REMARKS ON COMPACT GROUPS WITH ORDERED DUALS

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INTRODUCTION

Here we will study some of the results and problems which were discusseed by Helson in [3]. Our aims are two-fold.

In the first part we extract from Helson's work a proof that H^∞ $(d\sigma)$ is a weak-star maximal subalgebra of L^∞ $(d\sigma)$. This result belongs to a circle of theorems which are of a simpler nature than the classification of invariant subspaces in terms of cocycles. The theorems in the chain leading to the maximality theorem can be obtained, in turn, as simple consequences of the maximality theorem. It would be useful to construct a proof of the maximality theorem analogous to the Hoffman-Singer proof for the disc, and obtain these other theorems as corollaries.

None of the ideas in this first part are new — they are all due to Helson or to Helson and Lowdenslager in conjunction. We are interested here in the organization of these results, and in providing the setting for the discussion of cocycles.

In the second part we discuss the classification of invariant subspaces via cocycles. Here the main result is a canonical expression for a cocycle, modulo coboundaries, which is contained in theorem 6. This is used to settle affirmatively a question of Helson as to whether there exist real cocycles which are not coboundaries. We also identify the cocycle corresponding to the invariant subspace generated by an arbitrary element $f \in L^a$ $(d\sigma)$.

2. *Weak-star maximality*

Let A be a uniform algebra on X, and let ϕ be a non-zero complex-valued homomorphism of $A . A_0$ will be the kernel of ϕ. A

Szegö measure for ϕ is a representing measure dm on X for ϕ such that Szegö's theorem is valid: For any function $0 \leqslant w \in L^1(dm)$,

$$\inf_{f \ni A_0} \int |1-f|^2 w \, d m = \exp \{ \int \log w \, dm \} .$$

dm is a Szegö measure for ϕ if and only if dm has the following property: If $d\mu$ is a representing measure for ϕ which is absolutely continuous with respect to dm, then $d\mu = dm$.

For $0 < p < \infty$, let $H^p(dm)$ be the closure of A in $L^p(dm)$. $H^\infty(dm)$ is the weak-star closure of A in $L^\infty(dm)$. $H^\infty(dm)$ can be considered as an algebra of functions on the spectrum of $L^\infty(dm)$. dm is a Szegö measure if and only if $H^\infty(dm)$ is a logmodular algebra. In particular, if dm is a Szegö measure, the H^p-theory developed in [7] applies.

Also, everey Szegö measure is a Jensen measure, *i.e.*,

$$\log \left| \int f dm \right| \leqslant \int \log |f| \, dm \tag{1}$$

for all $f \in A$. A standard approximation argument shows that (1) continues to hold for functions $f \in H^1(dm)$.

Theorem 1: Suppose that dm is a Szegö measure for ϕ. Then the following conditions are equivalent:

(i) $H^\infty(dm)$ is a maximal weak-star closed subalgebra of $L^\infty(dm)$.

(ii) Whenever $h \in L^\infty(dm)$ and $f \not\equiv 0$ are such that $h^n f \in H^1(dm)$ for all integers $n \geqslant 0$, then $h \in H^\infty(dm)$.

Proof: Actually, the only fact we will need about Szegö measures is that if $g \in L^1(dm)$ is such that $\int ghdm = 0$ for all $h \in A$, then $g \in L^1 A dm)$.

Suppose that $H^\infty(dm)$ satisfies (i), and let h and f be as in (ii). Let B be the subalgebra of $L^\infty(dm)$ generated by $H^\infty(dm)$ and h. Since $\int kfgdm = \int kfdm \int gdm = 0$ for all $k \in B$ and $g \in A_0$, B is not weak-star dense in $L^\infty(dm)$. Hence $B = H^\infty(dm)$, and $h \in H^\infty(dm)$.

Now suppose that the condition (ii) is valid. Let B be a subalgebra of $L^\infty(dm)$ which contains $H^\infty(dm)$ but is not weak-star dense in $L^\infty(dm)$. Choose $0 \not\equiv f \in L^1(dm)$ such that $\int fkdm = 0$ for all $k \in B$. If $h \in B$ and $n \geqslant 0$ is an integer, then $\int fh^n kdm = 0$, $k \in B$. In particular, $h^n f \in H^1(dm)$ for all $n \geqslant 0$. Applying (ii), $h \in H^\infty(dm)$ and $B = H^\infty(dm)$. Q.E.D.

A closed subspace M of $L^2(dm)$ *is invariant* if $AM \subseteq M$. If E is a measurable subset of X, and χ_E is the characteristic function of E, then $\chi_E L^2(dm)$ is evidently an invariant subspace of $L^2(dm)$.

Theorem 2: Suppose $H^\infty(dm)$ is a maximal weak-star closed subalgebra of $L^\infty(dm)$. Suppose M is an invariant subspace of $L^2(dm)$ which is not of the form $\chi_E L^2(dm)$. If $h \in L^\infty(dm)$ and $hM \subseteq M$, then $h \in H^\infty(dm)$.

Proof: The family B of functions $h \in L^\infty(dm)$ such that $hM \subseteq M$ forms a weak-star closed subalgebra of $L^\infty(dm)$ containing $H^\infty(dm)$. If $B = L^\infty(dm)$, evidently M is of the form $\chi_E L^2(dm)$. In the other case, $B = H^\infty(dm)$, and $h \in H^\infty(dm)$. Q.E. D.

3. *Almost periodic functions*

Let Γ be a subgroup of R_d, the real numbers with the discrete topology, such that Γ is dense in the real numbers with the ordinary topology. Let G be the compact character group of Γ. Let χ_λ be the character of G determined by $\lambda \in \Gamma$, and let A be the closed subalgebra of $C(G)$ generated by the set of characters $\{\chi_\lambda : \lambda \in \Gamma, \lambda \geqslant 0\}$.

The spectrum $\Sigma(A)$ of A is topologically the product $G \times [0,1]$, with the slice $G \times \{0\}$ identified to a point. For $0 < r \leqslant 1$, the homomorphism corresponding to (x, r) is given by $\phi_{rx}(\chi_\lambda) = r^\lambda \chi_\lambda(x)$. The homomorphism corresponding to $r = 0$ is the Haar homomorphism $\phi_0(f) = \int f \, d\sigma$.

Each real number s determines a character $e_s \in G$ defined by $e_s(\lambda) = e^{i\lambda s} = \chi_\lambda(e_s)$. The correspondence $s \rightarrow e_s$ embeds the real line R isomorphically as a dense subgroup L of G.

Considered as functions on the line L, each of the functions $f \in A$ can be extanded to be analytic and bounded in the upper half-plane, and

$$f(s + it) = \frac{t}{\pi} \int_{-\infty}^{\infty} f(s + u) \frac{du}{t^2 + u^2} = (\mu_r * f)(s) = \phi_{re_s}(f),$$

where $d\mu_r = \dfrac{t\,du}{\pi(t^2 + u^2)}$ and $t = -\log r$.

Trough each point $x \in G$ passes the coset $x + L$, and the functions $f \in A$ are analytic on the half-plane "above" $x + L$. The representing measure for ϕ_{rx} is supported on the line $x + L$, and

$$\phi_{rx}(f) = \frac{t}{\pi}\int_{-\infty}^{\infty} f(x + e_u)\frac{du}{t^2 + u^2} = (\mu_r * f)(x)$$

If f_r is the restriction of f to the slice corresponding to r, then

$$f_r = \mu_r * f. \tag{2}$$

Since $A + \bar{A}$ is dense in $C(G)$, every $\phi \in \Sigma(A)$ has a unique representing measure on G. So each of the measures μ_{rx} and σ is in particular a Szegö measure and a Jensen measure. From (2) and Jensen's inequality, we obtain

$$\log |f_r| \leqslant \mu_r * \log |f|, \ f \in A. \tag{3}$$

Multiplying Γ by a constant, if necessary, we can and will assume that $2\pi \in \Gamma$. Set $K = \{ y \in G : y(2\pi) = 1 \}$. K is a compact subgroup of G. Since $e_s(2\pi) = e^{2\pi is}$, e_s belongs to K if and only if s is an integer.

Consider the map T from the locally compact abelian group $K \times R$ to G, defined by $T(y, s) = y + e_s$. T is a homomorphism from $K \times R$ onto G, and the kernel of T is the discrete subgroup $\{ (e_{-n}, n) : n = 0, 1, 2, \dots \}$. T is a covering map with fundamental domain $D = \{ (y, s) : y \in K, \ 0 \leqslant s < 1 \}$. K is obtained from $\bar{D}$ by identifying $(y, 1)$ with $(y + e_1, 0)$, $y \in K$. If $d\tau$ is the Haar measure for K, then the Haar measure $d\sigma$ of G, regarded as a measure on D, is the product measure $d\sigma = d\tau \times ds$.

On $K \times R$ we define the finite measure $d\nu = d\tau \times \dfrac{ds}{1 + s^2}$.
We say that a measurable function f on $K \times R$ is *automorphic* if $f(y + e_1, s - 1) = f(y, s) a.e. d\nu$. f is then determined almost everywhere by its values on D. Conversely, every measurable function on D determines an automorphic measurable function on $K \times R$.

If $f \in L^p(d\nu)$, then its restriction to D belongs to $L^p(d\sigma)$. Conversely, if $f \in L^p(d\sigma)$, then its automorphic extension is in $L^p(d\nu)$. The norms of $L^p(d\sigma)$ and $L^p(d\nu)$ are equivalent on the subspace of automorphic functions, so we can regard $L^p(d\sigma)$ as a subspace of $L^p(d\nu)$.

In [4], p. 186, it is shown that if g nom the characteristic function of a set of positive Haar measure on G, then $\mu_r * g > 0$ $a.e.d\,\sigma$. This is equivalent to the following, which is fundamental in our development.

Basic lemma: Let f be a measurable automorphic function on $K \times R$ such that for almost all $y \epsilon K$, $f(y, .)$ is constant $a.e.ds$. Then f is constant $a.e.d\,\tau \times ds$.

4. *H^p - spaces*

We denote by $H^p(x+L)$ the H^p - spaces associated with the representing measure μ_{rx}. $H^p(x+L)$ does not depend on $r \epsilon (0,1)$. It depends only on the line $x+L$. Each of the functions in H^p $(x+L)$ has analytic extensions to the upper half-plane above $x+L$ satisfying certain growth conditions $H^p(x+L)$ coincides with the transplant of the usual H^p-spaces of the unit disc under the conformal map of the disc onto the upper half-plane. In particular, $H^\infty(x+L)$ consists of hall bounded analytic functions in the upper half-plane. According to a theorem of Hoffman and Singer, $H^\infty(x+L)$ is a maximal weak-star closed subalgebra of $L^\infty(d\,\mu_{rx})$.

By "almost all lines" we mean lines of the form $x+L$ for σ- almost all $X \epsilon G$, or equivalently, lines which pass through K in a subset of τ - measure 1.

If $f_n \epsilon A$ converges to f in $L^p(d\,\sigma)$, then f_n converges to f in $L^p(x+L)$ for almost all limes, and $f \epsilon H^p(x+L)$ for almost all lines. In particular, if $f \epsilon H^p(d\,\sigma)$ we obtain, in view of Jensen's inequality (1) for the measure μ_{rx}, the following extension of (3):

$$\log |\, \mu_r * f \,| \leqslant \mu_r * \log |\, f \,| \text{ a.e., } f \epsilon H^1(d\,\sigma).$$

This is Malliavin's inequality.

The converse of the above statement is true, as proved by Helson in [3]. We will give another more direct proof, which is also due to Helson.

Theorem 3: Let $0 < p \leqslant \infty$, and let $f \epsilon L^p(d\,\sigma)$. Then $f \epsilon H^p(d\,\sigma)$ if and only if $f \epsilon H^p(x+L)$ for almost all lines.

Proof: We have already discussed the forward implication.

Helson's proof of the reverse implication operates in $L^2(d\,\sigma)$. Let $f \epsilon L^2(d\,\sigma)$ such that $f \epsilon H^p$ $(x+L)$ for almost all lines. Let g be the orthogonal projection of f on $H^2(d\,\sigma)$, g belongs to $H^2(x+L)$ for almost all lines. Since $f\text{-}g \perp H^2(d\,\sigma)$, it follows that $f\text{-}g \epsilon H^2$ $(d\,\sigma)$, and f-g also belongs to $H^2(+L)$ for almost all lines. Hence f-g is constant a.e., on almost all lines. By the basie lemma, f-g is a constant. Hence f belongs to $H^2(d\sigma)$.

Now let $f \epsilon L^p(d\sigma)$ be such that $f \epsilon H^2(x+L)$ for almost all lines. Fix $x \epsilon G$ such that $|f(x)| < 1/2$. If $|t| < 1/2$, $\log |f x) + t| < 0$.

Also $\int_{-1/2}^{1/2} \log |f(x)+t|\, dt > \int_{1-}^{1} \log |s|\, ds = -2$, and so $\int_{|f(x)|<1/2} \int_{|t|<1/2} \log |f(x)+t|\, dt\, d\sigma(x) > -2$.

Interchanging orders of integration, we see that for almost all t, $\log |f(x)+t|$ is σ-integrable over the set where $|f(x)| \leqslant 1/2$. It is always σ-integrable ower the set where $|f(x)| \geqslant 1/2$. Hence there always exists a real number t such that $\log |f+t|$ is σ-integrable. Replacing f by $f+t$, we can assume that $\log |f|$ is integrable.

Let g be the outer function in $H^\infty(d\sigma)$ whose modulus is $|g| = min\ (1, |f|)$. $gf \epsilon L^\infty(d\sigma)$, and gf belongs to $H^\infty(x+L)$ for almost all lines. By the L^2-theorem, $gf \epsilon H^\infty(d\sigma)$. Since g is the pointwise limit of a sequence of invertible functions $g_n \epsilon H^\infty(d\sigma)$ such that $|g_n| \geqslant |f|$, $f = \lim fg/g_n$ belongs to $H^p(d\sigma)$. *Q.E.D.*

Theorem 4: $H^\infty(d\sigma)$ is a maximal weak-star closed subalgebra of $L^\infty(d\sigma)$.

Proof: We will verify condition (ii) of theorem 1.

Suppose $h \epsilon L^\infty(d\sigma)$ and $0 \not\equiv f \epsilon H^1(d\sigma)$ satisfy $hf \epsilon H^1(d\sigma)$ for all positive integers n. Then $h^n f \epsilon H^1(x+L)$ for almost all lines. Since $H^\infty(x+L)$ is weak-star maximal, $h \epsilon H^\infty(x+L)$ for almost all lines. Hence $h \epsilon H^\infty(d\sigma)$. *Q. E. D.*

5. *Invariant subspaces*

Invariant subspaces are of essentially three kinds. We say that M is *doubly invariant* if $\chi_\lambda M \subseteq M$ for all $\lambda \epsilon \Gamma$. In this case, $M = \chi_E L^2(d\sigma)$ for some measurable set E. Otherwise, we will say that M is *singly invariant*. Let F be a measurable function which is of modulus 1 a. e. Then FH^2 and FH_0^2 are singly invariant subspaces. FH^2 are FH^2_0 are regarded as different versions of the same invariant subspaces. We shall call subspaces of this type *discontinuous*. Singly invariant subspaces which are not discontinuous we call *continuous*.

One way to construct singly invariant subspaces is as follows. We start with a function F on G such that, on every line, F is measurable and $|F| = 1$ a.e. Let M_F be the set of all $f \in L^2\ (d\sigma)$ such that on each line $x + L$, $Ff \in H^2\ (x + L)$. M_F is then an invariant subspace of $L^2\ (d\sigma)$. If F is σ - measurable, then $M_F = FH^2\ (d\sigma)$, in view of theorem 3.

It will be convenient to fix determinations of all functions of modulus 1 a.e. so that they are of modulus 1 everywhere. Measurability will always be in the sense of Borel.

Given a function of F of modulus 1, we assign to each $x \in G$ the function determined by F on the line through x, but normalized to assume the value 1 at x. This function is $B\ (x, t) = \overline{F\ (x)}\ F\ (x + e_t)$. B satisfies the following conditions:

(i) $|B\ (x, t)| = 1, x \in G, t \in R$

(ii) $B\ (x, 0) = 1, x \in G$

(iii) $B\ (x + e_s, t) = \overline{B\ (x, s)}\ B\ (x, s + t),\quad x \in G,\quad s, t \in R.$

A function B satisfying (i), (ii) and (iii) wil becalled a *cocycle.*

Every cocycle determines a function of modulus 1 on each line, providing we agree to identify two functions of modulus 1 when they are constant multiples of each other. If $B\ (x, s)$ is measurable as a function of s for each fixed x, we say that B is *linearly measurable*. Each linearly measurable cocycle B comes from a function F of modulus 1 which is measurable on each line. The invariant subspace it determines will be denoted by M_B.

We say that a cocycle B is *measurable* if the map $(x, t) \to B(x, t)$ of $G \times R$ into the circle is measurable. In particular, a measurable cocycle is linearly measurable.

Lemma: If B is a measurable cocycle, the functions $B\ (., t)$ move continuously with t in $L^2\ (d\sigma)$.

Proof: The distance between $B(\cdot, t)$ and $B\ (\cdot, t_0)$ in $L^2\ (d\sigma)$ is given, with the aid of Fubni's theorem, by.

$$\int_K \int_0^1 |B\ (y + e_s, t) - B\ (y + e_s, t_0)|^2\, d s\, d\tau\ (y).$$

Using (i) and (iii) this expression becomes

$$\int_K \int_0^1 |B(y, s + t) - B\ (y, s + t_0)|^2\, d s\, d\tau\ (y)$$

For fixed $y \in K$, the inner integral tends to zero as $t \to t_0$. By the bounded pointwise convergence theorem, the double integral tends to zero. *Q.E.D.*

This shows that measurable cocycles are cocycles in the sense of Helson. Hence every invariant subspace of $L^2\ (d\sigma)$ which is not doubly-invariant is M_B for some uniquely determined (up to sets of measure zero) measurable cocycle B.

The product of two cocycles is a cocycle that is measurable if the factors are measurable, and linearly measurable if the factors are linearly measurable.

A cocycle is *analytic* if $B(x, \cdot) \in H^\infty (x + L)$ for almost all lines.

The following lemma is a slight generalization of one of Helson's theorems in the sense that it allows one of the cocycles to not be measurable.

Lemma: Let A be a linearly measurable cocycle and B a measurable cocycle. Then $M_A \supseteq M_B$ if and only if $A\bar{B}$ is analytic.

Proof: Analyticity of $A\bar{B}$ implies inmediately that $M_A \supseteq M_B$.

Suppose that $M_A \supseteq M_B$. $M_{A\bar{B}}$ contains all products of bounded functions in M_A with bounded functions in $M_{\bar{B}}$. According *to* [3], products of bounded functions in M_B with bounded functions in $M_{\bar{B}}$ are dense in $H_0^2 (d\sigma)$. Hence $M_{A\bar{B}} \supseteq H_0^2 (d\sigma)$. It suffices then to prove that if C is a linearly measurable cocycle such that $M_C \supseteq H_0^2 (d\sigma)$, then C is analytic.

For each $0 < \lambda \in \Gamma$, $\chi_\lambda \in M_c$, so $C(x, t)\, e^{i\lambda t}$ extends analytically to the upper half-plane, providing x does not lie in a set $E(\lambda)$ of measure zero. If $x \bar{\in} E(\lambda)$, then

$$\int_{-\infty}^{\infty} C(x,t)\ (1 - it)^{-1}\ e^{iut}\ dt = 0 \tag{4}$$

for almost all values of $u > \lambda$. Now let $0 < \lambda_n \in \Gamma$ be a sequence such that $\lambda_n \rightarrow 0$, and let $E = \bigcup E(\lambda_n)$. For each $x \bar{\in} E$, (4) *holds* for almost all $u > 0$. Hence except for x in a set of measure zero, $C(x, .) \in H^2 (x + L)$. *Q.E.D.*

Now let f be a function in $L^2 (d\sigma)$ such that $\mu_r * \log |f| > -\infty$ *a.e. i.e.*, such that $\log |f| \in L^1(x + L)$ for almost all lines. On almost all lines, we can write $f = F g$, where $|F| = 1$ and g is an outer function in $H^2 (x + L)$. F is uniquely determined, up to a constant multiple of modulus 1 on each line. So F uniquely determines a cocycle, also denoted by F, which we call the *inner part* of f. F is a linearly measurable cocycle.

M_f will denote the invariant subspace generated by f, *i.e.*, the closure of fA in $L^2 (d\sigma)$.

Theorem 5: Let $f \in L^2 (d\sigma)$ be such that $\mu_r * \log |f| > -\infty$ *a.e.* Then the inner part F of f is *a* measurable cocycle, and $M_f = M_F$.

Proof: Suppose $M_f = M_B$ for the measurable cocycle B. Since $F\,g$ is analytic on each line for all $g \in M_f$, $M_F \supseteq M_B$. By the lemma, $F\,\overline{B}$ is analytic. Since $f \in M_B$. Bf is analytic on almost all lines, so $B\,\overline{F}$ is analytic. Hence $F\,\overline{B}$ is constant on almost all lines, so $F = B$. *Q.E.D.*

6. *Structure of cocycles*

The measurable cocycles with operation multiplication form a group, which we will denote by $\mathcal{C}$. The cocycles which correspond to discontinuous invariant subspaces are those cocycles of the form

$$B(x,t) = \overline{F(x)}\,F(x + e_t) \tag{5}$$

for a measurable unit function F on G. Such cocycles we call *coboundarias.* The set $\mathcal{B}$ of coboundaries forms a subgroup of $\mathcal{C}$. If two measurable cocycles belong to the same coset of $\mathcal{C}/\mathcal{B}$ we say the cocycles are *cohomologous.* Cohomologous cocycles correspond to invariant subspaces which are equivalent, in the sense that one can be obtained from the other by multiplication by a unit function.

If B is a cocycle, then in particular B satisfies the following conditions:

(i)' $|B(y,t)| = 1, \; y \in K, \; t \in R$

(ii)' $B(y,0) = 1, \; y \in K$

(iii)' $B(y + e_1, t) = \overline{B(y,1)}\;B(y, t+1), \; y \in K, \; t \in R.$

A function on $K \times R$ satisfying (i)', (ii)' and (iii)' will be called *a cocycle on K.* A cocycle B on K determines on each line a function of modulus 1 which is unique up to a constant multiple. Hence B determines a cocycle on G, which is realized explicitly by defining $B(y + e_s, t) = \overline{B(y,s)}\;B(y, s+t), \; y \in K, s, t \in R$. Hence we can regard cocycles as being either cocycles on K or on G. B is measurable, es a cocycle on G, if and only if B is measurable as a cocycle on K.

If B is a coboundary, then from (5) we obtain

$$B(y,t) = \overline{F(y)}\;F(y + e_t), \; y \in K, t \in R, \tag{6}$$

for some measurable unit function F on G. Conversely, if there is a measurable unit function F on G which satisfies (6), B is the coboundary of F.

It will be convenient in the future not to distinguish between cocycles on G and cocycles on K.

Let $\mathcal{U}$ be the family of measurable unit functions on K, identifying as usual two functions which agree almost everywhere. Endowed with the operation multiplication, $\mathcal{U}$ becomes a group. Let $\mathcal{P}$ be the subgroup of $\mathcal{U}$ of functions of the form $h(y) = \overline{F}(y)$ $f(y + e_1)$ for some $f \in \mathcal{U}$.

Every $h \in \mathcal{U}$ determines a cocycle B_h via the formula

$$B_h(y, t) = \begin{cases} \Pi_{d=0}^{[t]-1} h(y + e_j), t \geqslant 1, y \in K, 0 \leqslant t < 1, y \in K \\ 1 \\ \Pi_{j=[i]-1}^{-1} h(y + e_j), t < 0, y \in K. 0 s t < |, y \in K. \end{cases}$$

Here $[t]$ is the largest integer which does not exced t. B_h is a measurable cocycle which is constant on each interval $n \leqslant t < n + 1$, n an integer.

The correspondence $h \to B_h$ is a monomorphism of $\mathcal{U}$ into $\mathcal{C}$. If B_h is the coboundary of F, then $F(y, t)$ is also constant on each interval $n \leqslant t < n + 1$. In particular, if $f(y) = F(y, 0)$, then $h(y) = B_h(y, 1) = \overline{f(y)}\, f(y + e_1)$. Hence $h \in \mathcal{P}$. Conversely, if $h \in \mathcal{P}$, B_h is the coboundary of the function F *defined by* $F(y + e_s) = f(y)$, $y \in K$, $0 \leqslant s < 1$.

Theorem 6: The monomorphism $h \to B_h$ induces an isomorphism of $\mathcal{C}/\mathcal{B}$ and $\mathcal{U}/\mathcal{P}$.

Proof: It suffices to show that every measurable cocycle A is cohomologous to a cocycle of the form B_h. For this we define $C(y, t) = \overline{A(y, [t])}\, A(y, t)$, $y \in K$, $t \in R$. One verifies immediately that C satisfies (i)', (ii)' and (iii)', so C is measurable cocycle. Also $C(y, 1) = 1$, $y \in K$, so $C(y + e_1, t) = C(y, t + 1)$. Hence the function $F(y + e_t) = C(y, t)$, $y \in K$, $t \in R$, is well defined and measurable on G. Since $F(y) = 1$, $y \in K$, we see from (6) that C is the coboundary of F.

Hence $B(y,t) = A(y,[t])$, $y \in K$, $t \in R$, defines a ccoycle which is cohomologous to A. B is measurable, and B is constant on each interval of the form $n \leqslant t < n + 1$. If $h(y) = B(y, 1)$, then $h \in \mathcal{U}$ and $B = B_h$. *Q.E.D.*

7. *Real cocycles*

If we restrict ourselves to real cocycles, we can carry this isomorphism one step further. First we note that if h is a real coboundary, then h is the coboundary of a real unit function p on K. Indeed, if q is a unit function such that $h(y) = \overline{q(y)}\, q(y + e_1)$, we can define.

$$p(y) = \begin{cases} 1, 0 \leqslant arg\, q < 2 \\ -1, 2 \leqslant arg\, q < 2 \end{cases}$$

Then $h(y) = p(y)\, p(y + e_1)$.

Let $\mathcal{U}_r$ denote the real functions in $\mathcal{U}$, and $\mathcal{P}_r$ the real functions in $\mathcal{P}$. The preceeding remark shows that the cosets of $\mathcal{U}_r / \mathcal{P}_r$ yield non-equivalent invariant subspaces of $L^2\ (d\sigma)$.

Let now $\mathcal{M}$ be the family of measurable subsets of K, modulo null sets. $\mathcal{M}$ is a group with operation $E_0 \triangle E_1 = (E_0 \cap E_1^c) \cup (E_1 \cup E_0^c)$. Sets of the form $D \triangle (D + e_1)$, where $D \in \mathcal{M}$, will be called *caboundaries*. They form a subgroup of $\mathcal{M}$ which we will denote by $\mathcal{N}$.

For each $E \in \mathcal{M}$ we define the function $h_E \in \mathcal{U}^r$ by

$$h_E(y) = \begin{cases} -1\ ,\ y \,\bar{\in}\, E \\ \ \ 1\ ,\ y \,\bar{\in}\, E \end{cases}$$

Theorem 7: The correspondence $E \rightarrow h_E$ is an isomorphism of $\mathcal{M}$ and $\mathcal{U}_r$. It induces an isomorphism of $\mathcal{M}/\mathcal{N}$ and $\mathcal{U}_r/\mathcal{P}_r$

Proof: Since $h_D\, h_E$ is -1 on $D \triangle E$, the correspondence is a group homomorphism. Since $h \in \mathcal{U}_r$ is uniquely determined by the set where it is -1, $E \rightarrow h_E$ is an isomorphism. One verifies immediately that the image of $\mathcal{N}$ is $\mathcal{P}_r$. *Q.E.D.*

Theorem 8: Let K be a compact infinite abelian group with Haar Measure $d\mu$. Let $e \in K$ generate a subgroup J which is dense in K. Then there is no measurable set $F \subseteq K$ such that $K = F \triangle (F - e)$.

Proof: Let H be the quotient of $K \times R$ and the discrete subgroup $H_0 = \{\ (-2ne, 2n)\ : n = 0\ , \pm 1\ , \ldots\ \}$. Topologically, H is obtained from $K \times [0,2]$ by identifying $(y, 2)$ with $(y + 2e, 0)$. The line $s \rightarrow (0, s)/H_0$ is dense in H. Hence the character group of H can be realized as a subgroup of R, and H is a group of the type that we have been considering. In particular, we can apply the basic lemma to H.

Now suppose $K = F \triangle (F - e)$, where F is measurable. In other words, $K = F \cup (F - e)$, and $F \cap (F - e)$ is empty. In particular we have $\tau(F) = \tau(F - e) = 1/2$.

Since $F - 2e = F$, every line $x + L$ either always meets K in a point of F or never meets K in a point of F. The set $\{ (y, s) : y \in F,\ 0 \leqslant s < 2\}$ is then measurable with measure 1/2, and it contains every line that it intersects. This contradicts the basic lemma. *Q.E.D.*

Corollary: There exist real measurable cocycles whose invariant subspaces are continous. One such cocycle is given by $B(x, t) = (-1)^{[s+t]}$, where $x = y + e_s$ with and $0 \leqslant s < 1$.

If E is a coboundary, then $E \triangle K = E^c$ is cohomologous with K. Hence if E is any measurable subset of K, either E or E^c is not a coboundary.

In order that E be cohomologous with K, it is necessary and sufficient that E^c be a coboundary. To find sets E which are neither coboundaries nor cohomologous to K, we must find E such that neither E nor E^c is a coboundary. To construct such sets, it suffices to find a measurable set E and *a* $y \in K$ such that $E \triangle (y + E) = K$. This can always be accomplished. Clearly E and $y + E$ are then simultaneously coboundaries or not coboundaries. Since $E^c = (y + E)$, *at least one of them is not a coboundary.* Hence E is a set of the desired type.

It appears that $\mathcal{U}_r\ \mathcal{P}_r$ is quite large.

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