda g)$ into a power series and using the linear and multiplicative properties of the homomorphism $0 \in M$ we arrive at the fact that

$$\int_X \exp(\pm i \lambda g)\, d\mu = \exp(\pm i \lambda \hat{g}(0)) = \exp(\pm i \lambda \hat{u}(0)).$$

Taking real parts we get

$$\int_X \cos \lambda u \exp (\pm \lambda Cu) \, d\mu = \cos \lambda \hat{u} (0) \leqq 1.$$

Since $\exp \lambda | Cu | \leqq \exp \lambda Cu + \exp - \lambda Cu$ and $\cos \lambda \leqq \cos \lambda u$ for $| u | \leqq 1$ and $0 \leqq \lambda < \pi/2$, it follows that

$$\int_X \exp \lambda | Cu | \, d\mu \leqq 2 / \cos \lambda.$$

Suppose now that u is real and measurable and $| u | \leqq 1$. Since $d\mu$ is regular we may find a sequence $\{u_n\} \subseteq ReA$ such that $| u_n | \leqq 1$ and $u_n \to u$ almost everywhere. Then by Fatou's lemma

$$\int_X \exp \lambda | Cu | \, d\mu = \int_X \lim_{n \to \infty} \exp \lambda | Cu_n | \, d\mu \leqq$$
$$\underline{\lim}_{n \to \infty} \int_X \exp \lambda | | Cu_n | \, d\mu \leqq 2 / \cos \lambda .$$

(b) If u is real and continuous, given $\epsilon > 0$ there exist a $w \in ReA$ such that $| w - u | \leqq \epsilon$. Hence, by part (a) of this theorem we have

$$\int_X \exp \lambda | Cw - Cu | \, d\mu \leqq B\lambda$$

for $0 \leqq \lambda < \pi / 2$. Now, $| Cw |$ is bounded, say by N, and hence

$$\exp \lambda | Cw - Cu | \geqq \exp \lambda | Cu | \exp - \lambda N$$

which shows that

$$\int_X \exp \lambda | Cu | \, d\mu \leqq B\lambda \exp \lambda N.$$

Since ϵ is arbitrary this gives the result.

5. As we have mentioned before, we do not know whether the Fourier-Gelfand transform of an outer factor in A vanishes at any point of the homomorphism space M. Nevertheless, as an application of the results we have writen down in sections II and III we shall show that outer factors in H^1 spaces generated by general Dirichlet algebras behave very much like outer factors on the circle group. Our first theorem in this direction says, in effect, that we may take roots and logarithms of outer factors and obtain "analytic functions."

Theorem 11. *The function $f \in L^1$ is, up to a constant multiplicative factor of unit modulus, an outer factor in H^1 if and only if*

$$f = \exp (u + iCu + i\alpha)$$

where u is real and in L^1 and α is a real constant.

Proof. (a) Suppose $f \in L^1$ and

$$f = \exp\,(u + iCu)$$

where u is real and in L^1. Suppose first that u is bounded, say $|u| \leqq N$. Since $d\mu$ is regular we may find a sequence $\{u_n\} \subseteq A$ such that $u_n \to u$ in L^1 and $|u_n| \leq N$. By theorem 7 we find (taking a subsequence if necessary), $u_n \to u$ and $C u_n \to Cu$ almost everywhere.

Set

$$f_n = \exp\,(u_n + iCu_n)\ ;$$

then f_n and $1/f_n$ are elements in A and therefore, by theorem 5, f_n is, up to a constant factor of unit modulus, an outer factor. Since $|f_n| \leqq \exp N \in L^1$, the use of Lebesgue's dominated convergence theorem shows that $f_n \to f$ in L^1 and hence $f \in H^1$. Further,

$$\log|\hat{f}_n(0)| = \int_X u_n\, d\mu$$

and letting $n \to \infty$ we get

$$\log|\hat{f}(0)| = \int_X u\, d\mu = \int_X \log|\exp\,(u + iCu)|\, d\mu,$$

which shows f is, up to a constant of unit modulus, an outer factor.

If u is not bounded, set

$$u_N = \begin{cases} u & \text{when } |u| \leqq N \\ N & \text{when } u > N \\ -N & \text{when } u < -N\,. \end{cases}$$

By the previous paragraph, if we set $f_N = \exp\,(u_N + iCu_N)$, then f_N is outer in H^1. Now, $|u_N| \leq |u|$ and hence (taking a subsequence if necessary) $u_N \to u$ and $Cu_N \to Cu$ almost everywhere. Further, $|f_N| \leqq 1 + \exp u$ which is in L^1 by the hypothesis that $f \in L^1$. Hence, using the Lebesgue dominated convergence theorem $f_N \to f$ in L^1 and hence in H^1. Also, by the same argument as above f is, up to a constant factor, outer.

(b) Conversely, suppose f is an outer factor in H^1. Let $\epsilon > 0$ and let f_ε be the outer factor in H^1 such that $|f_\varepsilon| = |f| + \epsilon$. It follows *from the proofs* of theorems 2 and 3 that $f_\varepsilon \to f$ in H^1 as

$\epsilon \to 0$. Set $u_\varepsilon = \log(|f| + \varepsilon)$; then $u_\varepsilon \to u = \log|f|$ in L^1 and hence (choosing a subsequence if necessary) $Cu_\varepsilon \to Cu$ almost everywhere. By part (a) of this proof,

$$F_\varepsilon = \exp(u_\varepsilon + iCu_\varepsilon)$$

is, up to a constant of unit modulus, an outer factor and hence, by theorem 5, so is $F_\varepsilon / f_\varepsilon$. But $|F_\varepsilon / f_\varepsilon| \equiv 1$, which means it is an inner factor and consequently

$$f_\varepsilon = \exp(u_\varepsilon + iCu + i\alpha_\varepsilon), \quad \alpha_\varepsilon \text{ constant and real.}$$

Letting $\epsilon \to 0$ we get, $\exp i\alpha_\varepsilon \to \exp i\alpha$ and

$$f = \exp(u + iCu + i\alpha), \quad \alpha \text{ constant and real.}$$

Our next theorem says that any element in H^1 whose range is in any half section of the complex plane is an outer factor. We formulate it as follows.

Theorem 12. *If* $f \in H^1$ and $Re f \geqq 0$, *then* f *is, up to a constant factor of unit modulus, an outer factor.*

We shall need several elementary statements which we record as lemmas.

Lemma 1. *If* $f \in H^1$, $f = u + iv + id$, u, v, d *real,* $\hat{v}(0) = 0$, *then,* $v = Cu$.

Proof. Let $f_n = u_n + iCu_n + id_n \in A$ such that $f_n \to f$ in H^1. Since $u_n \to u$ in L^1, by theorem 7 $Cu_n \to Cu$ in L^p, $0 < p < 1$. Therefore (taking a subsequence if necessary) $Cu_n \to Cu$ almost everywhere. But $Cu_n + d_n \to v + d$ almost everywhere and since $d_n = \int Im f_n \, d\mu \to d = \int Im f \, d\mu$ we see that $Cu = v$.

Lemma 2. *If* $f \in H^1$ and $Re f > 0$ *then there exist* $f_n \in A$ *with* $Re f_n > 0$ *such that* $f_n \to f$ *almost everywhere.*

Proof. Let $f = u + iCu + id$, $u > 0$, d real. Since $u > 0$ and $d\mu$ is regular there exist $\{u_n\} \subseteq Re A$ such that $u_n > 0$ and $u_p \to u$ in L^1. Hence, by theorem 7 (taking a subsequence if necessary) $Cu_n \to Cu$ almost everywhere and consequently $f_n = u_n + iCu_n + id \to f$ almost everywhere.

Lemma 3. *If* $f \in H^1$, $Re f > 0$, *then* $\exp tf \in H^\infty$ *for* $t \leqq 0$.

Proof. By lemma 2 there, exist $f_n \in A$ such that $Re f_n > 0$ and $f_n \to f$ almost everywhere. Now $\exp tf_n \in A$ and $\exp tf_n > \exp tf$ almost everywhere. Further, $|\exp tf_n| \leqq 1$ and hence by Lebesgue's dominated convergence theorem

$$\int_X |\exp \,|\exp tf_n - \exp tf|\, d\mu \to 0.$$

Lemma 4. *If* $f \in H^1$, $1/Ref \in L^1$ *and* $Ref > 0$, *then* f *is, up to a constant factor of unit modulus, outer.*

Proof. By theorem 5 it is enough to show that $1/f \in H^1$. Now, for $t \geqq 0$, $h_t = \exp(-tf)$ is a continuous function of t in the topology of H^1 and hence

$$\int_0^R h_t\, dt$$

exists as a limit of Riemann sums in the topology of H^1. Because of this we have for almost all $x \in X$,

$$\left(\int_0^R h_t dt\right)(x) = \int_0^R h_t(x)\, dt.$$

Now, for every $x \in X$,

$$\int_0^R h_t(x)\, dt \to \int_0^\infty h_t(x) dt = 1/f(x) \text{ as } R \to \infty.$$

Further,

$$\left|\int_0^R h_t(x) dt\right| \leqq \int_0^\infty \exp\,(-tRef\,(x))\, dt = 1/Ref\,(x) \in L^1.$$

Therefore, taking a sequence $R_n \to \infty$ and using Lebesgue's dominated convergence theorem

$$\int_X \left|\int_0^{R_n} h_t\, dt - 1/f\right| d\mu \to 0$$

which shows $1/f \in H^1$.

Proof of theorem 12. If $Ref = 0$, then f is constant and hence is, up to a constant of unit modulus, outer. To see this write $f = iv$, where v is real. But then $v \in ReA \cap A$ which implies v is constant almost everywhere. Hence we may assume $Ref > 0$ on a set of positive measure and hence $\hat{f}(0) \neq 0$.

If $\epsilon > 0$, $f(x) + \epsilon$ satisfies all of the hypotheses of lemma 4 and hence

$$\log |\hat{f}(0) + \epsilon| = \int_X \log |f(x) + \epsilon| \, d\mu .$$

Since $Ref \geqq 0$, $|f(x) + \epsilon| \to |f(x)|$ as $\epsilon \to 0$ and hence $\log |f(x) + \epsilon| \to \log |f(h)|$ and

$$\log |\hat{f}(0)| = \int_X \log |f(x)| \, d\mu.$$

Theorem 13. *If* $f \in H^1$ *and* $Ref \geqq 0$, *then* Log $f \in H^1$, *where* Log z *is the principal branch of the logarithm.*

Proof. Assume first that $Ref \geqq \delta > 0$. Since f is, up to a multiplicative constant, outer, $1/f \in H^\infty$ and also Re $(1/f) \geqq 0$. For any $\epsilon > 0$, $(1/f) + \epsilon \in H^\infty$ and therefore using a Taylor series expansion

$$\mathrm{Log}\,(1/f + \epsilon) = \log |1/f + \epsilon| + i\mathrm{Arg}\,(1/f + \epsilon) \in H^\infty$$

Now, for every x, Log $(1/f(x) + \epsilon) \to$ Log $1/f(x)$ as $\epsilon \to 0$.

Further

$$|\mathrm{Arg}\,(1/f + \epsilon)| \leqq \pi/2 \text{ and as } \epsilon \to 0, \log |1/f + \epsilon| \to \log |1/f|.$$

Since f is up to a constant an outer factor and $1/f \in L^\infty$, it follows that $\log |1/f| \in L^1$. Given $\epsilon_0 > 0$, for $0 < \epsilon < \epsilon_0$, we have

$$||\log |1/f + \epsilon|| \leqq |\log |1/f|| + |\log |1/f + \epsilon_0|| .$$

By theorem 12, $1/f + \epsilon_0$ is, up to a constant of unit modulus, an outer factor since Re $(1/f + \epsilon_0) > 0$ and hence $\log |1/f + \epsilon_0| \in L^1$. Applying Lebesgue's dominated convergence theorem we see that Log $(1/f + \epsilon) \to$ Log $1/f$ in L^1 as $\epsilon \to 0$ and hence Log $1/f$ is in H^1. Hence —Log $1/f =$ Log $f \in H^1$.

Suppose now that f satisfied the hypotheses of the theorem. For any $\delta > 0$, $f + \delta$ satisfied the hypotheses of the first part of the proof and therefore $\text{Log}\,(f+\delta) \in H^1$. Letting $\delta \to 0$ and making the same type of argument as above we find that $\text{Log}\,(f+\delta) \to \text{Log}\, f$ in L^1 and therefore in H^1.

Corollary. *If f satisfied the hypotheses of theorem* 13 *then*

$$C\,\text{Arg}\ f = -\log|f| + \log|\hat{f}(0)|.$$

The proof is an immediate consequence of lemma 1 and theorem 13.

IV. THE HELSON-SZEGÖ CONJUGATE FUNCTION THEOREM

1. The problem posed by Helson and Szegö is as follows: Suppose A is a Dirichlet algebra and $d\nu$ is a bounded non-negative measure on the Borel field of X. When is it true that there exists a constant $B > 0$ such that for every $f \in ReA$.

$$\int_X |Cf|^2 d\nu \leq B \int_X |f|^2 d\nu\ ?$$

For the sake of simplicity we shall, at the outset, assume that $d\nu$ is absolutely continuous with respect to the representing measure $d\mu$; i.e., $d\nu = w\,d\mu$, where w is a non-negative function in L^1.

We shall designate by $H^2(w)$ the closure of A in $L^2(w\,d\mu)$ and by $\overline{H}_0{}^2(w)$ the closure of $\overline{A}_0$ in $L^2(w\,d\mu)$. Recall that A_0 is the collection of $g \in A$ such that $\hat{g}(0) = 0$. $\overline{A}_0$ is the collection of complex conjugates of elements in A_0.

2. Let H be a Hilbert space and R and S manifolds in H. Let

$$\rho = \sup\{|(f|g)|\ ;\ f \in R, g \in S, \|f\| = \|g\| = 1\}.$$

It is clear from Schwarz's inequality that $0 \leq \rho \leq 1$. If $\rho < 1$, the manifolds R and S are said to be at a *positive angle.*

The manifolds R and S are at a positive angle if and only if there exists a $0 \leq \sigma < 1$ such that for every $f \in R$, $g \in S$

$$2(1-\sigma)\|f\|\ \|g\| \leq \|f+g\|^2.$$

Indeed, suppose R and S are at a positive angle. Then from the facts that $\| f+g \|^2 = \| f \|^2 + 2\, Re(f \mid g) + \| g \|^2, 2 \| f \| \;\; \| g \| \leqq \| f \|^2 + \| g \|^2$ and $Re(f \mid g) \geqq -\rho \| f \| \;\; \| g \|$ we get $2(1-\rho) \| f \| \;\; \| g \| \leqq \| f + g \|^2$. Conversely, suppose there exists $0 \leqq \sigma < 1$ such that the above inequality is satisfied. Take $\| f \| = g \| = 1$ and suppose, for the moment, that $(f \mid g) \leqq 0$. We then get, by expanding the right side of the above displayed inequality, $\mid (f \mid g) \mid \leqq \sigma < 1$. In the general case where $(f \mid g)$ is complex, it is posible to multiply f by a constant of unit modulus so as to make the inner product non-positive. However, this will not affect the absolute value of the inner product of f and g.

Let us now specialize to the case where $H = L^2(wd\,\mu)$, $R = H^2(w)$ and $S = \overline{H}_0{}^2(w)$. Let P be the projection of $L^2(d\,\mu)$ onto H^2. A formula for this in terms of the conjugation operator can be obtained in the following way. Suppose at first that $u \in ReA$; then $f = u + iCu \in H^2$ and $u = [f + \hat{u}(0) + \overline{f} - \hat{u}(0)]\,/\,2$. But $f + \hat{u}(0) \in H^2$ and $\overline{f} - \hat{u}(0) \in (H^2)^{\top}$ and since the decomposition of u into components in orthogonal subspaces is unique we get $Pu = [f + \hat{u}(0)]\,/\,2$ or

$$Pu = \frac{1}{2}[u + \hat{u}(0) + iCu].$$

By linearity and continuity of P and C in L^2 this formula can be extended to all of L^2.

Restrict P to $ReA + iReA$. *$H^2(w)$ and $\overline{H}_0{}^2(w)$ are at a positive angle if and only if this restriction is bounded in $L^2(w\,d\,\mu)$*. Indeed, suppose the restriction of P is bounded. Then there exists a constant $B > 0$ such that

$$\| Pu \| \; \| (I-P)u \| \leqq B \| u \|^2.$$

Take $\sigma = (2B-1)/2B$, $f = Pu$ and $g = (I-P)u$ and we get $2(1-\sigma) \| f \| \;\; \| g \| \leq \| f + g \|^2$. Since the collection of such f and g are dense in $H^2(w)$ and $\overline{H}_0{}^2(w)$ respectively, these manifolds are at a positive angle. Conversely, suppose these manifolds are at a positive angle. Take $B = 1/2(1-\sigma)$, $f = Pu$, $g = (I-P)u$, where $\| u \| = 1$. We then get

$$\| Pu \| \mid 1 - \| Pu \| \mid \leqq B.$$

This shows that $\| Pu \| \leq \max(B, 2)$, which implies P is bounded.

From the formula connecting P and C we can easily show that *the restriction of C to $ReA + iReA$ is bounded in $L^2(w\, d\, \mu)$ if and only* if the restriction of P to this manifold is bounded in $L^2(w\, d\, \mu)$. Indeed, suppose P is bounded and $u \in ReA$. Then $Cu = 2ImPu$ which shows inmediately that Cu is bounded since $|\, Cu \,| \leq 2 \,|\, Pu \,|$. Conversely, suppose C is bounded. Take $u \in ReA + iReA$ and we find that $C^2u = - u + \hat{u}(0)$ which shows that $\hat{u}(0)$ is a continuous functional in $L^2(w\, d\, \mu)$. Now look at the formula connecting P and C and it results that P is bounded.

Putting the above statements together we have arrived at the following fact.

Lemma 5. *The conjugation operator C, restricted to $ReA + iReA$ is bounded in $L^2(w\, d\, \mu)$ if and only if $\overline{H}^2(w)$ and $\overline{H}_0{}^2(w)$ are at a positive angle.*

3. The question now arises as how to characterize w so that $H^2(w)$ and $\overline{H}_0{}^2(w)$ are at a positive angle in $L^2(w\, d\, \mu)$. We may as well assume from the outset, that $\int \log w\, d\, \mu > - \infty$. Otherwise, theorem 3 shows that $1 \epsilon H^2(w) \cap \overline{H}_0{}^2(w)$ which, from the definition, inmediately shows that these manifolds are not at a positive angle. Therefore, by theorem 2 we can write $w = |\, f \,|^2$ where f is an outer factor in H^2. Moreover, by theorem 6, the collection of functions of the form gf, $g \in A$ is dense in H^2 and more specifically the set of all such elements such that

$$\|\, g \,\|^2 = \int |\, gf \,|^2 d\, \mu \leqq 1$$

is dense in the unit ball of H^2.

Define the function $\exp - i\, \phi$ by equation $w = f^2 \exp - i\, \phi$.

The number ρ is given by

$$\rho = \sup \; \left| \int (gf)(hf) \exp - i\, \phi \; d\, \mu \right|$$

where the supremum is taken over all $g \in A$ and $h \in A_0$ such that $\|\, g \,\| \leqq 1$ and $\|\, h \,\| \leqq 1$. If $H_0{}^2$ is the subspace of H^2 consisting of elements with mean value zero then it is an inmediate consequence of theorem 6 that the set of elements hf, $h \in A_0$, $\|\, h \,\| \leqq 1$, is dense in the unit ball of $H_0{}^2$.

Before we proceed, it will be necessary for us to prove a fact, whose need and proof in the general situation was pointed out by I. I. Hirschman. We record this as

Lemma 6. *The set of functions* ghf^2 *with* $g \epsilon A$, $h \epsilon A_0$ *and* $\| g \| \leq 1$, $\| h \| \leq 1$ *range over a dense subset of the unit ball of* H_0^1 (*the subspace of elements in* H^1 *with mean value zero*).

Proof. Let k be an element of the open unit ball of H_0^1; i. e. $\int | k | d\mu \leq 1$. If $\int \log | k | d\mu > \infty$, then by theorem 4 we may write $k = l\, m$, where l is in the open unit ball of H^2 and m is in the open unit ball of H_0^2. Hence, by theorem 6, k may be approximated by the elements asserted in the lemma.

Suppose now that $\int \log | k | d\mu = -\infty$. For $\epsilon > 0$ set $k_\varepsilon = k + \varepsilon$, and suppose ϵ has been chosen small enough o that k_ε is still in the open ball units of H_0^1. Now $\hat{k}_\varepsilon(0) = \varepsilon > 0$ and hence by theorem 1, $\int \log \; |k_\varepsilon| \; d\mu > -\infty$. By theorem 4 we may write $|k_\varepsilon| = |l_\varepsilon{}^2|$ where l_ε is an outer factor in the open unit ball of H^2. Hence,

$$\log |\hat{l}_\varepsilon(0)|^2 = \int\limits_X \log |l_\varepsilon|^2 \, d\mu = \int\limits_X \log |k + \varepsilon| \, d\mu$$

and we see that $\hat{l}_\varepsilon(0) \to 0$ as $\epsilon \to 0$. Now write

$$k = n\,(l_\varepsilon - \hat{l}_\varepsilon(0))^2 + \hat{l}_\varepsilon(0)\; n\,(2\,l_\varepsilon - \hat{l}_\varepsilon(0)) - \varepsilon,$$

where n is an inner factor. Then term $2\,l_\varepsilon - \hat{l}_\varepsilon(0)$ remains uniformly bounded in H^1 as $\epsilon \to 0$ and $l_\varepsilon - \hat{l}_\varepsilon(0)$ is in the open unit ball of H_0^2 if ϵ is sufficiently small. Therefore, k can be approximated as asserted.

Using lemma 6 we may write

$$\rho = \sup \left| \int\limits_X h \exp - i\,\phi \, d\,\mu \right|$$

H_0^1 is a closed subspace of L^1 and ρ is the norm of a linear functional on H_0^1 defined by

$$\int\limits_X h \exp - i\,\phi \;\; d\,\mu, \; h \,\epsilon\, H_0^1.$$

The Hahn-Banach theorem tells us we can extend this linear functional to all of L^1 with preservation of norm. Every continuous linear functional on L^1 may be identified with an element of L^∞, and we shall write any element which extends the above functional with preservation of norm as $e^{-i\phi} - g_0$. It is clear that

$$\int hg_0 d\mu = 0, \text{ all } h \epsilon A_0,$$

and this means $g_0 \epsilon H^\infty$. Further, $\rho = \| g_0 - \exp - i\phi \|_\infty \leqq \| g - - \exp - i\phi \|_\infty$ for any $g \epsilon H^\infty$, since $(g - \exp - i\phi)$ may be identified with an extension of the given linear functional.

Theorem 14. $\rho < 1$ *if and only if there exists a* $g \epsilon H^\infty$ *such that* $1/g \epsilon H^\infty$ *and* $\| \operatorname{Arg} gf^2 \|_\infty < \pi/2$, *where* $-\pi \leqq \operatorname{Arg} z < \pi$.

Proof. If $\rho < 1$, pick $g = g_0$, where by our previous remarks $\| g_0 - \exp - i\phi \|_\infty = \rho < 1$. It is clear by a simple geometric consideration (draw a circle around $e^{-i\phi}$ of radius ρ) that $1/g \epsilon L^\infty$ and $\| \operatorname{Arg} gf^2 \|_\infty < \pi/2$. Now, $gf^2 \epsilon H^1$ and $Re\, gf^2 \geqq 0$ and hence by theorem 12, gf^2 is, up to a constant o funit modulus, an outer factor. Hence, the same must be true for g which makes $1/g \epsilon H^\infty$.

Conversely, suppose g satisfies the conditions of the theorem. In order to fix matters look at the function $g \exp i\phi$ which has the same essential supremum and infimum as g and the same arguments as gf^2. Let $\alpha = \| \operatorname{Arg} gf^2 \|_\infty < \pi/2$; then if $|\theta| \leqq \alpha$, the length of the line segment from the origin to the circle of unit radius about 1 and along the line $\operatorname{Arg} z = \theta$ is $2 \cos\theta \geqq 2 \cos \alpha$. Let $\lambda = \cos\alpha / \| g \|_z$; then since the essential infimum of g is bounded away from zero, there exists a $0 < \delta < \cos\alpha$ so that $|\lambda g| \geqq \delta$ and $\lambda g \exp i\phi$ is inside the circle about 1 of unit radius. The latter statement follows from the fact that $|\theta| = \operatorname{Arg} \lambda g e^{i\phi} \,| \leqq \alpha$ and $|\, \lambda g e^{i\phi} \,| \leqq \cos\alpha \leqq \cos\theta$. If we take now, say, $\delta^2 - 2\delta\cos\alpha + 1 < \sigma^2 < 1$ then $\lambda g \exp i\phi$ is contained in the circle about 1 of radius σ. Therefore

$$\rho \leqq \| \lambda g - \exp - i\phi \|_\infty \leqq \sigma < 1.$$

Theorem 15. *The conjugation operator* C, *restricted to* $ReA + iReA$, *is bounded in* $L^2(w\, d\mu)$ *if and only if*

$$w = \exp(u + Cv),$$

where u *and* v *are real,* $\| u \|_\infty < \infty$ and $\| v \|_\infty < \pi/2$.

Proof. Suppose C is bounded in $L^2(w\, d\mu)$. Then, as we have noted, we may write $w = |h|$ where h is outer in H^1. Lemma 5 says that $H^2(w)$ and $\overline{H}_0{}^2(w)$ are at a positive angle and theorem 14 says that there exists a $g \in H^\infty$, such that $1/g \in H^\infty$ and $\| \operatorname{Arg} gh \|_\infty < \pi/2$. Therefore, we have

$$w = |g^{-1}| \; |gh| = \exp(\log |g^{-1}| + \log |gh|).$$

Now, g and g^{-1} are bounded and hence $\log |g^{-1}| \in L^\infty$. Further, $Re gh \geqq 0$ and hence by the corollary to theorem 13, $C \operatorname{Arg} gh = -$ $-\log |gh| + \log |\hat{gh}(0)|$. Take $u = \log |g^{-1}| + \log |\hat{gh}(0)|$ and $v = -C \operatorname{Arg} gh$.

Conversely suppose w is of the form as shown in the theorem. If we set $w_1 = w \exp -u$, then $H^2(w_1)$ and $\overline{H}_0{}^2(w_1)$ are at a positive angle if and only if $H^2(w)$ and $\overline{H}_0{}^2(w)$ are at a positive angle. This follows immediately from the fact that $\exp -u$ is bounded away from zero and hence the projection operator P is simultaneously bounded of unbounded in $L^1(w_1 d\mu)$ and $L^1(w d\mu)$. Hence, we may as well suppose $u = 0$.

Set

$$h = \exp(Cv - iv + i\,a);$$

then $w = |h|$ and theorems 10 and 11 tell us that if we choose a correctly h is an outer factor in H^p for some $p > 1$ and therefore h is the function of theorem 14. Take $g = \exp(-i\,a)$ and we see that $\| \operatorname{Arg} gh \|_\infty = \| v \|_\infty < \pi/2$ and the result follows by theorem 14.

SELECTED REFERENCES

1. Arens, R. and Singer, I. M., *Function values as boundary integrals*, Proc. Amer. Math. Soc., 5 (1954), 735-745.
2. — —, *Generalized analytic functions*, Trans. Amer. Math. Soc., 81 (1956), 379-395.
3. Beurling, A., *On two problems concerning linear transformations in Hilbert space*, Acta Math., 81 (1949), 239-255.
4. Bochner, S., *Additive set functions on groups*, Ann. of Math. 40 (1939), 769-799.
5. — —, *Generalized conjugate and analytic functions without expansions*, Proc. Nat. Acad. Sci., 45 (1959), 855-857.
6. Devinatz, A., *The factorization of operator valued functions*, Ann. of Math., 73 (7961), 458-495.

6a. — —, *Toeplitz operators on H^2 spaces,* Trans. Amer. Math. Soc., 1964.

7. GLEASON, A., *Function Algebras,* Seminars on Analytic Functions, Institute for Advanced Study, 1957.

8. HALMOS, P., *Shifts on Hilbert spaces,* J. Reine Angew. Math., 208 (1961), 102-112.

9. HELSON, H. and LOWDENSLAGER, D., *Prediction theory and Fourier series in several variables,* Acta Math., 89 (1958), 165-202.

10. HELSON, N. and SZEGÖ, G., *A problem in prediction theory,* Annali di Mat. LI (1960), 107-138.

11. HOFFMAN, K., *Analytic functions and logmodular Banach algebras,* Acta Math., 108, 1962), 271-317.

12. — —, "Banach Spaces of Analytic Functions", Prentice-Hall, Englewood Cliffs, N. J., 1962.

13. LAX, P., *Translation invariant subspaces,* Acta Math., 101 (1959), 163-178.

14. SZEGÖ, G., *Beiträge zur Theorie der Toeplitzchen Formen,* 1, II, Math. Zeit., 6 (1920), 167-202; 9 (1921), 167-190.

15. — —, *Über die Randwerte einer analytischen Funktion,* Math. Ann., 84 (1921), 232-244.

16. WERMER, J., *Dirichlet algebras,* Duke J. Math., 27 (1960), 373-382.

17. — —, *Banach algebras and analytic functions,* Advances in Mathematics, Academic Press, New York, 1961.

18. WIENER, N .and MASANI, P., *The prediction theory of multivariate stochastic processes,* Acta Math., 98 (1957), 111-150; II 99 (1958), 93-137.

19. ZYGMUND, A., "Trigonometric Series I, II", Cambridge University Press, 1959.