ON JORDAN OPERATORS AND RIGIDITY OF LINEAR CONTROL SYSTEMS

by H. O. FATTORINI *
University of Buenos Aires

INTRODUCTION

Let E be a vector space over a field K. A linear operator A in E will be called a *Jordan operator* if there exists a non-null polynomial $P(\lambda) = a_0 + a_1 \lambda + \ldots$ with coefficients in K such that

$$P(A) = a_0 + a_1 A + \ldots = 0 \tag{1}$$

We present in this paper a result on these operators (Theorem 1.2). It is established for the case $K =$ real or complex numbers, E a Banach space, A a bounded operator, although it is easily seen to be valid, with an additional assumption, for general K, E and A. (See the observations after the proof). Theorem 1.2 is proved with the help of a result in [5] (Theorem 15) which, for the sake of completeness, is included here together with Lemma 14 as Theorem 1.1 and Remark 1 respectively. We establish next a version of Theorem 1.2 for certain unbounded operators A (Theorem 2.2) and point out its connections with control theory. Theorem 2.2 is a generalization of Theorem 2.2 of [4] from the case in which the "control" space F has dimension 1 to the case of arbitrary finite dimension.

Paragraph § 1 is fairly self contained and makes use only of elementary notions of linear topological algebra; paragraph § 2 is

* Most of the results of this paper were obtained at the Courant Institute of Mathematical Sciences, New York University, with the support of a Ford Foundation Pre-Doctoral Fellowship.

closely related to [4], Section § 2 and uses notations, definitions and results in that paper.

§ 1. *The case of bounded A*

We shall suppose throughout this paragraph (unless otherwise stated) that K is the field of real (complex) numbers, E is a real (complex) Banach space and A is a bounded operator.

Theorem 1.1. Assume that for every $u \in E$ there exists a polynomial $p(\lambda) = p(u;\lambda) \neq 0$ such that $p(A)u = 0$. Then A is a Jordan operator.

Proof: Let $p(u;\lambda)$ be the *minimal* polynomial of A at u, i.e. the generator of the ideal I_u (of the ring of polynomials in one indeterminate with coefficients in K) consisting of all polynomials $p(\lambda)$ with $p(A)u = 0$. Recall that $p(u;\lambda)$ is uniquely defined, save by multiplication by a nonzero element of K. We have

$$p(cu;\lambda) = p(u;\lambda) \ , \ c \in K \ , \ c \neq 0 \tag{1.1}$$

$$p(u;\lambda)\, p\,(v;\lambda) \text{ is divisible by } p\,(u+v;\lambda) \tag{1.2}$$

(1.1) is clear; (1.2) follows from the relation

$$p(u;A)\, p\,(v;A)\,(u+v) = p(v;A)\, p(u;A)u + \\ + p(u;A)p\,(v;A)v = 0$$

Let us observe next that the degree of $p(u;\lambda)$ *is* bounded independently of u. In fact, let
$E_N = \{\ u \in E \mid \deg\ p(u;\lambda) \leqq N \}$, and let $\{\ u_n \}$ be a sequence in some E_N convergent to some element $u \in E$.
Normalize $p(u_n;\lambda) = a_{0,n} + a_{1,n}\lambda + \dots$ by, say, the condition $|a_{0,n}| + |a_{1,n}| + \dots. = 1$. By passing, if necessary to a subsequence we can suppose that $a_{k,n} \to a_k$ as $n \to \infty$; by the normalization condition $|a_0| + |a_1| + \dots = 1$ and therefore $p(\lambda) = a_0 + a_1\lambda + \dots \neq 0$. But

$$p(A)u = \lim p(u_n;A)u_n = 0$$

hence $\deg\ p(u;\lambda) \leqq \deg\ p(\lambda)$ and $u \in E_N$. This shows that each E_N is closed. Since $\bigcup_N E_N = E$ the category theorem of Baire implies that some E_N contains a sphere, say $\{ u \epsilon E \mid |u - u_0| \leqq \rho \}$ But if v is any element of E $p(v;\lambda) = p(\rho v/|v|;\lambda)$ divides $p(u_0 - \rho v/|v|;\lambda)\, p(u_0;\lambda)$ which shows that

$$\deg \mathrm{p}(v;\lambda) \leqq 2N.$$

Let us pass now to the construction of the polynomial P in (1). Choose $u \epsilon E$ such that

$$\deg p(u;\lambda) = \sup \{ \deg p(v;\lambda) \ ; \ v \epsilon E \} \tag{1.3}$$

We shall show that $p(u;\lambda) = P(\lambda)$. In fact, let w be any element of E such that $p(w;\lambda) = p_0(\lambda)^m$, $m \geqq 1$ where $p_0(\lambda)$ is an *irreducible* polynomial. In view of (1.2) we have

$$p(u+w;\lambda)p(w;\lambda) = p(u;\lambda)\, q(\lambda) \tag{1.4}$$

$$p(u;\lambda)\, p(w;\lambda) = p(u+w;\lambda)\, r(\lambda) \tag{1.5}$$

where q, r, are polynomials. We get from (1.4) and (1.5) that

$$q(\lambda)\, r(\lambda) = p(w;\lambda)^2 = p_0(\lambda)^{2m}$$

$$so\ \ q(\lambda) = p_0(\lambda)^k,\ r(\lambda) = p_0(\lambda)^j,\ k, j \geqq 0, k + j = 2m$$

Then

$$p(u;\lambda) = p(u+w;\lambda)\, p_0(\lambda)^h,$$

$-m \leqq h \leqq m$. By virtue of (1.3) $h \geqq 0$. But then $p(u;A)\, w = p_0(A)^h p(u+w;A)\,(u+w) - p(u;A)\, u = 0$, so $p(u;\lambda)$ is divisible by $p(w;\lambda)$.

Let now v be any element of E, $p(v;\lambda) = \Pi^n_{k=1} p_k(\lambda)^{m_k}$ where $p_1, \dots p_n$ are different irreducible polynomials. It is plain that if $w = \Pi_{k \neq j} p_k(\lambda)^{m_k}$, $p(w;\lambda) = p_j(\lambda)^{m_k}$ By virtue of the precedings considerations $p(u;\lambda)$is divisible by all the polynomials $p_j(\lambda)^{m_j}$, and hence by $p(v;\lambda)$ itself. This ends the proof of Theorem 1.

Remark 1 Clearly, Theorem 1.1 remains valid for general K, E and A if we *assume*

$$\sup \{ \deg p(u;\lambda) \ ; \ u \epsilon E \} < \infty \tag{1.6}$$

On the other hand, if (1.6) is false the conclusion of Theorem 1.1 might not hold. In fact, let E consist of all sequences $\{ a_0, a_1, \dots \}$ of elements of K such that $a_k = 0$ except for a finite number of indices, $A \{ a_0, a_1, \dots \} = \{ a_1, a_2, \dots \}$. Then for each $u \in E$ there exists $n = n(u)$ such that $A^n u = 0$; however, it is easy to see that A is not a Jordan operator.

Remark 2 We only need to assume in Theorem 1.1 the existence of a function $f(u; \lambda)$ for each $u \in E$, analytic in $\sigma(A)$ such *that* $f(A)u = 0$ (2). In fact, any such f can be written $f = gp$, where g has no zeros in $\sigma(A)$ and p is a polynomial. Then $f(A) = g(A)p(A)$ and, since $g(A)$ is one-to-one $f(A)u = 0$ implies $p(A)u = 0$.

Remark 3 It is clear from the proof of Theorem 1.1 that we need to assume the existence of $p(u; \lambda)$ (or $f(u; \lambda)$, see Remark 2) only for u in a subspace of the second category of E.

Theorem 1.2 Let $m \geqq 1$. *Assume that for every m-ple* $(u_1, u_2, \dots u_m)$ *there exists a m-ple of polynomials* $(p_1, \dots p_m)$ *not all zero such that* $\Sigma^m_{k=1} p_k (A) u_k = 0$. *Then A is a Jordan operator.*

Proof: Let E^m be the Banach space of all m-ples $(u_1, u_2, \dots u_m)$ of elements of E (pointwise operations) normed with, say, $|(u_1, u_2, \dots u_m)| = \max (|u_1|, |u_2|, \dots |u_m|)$. *Let* $E_N{}^m = \{ (u_1, u_2, \dots u_m) \in E^m$ such that there exists polynomials $p_1, p_2, \dots p_m$ not all zero with $\Sigma^m_{k=1} p_k(A)u_k = 0$ and $\max_k \deg p_k \leqq N \}$. It is easy to show like in the proof of Theorem 1.1 that each $E_N{}^m$ is closed; thus by Baire's cathegory theorem some $E_N{}^m$ contains a sphere. This implies again that the degree of the polynomials $p_1, p_2, \dots p_m$ in the statement of Theorem 1.2 can be supposed bounded by a constant N independent of $(u_1, u_2, \dots u_m)$.

We end now the proof by induction. If $m = 1$ we are in the case considered in Theorem 1.1. Let $m > 1$ and let $(u_1, u_2, \dots u_{m-1})$ be any $(m-1)$ — ple of elements of E.

Consider the m-ple

$$(u_1, u_2, \dots, A^{N+1} u_{m-1}, u_{m-1})$$

By the preceding considerations, there exists a m-ple $(p_1, p_2, \dots p_m)$ of polynomials, not all zero and such that $\Sigma_{k=1}^{m-2} p_k(A) u_k + (p_{m-1}(A) A^{N+1} + p_m(A)) u_{m-1} = 0$, $\max_k \deg p_k \leqq N$. Since

(2) See [2], VII for the necessary notions of operational calculus.

deg $p_m \leqq N$ the polynomials above cannot be all zero, and thus our inductive step is achieved. Theorem 1.2 is proved.

Remarks 1 and 3 after Theorem 1.1 have evident generalizations to this case. As regards to Remark 2 we only need to assume in Theorem 1.2 for each $(u_1, \ldots u_m) \in E^m$ the existence of m functions $f_1, \ldots f_m$, analytic in a domain $D \supset \sigma(A)$ (independent of $(u_1, \ldots u_m)$), not all zero, such that $\Sigma f_k(A) u_k = 0$. The proof is substantially similar to that of Theorem 2.2 below.

§ 2. *Rigidity of linear control systems*

We consider in this paragraph linear control systems

$$u'(t) = Au(t) + Bf(t), \; t \geqq 0 \tag{2.1}$$

Here A is the infinitesimal generator of a strongly continuous *semigroup* $T(t)$ of bounded operators in the complex Banach space E, $u(t)$ is a point in the space E describing the state of the system at the time t, $f(t)$ is a function (the *input* or *control*) with values in some other Banach space F and the linear bounded operator $B : F \rightarrow E$ is a "transmission mechanism" through which f acts on (2.1).

We shall understand by a solution of (2.1) with initial data $u(0) = u \epsilon E$ and input f in some space $L^p(0, \infty; F)$, $1 \leqq p \leqq \infty$, the expression

$$u(t) = T(t)u + \int_0^t T(t-s)\, Bf(s)\, ds \tag{2.2}$$

where $T(t)$ is the semigroup generated by A (see [4])

A point $v \in E$ will be called *reachable from* u if there exists f such that the solution $u(t)$ of (2.1) starting at u (say, for $t = 0$) satisfies $u(t) = v$ for some $t \geqq 0$.

Definition The linear control system (2.1) will be called *rigid* if any point v, reachable from another point u in the time t by means of some control f is not reachable from u in the same time by any control different from f.

It follows easily from the representation (2.2) for the solution of (2.1) (and the replacement of $t-s$ by s in the integral) that the system (2.1) will be rigid if and only if the map

$$f \rightarrow \int_0^t T(s)\, Bf(s)\,.ds \tag{2.3}$$

from $L^p(0, t; F)$ to E is one-to-one for all $t > 0$

Let us pass now to establish the relation between these notions and the results in § 1. In view of the last observation in the proof of Theorem 2.2 in [4] we need only to consider the case $p = 2$. Observe next that if F is m-dimensional unitary space, the space $\mathcal{L}\ (F; E)$ [3] of all linear bounded operators from F to E can be algebraically and topologically identified with the space E^m defined in the proof of Theorem 1.2 by means of the correspondence that assigns to the element $(u_1, \ldots, u_m) \in E^m$ the operator in $\mathcal{L}\ (F; E)$

$$B(x_1, \ldots, x_m) = \Sigma_{k=1}^m x_k\, u_k\,, \quad (x_1, \ldots, x_m) \in F \tag{2.4}$$

It is a consequence of the functional calculus for infinitesimal generators (see [4], § 2) that if

$$f(s) = (f_1(s), \ \ldots, f_m(\mathrm{s})) \in L^2\ (0, \infty; F)$$

and B is the operator (2.4)

$$\int_0^t T(s)\, Bf(s)\, ds = \Sigma_{k=1}^m \hat{f}_k(A) u_k$$

where the functions $\hat{f}_k$ (the Fourier transforms $\hat{f}_k(\lambda) = \int f_k(s) \exp(\lambda s)\, ds$ of f_k) belong to the space H^2 of the left half-plane (see [4], § 2 and [3])

Finally, let us recall the notion of *operator of admissible meromorphic type*, generalization of that of Jordan operator for the unbounded case. An infinitesimal generator A is said to be of admissible meromorphic type if the resolvent $R(\lambda; A)$ is a meromorphic function with poles of order m_k at points λ_k and

$$-\Sigma\, m_k\, Re\, \lambda_k \,/\, (1 + |\, \lambda_k\,|^2) < \infty$$

(see again [4], § 2). The preceding considerations make clear the equivalence of

(3) We endow $\mathcal{L}\ (F; E)$ with the uniform topology of operators.

Theorem 2.1. Let A be an infinitesimal generator satisfying conditions (2.1.*a*), (2.1.*b*) *of* [4], § 2. *Assume A is not of admissible meromorphic type. Then the linear control system* (2.1) *is rigid for all operators $B \in \mathcal{L}(F; E)$ except for those in a subset of the first category of $\mathcal{L}(F; E)$*
and

Auxiliary Theorem 2.2 Let A satisfy the same conditions of Theorem 2.1. Assume there exists a subset L of the second category of E^m such that for every $(u_1, \ldots, u_m) \in L$ there exist m functions $f_1, \ldots, f_m$ in H^2, not all zero and such that $\Sigma_{k=1}^{m} f_k(A) u_k = 0$. Then A is of admissible meromorphic type.

For the proof, we shall make use of

Lemma 2.3. Let $\{ f_n \}$ be a sequence in H^2 of the half-plane $Re\, \lambda \leqq 0$ such that $| f_n |_{H^2} \leqq 1$. Then there exists a subsequence $\{ f_m \}$ such that:

(a) *$\{ f_m \}$ converges weakly to a function $f \in H^2$, $| f |_{H^2} \leqq$.*v

(b) *$f_m(A)$ converges to $f(A)$ in the uniform topology of operators.*

Proof: The fact that there exists a subsequence $\{ f_m \}$ satisfying (*a*) is an elementary fact of the theory of H^2 (in fact, Hilbert) spaces. To show (*b*), let us consider the representation (2.11) of [4]

$$f_m(A) = \frac{1}{2\pi i} \int_{P(c,\theta)} f_m(\lambda)\, R(\lambda; A)\, d\lambda \tag{2.5}$$

where $P(c, \theta)$ is the contour $c + |y| \operatorname{cotg} \theta + iy$, $-\infty < y < \infty$ for suitable $c < 0$, $\theta > \pi/2$ (see [4], § 2). Cauchy's formula

$$f_m(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f_m(it)}{it - \lambda}\, dt \tag{2.6}$$

and the weak convergence of $\{ f_m \}$ imply that $\{ f_m \}$ converges uniformly on compacts of $Re\, \lambda < 0$ to f. It is then clear that (*b*) will hold for $\{ f_m \}$ if we can show

$$\lim_{n \to \infty} \left| \int_{P(c,n,\theta)} f_m(\lambda)\, R(\lambda; A)\, d\lambda \right| = 0 \tag{2.7}$$

uniformly with respect to m, where $P(c, n, \theta)$ is the intersection of $P(c, \theta)$ with the region $|\lambda| \geqq n$. But in view of (2.1.b) of [4]

$$|R(\lambda; A)| \leqq C/|\lambda|$$

for $\lambda \in P(c, \theta)$ and some constant C. Furthermore, (2.6) and the Cauchy-Schwarz inequality imply

$$\| f_m(\lambda) | \leqq C/|\lambda|^{1/2} \tag{2.9}$$

for $\lambda \in P(c, \theta)$ and some constant C, uniformly with respect to m. (2.8) together with (2.9) imply 2.7) and, a fortiori, Lemma 2.3

Proof of Theorem 2.2. Define subsets $L_{M,N}$ $(M = 1, 2, 3, \ldots$ $N = 1, 2, \ldots m)$ of L as follows: $L_{M,N} =$ $(u_1, \ldots, u_m) \in L$ such that there exist functions $f_1, f_2, \ldots, f_m$ in H^2, not all zero and such that (a) max $_k |f_k|_{H^2} \leqq 1$ (b) $|f_N(c)| \geqq 1/M$, c a fixed point outside $\sigma(A)$, (c) $\Sigma_{k=1}^{m} f_k(A) u_k = 0$. It is easy to see that every $(u_1, \ldots, u_m) \in L$ belongs to some $L_{M,N}$ (if the corresponding functions $f_1, \ldots, f_m$ all vanish at c multiply them by a convenient power of $(\lambda - c)^{-1}$) and that each $L_{M,N}$ is closed (to do this we proceed in a way similar to that of Theorem 1.2 and make use of Lemma 2.3) Again by an application of Baire's theorem we deduce that some $L_{M,N}$ has an interior point, and this can be easily seen to imply (possibly after a rearrangement of indices) that the functions $f_1, \ldots, f_m$ in the statement of Theorem 2.2 can be chosen in such a way that $f_m(c) \neq 0$.

The proof ends now like that of Theorem 1.2. Let $(u_1, \ldots u_{m-1})$ be any $(m-1)$ — ple of elements of E, and let $g(\lambda) = (\lambda - c)$ $(\lambda + c)^{-2} \in H^2$. Then, if $f_1, \ldots, f_m$ are the functions corresponding to the m-ple

$$(u_1, \ldots, u_{m-2}, g(A) u_{m-1}, u_{m-1})$$

we have $\Sigma_{k=1}^{m-2} f_k(A) u_k + (f_{m-1}(A)\, g(A) + f_m(A)\, u_{m-1} = 0$, the functions $f_1, \ldots, f_{m-2}, f_{m-1} g + f_m$ not all zero. This allows us to reduce the case of m-ples to the case of $(m-1)$ ples, and when $m = 1$ Theorem 2.2 reduces to Theorem 2.2 of [4].

Remark Theorem 1.2 states that when A is not of admissible meromorphic type and F is finite-dimensional then (2.1) is rigid for "most" operators B in $\mathcal{L}$ $(F; E)$. The situation changes when

F is of infinite dimension; for instance, if $E = F$ it is not difficult to see that (2.1) is not rigid when B has a bounded inverse or is not one-to-one.

BIBLIOGRAPHY

[1] A. E. TAYLOR, *Introduction to Functional Analysis*, Wiley, New York, 1958.

[2] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators*, vol. 1, Interscience, New York, 1957.

[3] K. HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall Inc., Englewood Cliffs, 1962.

[4] H. O. FATTORINI, *Control in finite time of differential equations in Banach space*, Comm. Pure Appl. Math. XIX (1966) 17 - 34.

[5] I. KAPLANSKY, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor, 1954.