NOTES ON THE MEASURE EXTENSION PROBLEM,

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1. INTRODUCCIÓN. The so called measure problem has been proposed by Lebesgue in 1904 ([*L*]) and solved by Vitali ([*V*]) the next year, and asks for a traslation invariant finite measure in $\mathcal{P}$ ([0,1)), (see next paragraph for the nomenclature).

Vitali's theorem asserts that the only solution is $m = 0$, (a proof of it can be seen in [*H*], p. 70). Observing that the conditions on which is enunciated the problem imply that every point must have measure zero, Banach and Kuratowski proposed *the generalized measure problem*: define in $\mathcal{P}$ ([0,1)) a real valued, signed measure, null on every point. The same authors proved ([*BK*]) the following: under the continuum hypothesis the only solution is $m = 0$.

The problem was generalized even more by Ulam who proved: *if X is a set whose power is weakly accesible, then, on* $\mathcal{P}$ (X), *may be defined only one real valued, signed measure, vanishing at each point*: $m = 0$, (cf. [*U*] and [*B*]).

Something more can be said if we impose more restrictive conditions on *m*, and precisely: if X is a set whose power is strongly accesible, it cannot be defined on $\mathcal{P}$ (X) a$0 - 1$ valued measure, vanishing at every point and non trivial, (Ulam-Tarski, [*U*]).

The problem can be more generally posed as follows, (cf. [*B*1] and [*LM*]): *let* $(X, \mathcal{B}, P)$ *be a probability space and* $\mathcal{A}$ *a* σ*-algebra of* $\mathcal{P}$ (X) *containing* $\mathcal{B}$. *Does there exist a measure (hence, a probability)* $\bar{P}$ *on* $\mathcal{A}$ *such that* $\bar{P} = P$ *on* $\mathcal{B}$*?*

(No generatlity is lost considering probabilities instead of finite signed measures as it follows immediately from Jordan-Hahn decomposition theorem for signed measures.). When $\mathcal{B} = \{\phi, X\}$ and $\mathcal{A} = \mathcal{P}$ (X) we are in the Ulam's case and when besides $X = [0,1)$, in the case considered by Banach and Kuratowski. Therefore, the problem has in general no non-trivial solution. However, if $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$, where $\mathcal{C}$ is a finite partition of X, there are infinite solutions (Los-Marcewski), and the same holds when $\mathcal{C}$ is denumerable partition (Bierlein).

This paper is a set of notes on this problem and gives also a general view of if.

2. Nomenclature. By a measure we mean a σ-additive, non negative set function P defined on a σ-algebra of subsets of a set X and σ-finite. Whenever $P(X) = 1$, it will be called a probability.

A measure algebra is said to be purely atomic if the Boolean algebra of its sets mod. null sets (its Boolean algebra associated) is generated by a denumerable family of atoms. A measure will be called purely atomic (atomic) if its measure algebra associated is purely atomic (has atoms), and it will be called discret if all its mass is concentrated on a finite or countably infinite set of points. Finally, a measurable set S will be said indecomposable if and only if for every measurable $T \subseteq S$, $T = \phi$ or $T = S$.

If $\mathcal{B}$ and $\mathcal{C}$ are algebras of subsets of X, $\mathcal{B} \vee \mathcal{C}$ indicates the σ-algebra generated by them.

I will designate the half-closed unit interval $[0,1)$ and $\mathcal{P}(X)$ the family of subsets of the set X. βX will mean the Stone-Čech compactification of X, if X has a completely regular topology.

Aph$_\alpha$ will indicate the α-the infinite cardinal. We shall not enter into the definitions of weakly and strongly accesible cardinal numbers; it will suffice to us to observe shat Aph$_1$ is weakly accesible and every Aph$_\alpha \leqq c =$ the continuum power, is strongly accesible.

3. Auxiliary results. a) Assume $(\Omega_i, \mathcal{A}_i)$, $i = 1, 2$, are measurable spaces and $T : \Omega_1 \rightarrow \Omega_2$, is a measurable application such that $T^{-1}(\mathcal{A}_2) = \mathcal{A}_1$. Assume P_2 is a probability measure on Ω_2. Then, $P_1(T^{-1} B) = P_2(B)$, $B \in \mathcal{A}_2$, defines a probability P_1 on $\mathcal{A}_1$ if and only if $P_2^*(T(\Omega_1)) = 1$, (Doob).

The proof is straightforward.

b) Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{B}$ a σ-algebra, $\mathcal{B} \subseteq \mathcal{A}$ Suppose $(\Omega, \mathcal{B}, P)$ is not purely atomic. Let B be a subspace of $L^\infty(\Omega, \mathcal{A}, P)$ such that every function of $L_1(\Omega, \mathcal{B}, P)$ verifies:

$$\| f \|_1 = \operatorname*{supremum}_{\| b \|_\infty = 1,\, b \in B} \int f b \, dP \quad .$$

Then, there exists $b^* \epsilon B^*$ such that for any $f \epsilon L^1 (\Omega, \mathcal{A}, P)$, it is possible to find $b \epsilon B$ with :

$$b^* (b) \neq \int f\, b\, dP\,.$$

In other words, there exists a bounded linear functional in B^* not representable as a function of $L^1 (\Omega, \mathcal{A}, P)$,

The proof of this results is given in § 6, [*NP*], although the statement is slightly different.

c) If f_n is a non-negative submartingale sequence and $f \epsilon L^1$ closes it on the right, then f_n converges to f *a.e.* and in L^1. Besides, f_n is uniformly integrable if and only if f_n converges in probability to f which closes the submartingale on the right and $\int f_n\, dP \to \int f\, dP$. (Uniformly integrable means $\int f_n\, dP \to 0 \;\{ f_n \geqq a \}$ uniformly in n, when $a \to \infty$). (cf [*Le*] p. 394 and p. 528).

4. The measure problem for finitely additive measures. We want to exhibit now the several possibilities that appear combining algebras, σ-algebras, finitely additive an σ-additive measures.

Problem. *Let X be a set and Σ an algebra or σ-algebra of sets of X. Let Φ be another subalgebra of $\mathcal{P}$ (X) and $\overline{\Sigma}$ the algebra generated by Σ and Φ. Let m be a finitely additive or σ-additive signed measure of bounded variation defined on Σ. Define a signed measure of bounded variation $\overline{m}$ on $\overline{\Sigma}$ such that on Σ, $m = \overline{m}$.*

The problem has many subcases which we designate with we designate with $(a, b/c, d)$ where a, b, c, d, stand for the numbers 0,1, and with the following convention: $a(b)$ represents Σ $(\overline{\Sigma})$ and will be equal to 0 if Σ $(\overline{\Sigma})$ is an algebra and 1 if it is a σ-algebra; $c(d)$ represents $m(\overline{m})$ with value 1 or 0 depending whether or not m is σ-additive. The decomposition theorems associated to the names of Hahn and Jordan (cf. [*D S*] pp. 98 and 129; [*Le*], pp. 86-87) assert that for any values of a and b, it is sufficient to solve the problem for non-negative measures. Hence, we suppose that $\overline{m}$ and m are $\geqslant 0$, and besides that $m(X) = 1$.

Oustanding results in this situation are the following theorems.

Banach-Hausdorff theorem. Let R^n be the euclidean space of dimension n. For $n = 1(2)$, they can be defined on $\mathcal{P}$ (R^n) two finitely additive positive measures μ, $\overline{\nu}$, traslation (and rotation) invariant, taking the same values on intervals and such that μ is an extension of Lebesgue measure and $\overline{\nu}$ is not, ([*Bch*]); for

$n > 2$ the only finitely additive measure on $\mathcal{P}$ (R^n), vanishing at each point and rotation invariant, is the trivial one, ([*Hf*], p. 469).

ALEXANDROFF THEOREM. Let X be a compact space and m a finitely additive and regular measure defined on an algebra Σ, then m is σ-additive, and therefore admits a unique extension to the σ-algebra generated by Σ, (cf [*DS*], p. 138).

For a generalization of this result cf. *T.* 3 *B,* [*B* 1]. With the problem of determining in what case a finitely additive measure in a Boolean algebra is σ-additive deal Kelley's results (cf. [*K*] and [*Lu*], § 6). The same type of result is of great importance in the theory of measures in topological vector spaces. Cf. for example [*G V*], chapter IV.

CARATHEDORY THEOREM. Every σ-additive measure on an algebra admits a unique extension to the σ-algebra generated by Σ.

These theorems are examples of the cases (00/00), (00/01), and (01/11), respectively. Example for the case (11/11) is the result of Los and Marcewski already mentioned in the introduction.

Suppose that $d = 1$, i.e. $\overline{m} \in \sigma$-additive. Then, a case $(a\,0/c1)$ admits a solution whenever it is already a case type $(a1/c1)$, and in this last case it admits a solution if there is also a solution after replacing Σ by its generated σ-algebra (as one can see applying Caratheodory's extension theorem). That is, a case $(a1/c1)$ is always reduced to a case $(a\,1/11)$. Another application af Caratheodory's theorem shows that a case $(a\,1/11)$ can be reduced to a case (11/11).

Therefore, any case $(a\,0/c1)$ can be reduced to a case (11/11), and every case of this last type is also type $(a0/c1)$. Concluding, any measure problem with $d = 1$ is finally reduced to two problems. First, to determine that P on Σ is σ-additive as in Alexandroff, theorem, second, solve a problem of type (11/11). In this paper we shall be essentially concerned with the case (11/11).

The case $d = 0$ cannot be discused as before and it is vinculated with a set of astonishing results, cf. for example [*B T*], [*v N*], [*B c h*], [*H f*], and Hadwiger's book, [*H r*]. To show the difference of both cases it is enough to compare the theorem of Ulam and Tarski with the following result due also to Tarski: for any inifite set X there exists a non-discret, finitely additive, 0-1 valued measure, defined on $\mathcal{P}$ (X). Let us prove it.

THEOREM 1. *Le* $\mathcal{B}$ *be an algebra of subsets of* X *and* P *a finitely additive measure on* $\mathcal{B}$. *There exists a finitely additive measure* $\overline{P}$ *on* $\mathcal{P}(X)$ *such that* $\overline{P} = P$ *on* $\mathcal{B}$. *If* P *is* $0-1$ *valued,* $\overline{P}$ *can be chosen* $0-1$ *valued.*

Proof. There is a lattice isomorphism between the family of (real valued) bounded functions on X and the space of continuous functions in $C(\beta X)$. Under this isomorphism the space of bounded $\mathcal{B}$ measurable functions is in correspondence with a subspace S of $C(\beta X)$. P induces a bounded linear functional on $L^{\infty}(X, \mathcal{B}, P)$ and therefore on S, which can be extended to $C(\beta X)$. By the Riesz representation theorem this extension can be represented by a regular Borel measure μ. The restriction of μ to the clopen sets of βX is a finitely additive measure. By a result of Čech two sets of X are disjoint if and only if the clopens which are their closures are disjoint. Therefore, the restriction μ_0 of μ to the clopen sets can be also understood as defined on $\mathcal{P}(X)$, and defining there a finitely additive measure $\overline{\mathcal{P}}$. From the construction it follows that $\overline{P} = P$ on $\mathcal{B}$. Let K be the support of μ on βX. By definition of support, every clopen set intersecting K has μ-positive measure. If P is 0-1 valued, every clopen which is the closure of a set of $\mathcal{B}$, must contain K or be disjoint to it. Let ν be the δ-measure corresponding to a point of K. ν is also an extension of the linear functional associated to P and its restriction to $\mathcal{P}(X)$ is 0-1 valued.

Suppose B is an indecomposable infinite positive set of $\mathcal{B}$. Its closure on βX contains a point y in $\beta X - X$. Take a δ-measure concentrated on y and of magnitude $P(B)$ and proceed as above for P restricted to $X - B$. The union of these two partial extensions is an extension of P, $\overline{P}$, whose restriction is null on every point of B. (Naturally, it is not σ-additive). *Q.E.D.*

5. EXTREME CASE. We consider in this section the case with $\mathcal{B} = \{X, \phi\}$ *and* $\mathcal{A} = \mathcal{P}(X)$. We note with Ω the first infinite non-countable ordinal.

THOREM 2. *Assume* $|X| = Aph_1$.

I) *There exists a family of subsets of* $X, \{A^i_k\}, i, k = 1, 2, \ldots$, *such that* $A^i_k \wedge A^i_j = \phi$ if $k \neq j$, whatever be i; $\sum_{k=1}^{\infty} A^i_k = X$ and $\left| \bigcap_{i=1}^{\infty} (A^i_1 + \ldots + A^i_{k_i}) \right| \leq Aph_0$ *whatever be the sequence* $k_1, k_2, \ldots$, (Banach-Kuratowski).

II) *There exists a family* $\mathcal{F}$ *of sequences of positive integers with* $|\mathcal{F}| \geqq Aph_1$ *and such that for any sequence of positive integers* $S = (S_1, s_2, \ldots)$, *it holds*:

$|\{(t_1, t_2, \ldots) \in \mathcal{F};\ t_i \leqq s_i,$ *for every* $i\}| \leqq Aph_0$, (Banach-Kuratowski).

III) *There exists a sequence of functions* $\{f_n\}$, *defined on* X, *such that* $0 \leqq f_n \leqq 1$, $f_n(x)$ *converges to* 0 *for each* $x \in X$, *and if* $Y \subseteq X$ *is a set where* f_n *converges uniformly then* $|Y| \leqq Aph_0$, (Sierpinski).

1) *It does not exist a non-discret probability measure defined on* $\mathcal{P}(X)$, (Ulam).

2) *There exists a denumerable family of subsets of* X, $\{A_n\}$ *such that the generated* σ*-algebra* $\mathcal{B}(\{A_n\})$ *contains the one-point sets and cannot be the domain of definition of a non-discret probability measure,* (Bierlein).

3) *It is possible to define on* X *a real valued function* f *such that* $\mathcal{B}(f) =$ *the least* σ*-algebra on which* f *is measurable, contains the one-point sets and is not domain of definition of a non-discret probability measure.*

4) *Let* P *be a probability on* $\mathcal{B} \subset \mathcal{P}(X)$. *If* P *is extendable to* $\mathcal{P}(X)$ *there exists a subspace* B *of* $L^\infty(X, \mathcal{P}(X), P)$ *determiming for* $L^1(X, \mathcal{B}, P)$ (*) *such that dual* B^* *of* B *is equal to* $L^1(X, \mathcal{B}, P)$.

5) *For any* σ*-algebra* $\mathcal{B} \subset \mathcal{P}(X)$, *any measure on* $\mathcal{B}$ *extendable to* $\mathcal{P}(X)$ *is purely atomic.*

Then, the propositions I), II), and III), are equavalent and they imply 1), 2), 3) and 4), which are equivalent and true.

Asuming the continuum hypothesis, I) holds. The equivalence of I), II) and III) also holds for $|X| = c$.

Proof. I) $\leftrightarrow$ II) : cf. [*B K*], pp. 130-131. I) $\leftrightarrow$ III) : [*S*], p. 279. In the proofs no use is made of the magnitude of the power of X. I) holds assuming the continuum hypothesis: cf. [*B K*], p. 130. III) $\rightarrow$ 1) : suppose P is a non-discret probability on $\mathcal{P}(X)$. Without loss of generality we can assume that $P(\{x\}) = 0$ for every $x \in X$, and therefore, the countable sets will have measure zero. If $f_n \rightarrow 0$ pointwise, by Egoroff theorem, f_n converges uni-

(*) If means that $\|f\|_1 = \sup \left\{ \int f g \, dP \ ; \ \|g\|_\infty \leqq 1, \ g \in B \right\}$.

formly on a set of measure $1 - \epsilon$, and hence on a set of power Aph_1, contradicting III).

2) $\rightarrow$ 1) : trivial. 1) $\rightarrow$ 2) : cf. [*B* 1], p. 33.2) $\leftrightarrow$ 3) : cf. [*B*1], *Th.* 1*B*, p. 32, where the equivalence is proved using a useful result of Banach (see [*Mi*]). 1) is proved in [*U*]. Let us see that 4) is equivalent to 1) and 5).

1) $\rightarrow$ 4) : from the hypothesis, it follows that $(X, \mathcal{B}, P)$ is purely atomic, and therefore, there is a determining subspace B (in $L^\infty(X, \mathcal{B}, P)$) for $L^1(X, \mathcal{B}, P)$ such that $B^* = L^1$ ("the space of bounded sequences tending to zero when $n \rightarrow \infty$"). 4) $\rightarrow$ 5) $\rightarrow$ 1). Assume that P is a probabiliy measure on $\mathcal{B}$ and extendible to $\mathcal{P}(X)$ and suppose that it is not purely atomic. From § 3, *b*), it follows that every determining subspace for $L^1(X, \mathcal{B}, P)$ admits a linear functional which is not representable as a function of $L^1(\mathcal{B})$, contradicting 4). Hence $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ is purely atomic.

Extending it to a probability on $\mathcal{P}(X)$ and using the theorem of Ulam-Tarski mentioned at the introduction, it follows that P is a discret measure. (We can assume Ulam-Tarski theorem since its proof is independent of the proof of Ulam theorem).

6. General extreme case. It has been proved by Luxemburg (cf. [*Lu*], T. 4.5), that a complete Boolean algebra with the Egoroff property has at most a denumerable set of atoms, if it is assumed the continuum hypothesis. We shall not enter into the definition of Egoroff property.

It will suffice for us to say that every Boolean measure algebra has Egoroff property. A corollary of Luxemburg's result is that any complete Boolean measure algebra is isomorphic to the family of subsets of a denumerable set. However this result is trivial. The following theorem is in close connection with Luxemburg's result.

Theorem 3. *a) If* $(X, \mathcal{P}(X), P)$ *is a probability space and the continuum hypothesis holds(or better, c is weakly accesible), then* P *is purely atomic. b) If besides* X *is strongly accesible, then* P *is a discret measure.*

Proof. b) follows from a) using Ulam-Tarski theorem. Let us prove a). If P is not purely atomic, then we can assume, without loss of generality, that it is not atomic, *i.e.*, it has no atom. Therefore, it can be constructed a family of sets A_{rs}, r, s, rational numbers, $r < s$, which can be put in a one-to-one, measure and inclusion preserving, correspondence with the rational intervals of I, $\{[r, s]\}$. Let $\mathcal{D}$ be the family of sets D which are maximal with respect to the property of beeing contained or disjoint to every A_{rs}. Then, $|\mathcal{D}| \leq c$. Since P is defined on $\mathcal{P}(X)$, it is a fortiori defined on the family of subsets S having the property $D \in \mathcal{D}$ and $S \wedge D \neq \phi \rightarrow S \supseteq D$.

Hence, P induces a measure Q on $(Y = X/\mathcal{D}, \mathcal{P}(Y))$. From the construction it follows that there is a one-to-one correspondence τ between Y and a subset Z of real numbers, of Lebesgue exterior measure equal to one. Moreover, since Q is defined on $\mathcal{P}(Y)$, it means that the measure induced by the Lebesgue measure m on Z can be extended to $\mathcal{P}(Z)$. Now, defining $\overline{m}(A) = 0$ if A is a subset of $I - Z$ and $\overline{m}(A) = Q(\tau^{-1}(A))$ if $A \subseteq Z$, we get an extension to $\mathcal{P}(I)$ of the Lebesgue measure, (recall that $m^*(Z) = 1$). Then, a) follows from the assumption of weakly accessibility of c and the Ulam's theorem, Q.E.P.

From the proof it follows that the continuum hypothesis in theorem 3 could be omited, if the following problem had a negative solution.

PROBLEM 1. *Is it possible to find a probability measure which extends the Borel measure to* $\mathcal{P}(I)$?

And so, the problem for X, at least for non atomic measures, is reduced to the same problem for the unit interval. Observe that any sucho extension must be non-atomic and therefore taking into account Maharam theorem ($[M]$), it turns out that problem 1 is equivalent to the following;:

PROBLEM 2. *Is it possible to find a homogeneous probability which extends Lebesgue measure to* $\mathcal{P}(I)$?

We leave the details to the reader, (cf. Banach-Hausdorff theorem in § 4).

7. CRITERIONS FOR MEASURE EXTENSION. Let $(X, \mathcal{B}, P)$ be a probability space and $\mathcal{C}$ an algebra of subsets of X. Our purpose is now to discuss some methods to extend P to $\mathcal{C}$, in other words, to extend P to $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$.

In relation with the first and second method cf. [*B* 1], specially Ths. 3*A* and 2*B*.

We shall agree in this section that any σ-algebra $\mathcal{B}$ on which a measure is defined separates points, i.e., for any two points x, y, there exists $B \epsilon \mathcal{B}$ such that $x \epsilon \overline{B} \ni y$. This is not an essential restriction since except for atoms, the condition is satisfied by the completion of the measure and in a non--one-point atom several solutions are at hand. For example, if the atom has only a denumerable set of points we can add a non denumerable set of them problem and is of immediate application in the extreme cases.

7.1. CONSISTENCY CRITERION. $F(\mathcal{B})$ will design a family of $\mathcal{B}$-measurable functions such that the least σ-algebra on which the functions of $F(\mathcal{B})$ are measurable ir $\mathcal{B}$ itself. $\Pi(\mathcal{B})$ will design a product of real lines: $\Pi \{R_f;\ f \epsilon F(\mathcal{B})\}$. Let $\Phi(\mathcal{B})$ be the application related to $F(\mathcal{B})$ and $\Pi(\mathcal{B})$ defined by $\Phi(x) = (\ldots, f(x), \ldots) \in \Pi(\mathcal{B})$, $f \epsilon F(\mathcal{B})$. Then $(\Pi(\mathcal{B}), \mathcal{S}, P\Phi^{-1}(\mathcal{B})$ is a probability space, ($\mathcal{S}$ is the family of Borel stts).

THEOREM 4. *A necessary and sufficient condition for the existence of a probability* $\bar{P}$ *on* $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$ *with* $\bar{P}/\mathcal{B} = P$ *is the existence of a probability* μ *on* $(\Pi(\mathcal{B}) \times \Pi(\mathcal{C}), \mathcal{S})$ *such that*: 1) $\Phi(\mathcal{A})(X) = (\Phi(\mathcal{B}), \Phi(\mathcal{C}))(X)$ *is of exterior* μ*-measure one;* 2) *its projection on* $\Pi(\mathcal{B})$ *coincides with* $P\Phi^{-1}(\mathcal{B})$.

PROOF. The sufficiency follows from a), § 3. The necessity is trivial.

A necessary and sufficient condition for the existence of a probability μ on $\Pi(\mathcal{A})$ is the existence of consistent finite distributions on $\Pi^n_{i=1} R_{f_i} \times \Pi^m_{i=1} R_{g_i}$, $g_i \epsilon \mathcal{F}(\mathcal{C})$, $f_p \epsilon \mathcal{F}(\mathcal{B})$, (Kolmogoroff theorem). The projection of μ on $\Pi(\mathcal{B})$ will coincide with $P\Phi^{-1}(\mathcal{B})$ whenever the μ-probability of any set A defined on $\Pi^n_{i=1} R_{f_i}$, whatever be f_i and n, is equal to $P((f_1, \ldots, f_n)^{-1}(A))$.

COROLLARY. a) *Let* $(X, \mathcal{B}, P)$ *be a probability space and* A *a non-measurable subes tof* X. *There exists* (*infinitely many indeed*) *a probability* $\bar{P}$ *which extends* P *to* $\mathcal{B} \vee \{A\}$, (Los-Marcewski).

b) *Let* $\{A_\alpha\}$ *be a family of disjoint subsets of* X *such that the complement of a denumerable family* $\{A_n\}_{n=1}^{\infty}$, *is contained in a set* M *of interior measure zero. Then, there exist a measure* $\bar{P}$ *on* $\mathcal{B} \vee \{A_\alpha\}$ *such thet* $\bar{P}$. *extendeds* P, (*Bierlein*).

PROOF. Set V the sample space $(\Pi(\mathcal{B}), \mathcal{S}, P\Phi^{-1}(\mathcal{B})$ and $W = V \times R$, where R denotes the real line. Let us define μ and apply next the preceding theorem. Call $K(K')$ the measurable hull of A $(X - A)$. If $B \epsilon \mathcal{B}$, and $B \subseteq X - K$, put $\mu(B \times X\ \{0\}) =$

$= P(B)$, if $B \subseteq X - K'$, $\mu(B \times \{1\}) = P(B)$ and if $B \subseteq K \wedge K'$ define $\mu(B \times \{1\} = a P(B)$, $\mu(\times \{0\}) = (1-a)P(B)$, where a will be fixed at once. Condition 2) of theorem 4 is then fulfilled. Set now $\Phi(\mathcal{A}) = (\Phi(\mathcal{B}), \mathcal{X}_A)$, i.e., $\Phi(\mathcal{A})(x) = \Phi(\mathcal{B})(x) \times \{0\}$ for $x \epsilon A$, and $\Phi(\mathcal{A})(\mathcal{X}) = \Phi(\mathcal{B})(x) \times \{1\}$ for $x \epsilon A$.

Choosing now a verifying:
$P(K) = P^*(A) \geq \mu(K \times \{1\}) = a \geq P_*(A) = P(X - K')$, (**)
it follows condition 1) of Th. 4. Observe that (**) is a neceassary condition. The same procedure applies for corollary b). We shall restrict curselves to the definition of μ. One way of doing it is so. Set $\Phi(\mathcal{A})(x) = \Phi(\quad)(x) \times \{n\}$ if $x \epsilon A_n$, $n = 0,1,\ldots$, with $A_0 = M$. Let K_n be the measurable hull of A_n. If $B \epsilon \mathcal{B}$ and $B \subseteq K_n - \bigcup_1^{n-1} K_j$, $n = 2, 3, \ldots$, define $\mu(B \times \{n\}) = P(B)$, if $B \subseteq K_1$, $\mu(B \times \{1\}) = P(B)$.

7.2. Conditional Expectation Method. Suppose that $\mathcal{B} \subseteq \mathcal{A}$ and $(X, \mathcal{A}, P)$ is a probability space. Then, the conditional expectation operator $E(./\mathcal{B})$ has the following properties: 1) $E(./\mathcal{B}) : L^\infty(\mathcal{A}) \to L^\infty(\mathcal{B})$ is a contraction operator and preserves L^1-norms; 2) $\sum_{j=1}^{\infty} A_j = A_0$ implies $E(\Sigma A_j / \mathcal{B}) = \Sigma E(A_j / \mathcal{B}) = E(A_0/\mathcal{B})$; 3) $E(O/\mathcal{B}) = 0$. $E(1/\mathcal{B}) = 1$, *a.e.*.

Trying to use the conditional expectation operator to extend a measure, one gets:

Theorem 1. *Let $\mathcal{F}$ be a family of $\mathcal{B}$-measurable functions such that:*

a) $f \epsilon \mathcal{F} \to 1, \geq f \geq 0$ *a.e.*, b) *there exits an application* φ *from* $\mathcal{A} \supseteq \mathcal{B}$ *into* $\mathcal{F}$ *such that* $\varphi(\Omega) = 1$, $\varphi(\phi) = 0$ *and* $\varphi(\sum_1^\infty A_j) = \sum_1^\infty (A_j)$, c) $A \epsilon \mathcal{B} \to \varphi(A) = \chi(A)$, *a.e.. If there is a probability P on $\mathcal{B}$, then $\overline{P}(A) = \int \varphi(A)\, dP$ defines a probability P on $\mathcal{A}$ with $\overline{P}/\mathcal{B} = P$.*

The proof is trivial. From this theorem easily follows the corollary of section 7.1, (cf. [B1]).

7.3. Martingale criterion. Assume that $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$, $n = 1, 2, \ldots$, is a denumerable set of non-decreasing σ-subalgebras of $P(X)$ P_n, Q_n, are probabilities measures defined on $\mathcal{B}_n$. Suppose the P's admit a common extension $\bar{P}$ to $\mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{B}_n$. The problem is to establish conditions on the Q's so that they also admit a common extension to $\mathcal{A}$. It holds:

THEOREM 6. *Under the hypothesis mantioned above, if* $Q_{n+1}/\mathcal{B}_n = Q_n$ *and* $Q_n << P_n$, $n = 1, 2, 3, \ldots$, *then, the following propositions are equivalent.*

1) *there exist* $\bar{Q} << \bar{P}$ *defined on* $\mathcal{A}$ *such that* $\bar{Q}/\mathcal{B}_n = Q_n$ *for every* n.

2) $\{ f_n = dQ_n/dP_n \}$ *is martingale closed on the right with o closure function in* $L^1 (X, \mathcal{A}, P)$.

3) $\{ f_n \}$ *is fundamental in* L^1.

4) $\{ f_n \}$ *is uniformly integrale.*

In these cases, $f_n = dQ_n/dP_n$ *converges a.s. and in* $L^1 (X, \mathcal{A}, \bar{P})$ *to* $d\bar{Q}/d\bar{P}$.

PROOF. If $\{ f_n \}$ is L^1 -fundamental then it is uniformly integrable (cf. [*Le*], p. 163). Observe now that the conditional expectation $E(f_{n+1}/\mathcal{B}_n)$ is a.e. equal to f_n, and therefore that $\{ f_n \}$ is a martingale sequence. Using c), § 3, $f_n \xrightarrow[\bar{P}]{} f$ and f closes the martingale on the right and belongs to $L_1 (\bar{P})$. If $\{ f_n \}$ is a martingale closed on the right by an $L_1 (P)$ -function, then again by c), § 3, $f_n \xrightarrow{1} f$. Hence, 2), 3) and 4) are equivalent. Defining $\bar{Q}(A) = \int_A f \, d\bar{P}$, they, trivially, imply 1). Assume that 1) holds. Then, from f_n is closed on the right by $d\bar{Q}/d\bar{P} \in L^1 (\bar{P})$, *Q.E.D.*

Similar results can be obtained if one asks instead of $Q_{n+1}/\mathcal{B}_n = Q_n$ that $Q_{n+1} (B) \geq Q_n (B)$ for every $B \in \mathcal{B}$. In this case, $dQ_n = f_n$ is a submartingale sequence.

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