SOME REMARKS ON THE POINTWISE
CONVERGENCE OF SEQUENCES OF
MULTIPLIER OPERATORS

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Introduction. The purpose of the present paper is to give
some sufficient conditions for the pointwise convergence of op­
erators of the form
\[
\int_{\mathbb{R}^m} K_n(x - y) f(y) \, dy
\]
The kernels \( K_n \) need not belong to \( L^1 \). Some classes of singular in­
tegrals as well as a large family of Féjer-like kernels may be re­
garded as particular cases of theorems 1, 2, 3. The main results
are contained in theorems 1, 2 and 3 and some corollaries ard exam­
plles are given thereafter.

Definitions and Notation.

1) \( f \ast g \) will denote the convolution between \( f \) and \( g \); namely
\[
\int_{\mathbb{R}^m} f(x - y) g(y) \, dy = \int_{\mathbb{R}^m} f(x_1 - y_1, \ldots, x_m - y_m) \, dy_1 \cdots dy_m
\]
\( \mathbb{R}^m \) denotes the euclidean \( m \)-dimensional space.

2) By \( L^p \) we shall denote the class of all measurable functions
defined on \( \mathbb{R}^m \) such that \( \int_{\mathbb{R}^m} |f|^p \, dx = ||f||_p^p < \infty; p \geq 1 \).

3) By \( E(f > \lambda); f \geq 0 \), we shall denote the set of points of
\( \mathbb{R}^m \) such that \( f > \lambda \) and by \( |E(f > \lambda)| \) its measure.

4) A multiplier operator (*) acting from \( L^2 \) to \( L^p \) is a linear
operator such that \( T(f) = k \ast f \) for all \( f \in L^2 \).
\[
f(u) = \int_{\mathbb{R}^m} \exp (-i <u, x>) f(x) \, dx; <u, x> = \sum_{j=1}^{m} x_j u_j
\]

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(*) we shall be concerned with multiplier operators acting always from $L^2$ into $L^2$.

5) By $H_{\epsilon_0}$, $H$, $H_\alpha$ we shall denote respectively the classes of bounded functions $\phi$ defined on $\mathbb{R}^m$ homogeneous of degree zero, such that

a) $\phi \in C^\omega$ if $x \neq 0$; b) $\phi$ is the symbol of a singular integral operator $K$ acting continuously from $L^p$ into $L^p$ for all $p > 1$; c) No further restriction is imposed on $\phi$.

By $K(f)$ we shall always denote the operator defined by the multiplier $\phi$. In cases a) and b) the corresponding $K$ is a singular integral operator.

1) Theorem. Let $T_n = T$ be a sequence of multiplier operators and $k_n(u) = k_{n_1, \ldots, n_m}$ their corresponding multipliers; suppose that

i) There exists a function $\phi$, bounded and homogeneous of degree zero such that

\[
\int_{\mathbb{R}^m} |k_n(u) - \phi(u)|^2 |g(u)|^2 \, du \to 0 \quad \text{as } n \to \infty
\]

(for all $g$ belonging to $L^2$)

ii) There exists a sequence $\epsilon_n = (\epsilon_{n_1}, \ldots, \epsilon_{n_m})$ of real and positive parameters such that

a) $\epsilon_n \to \infty; \epsilon_{n_j} \neq 0$ for all $n_j$ and for all $j$.

b) The functions $\Psi_n(u) = k_{n_1, \ldots, n_m}(\epsilon_{n_1} u_1, \ldots, \epsilon_{n_m} u_m) \phi^{-1}(\epsilon_{n_1} u_1, \ldots, \epsilon_{n_m} u_m)$ have the following compactness property

\[
\left\| \frac{\partial^{s_1 + \ldots + s_m} \Psi_n}{\partial^{s_1} u_1 \cdots \partial^{s_m} u_m} \right\|_{p_0} \leq M
\]

for some $p_0$ such that $1 < p_0 \leq 2$ and for all $s = (s_1, \ldots, s_m)$ submitted to the conditions $0 \leq s_1 + \ldots + s_m \leq m$, where each $s_j$ can take the values 0 or 1 only. The constant $M$ does not depend on $n$ and the derivatives are taken in the distributions sense.
Under the above assumptions we have

A) If $p_0 < 2$ and $\phi$ belongs to the class $H_\phi$ or $H$ then

$A_1)$ \[ \| T_n f - K(f) \|_p \to 0 \text{ for all } f \in L^p; \quad p > p_0. \]

$A_2)$ If $f \in L^p$, $p > p_0$ then $T_n f$ converges a.e. to $K(f)$; furthermore the operator \[ \sup_{n_1, \ldots, n_m} |T_{n_1} \cdots T_{n_m} f| = T^* f \] verifies

$|| T^* f ||_p < C(p) \| f \|_{p_0} \| p > p_0$ where $C(p)$ depends on $p$ only.

B) If $\phi$ belongs to $H_\phi$ and $p_0 < 2$ then

$B_1)$ \[ \| T_n f - K(f) \|_2 \to 0 \text{ for all } f \in L^2. \]

$B_2)$ $T_n f$ converges a.e. to $K(f)$ and furthermore, if \[ \sup_n \| T_n f \| = T^* f \] we have \[ || T^* f ||_2 < C \| f \|_2. \]

C) Limiting Cases. Let $p = p_0$ and $\phi$ belong to $H$ or $H_\phi$; if $f \in L^p$ and \[ \| K(f) \|_{p_0} \log^+ (K(f))^{m-1} \text{ belongs locally to } L^1 \]

$C_1)$ \[ T_n f \text{ converges a.e. to } K(f). \]

$C_2)$ If $|E| < \infty$ we have, for $1 \leq s < p_0$

\[ \int_E |T^* f|^s \, dx \leq \int_E \left( \int_E |K(f)|^{p_0} (\log^+ |K(f)|)^{m-1} \right) \, dx. \]

$C_3)$ If \[ \| K(f) \|_{p_0} \log^+ (K(f))^{m} \text{ belongs locally to } L^1 \text{ and } |E| < \infty, \]

then \[ \int_E (T^* f)^{p_0} \, dx < A_1 (|E|) \]

$+ A_2 (|E|) \int_E |K(f)|^{p_0} \log^+ (K(f))^{m} \, dx. \]

D) If $p_0 = 2$ and $\phi$ belongs to $H_\phi$, the same conclusions of $C$ are valid.

E) If the sequence $\varepsilon_n = (\varepsilon_{n_1}, \varepsilon_{n_2}, \ldots, \varepsilon_{n_m})$ of ii) is such that each $\varepsilon_{n_j}$ is strictly increasing to $\infty$, then

$E_1)$ If $\phi$ belongs to $H$ or $H_\phi$, we have

\[ \| E \{ T^* f > \lambda \} \| < (C/\lambda^{p_0}) \int_{R^m} \| f \|_{p_0} \, dx, \quad f \in L^p_0 \]

and consequently $T_n f$ converges a.e. to $K(f)$.

$E_2)$ If $\phi$ belongs to $H_\phi$ and $p_0 = 2$ we have that the same conclusions of $E_1$ are valid.

F) For $\phi$ belonging to $H$ or $H_\phi$, $T_n f$ admits the representation

$F_1)$ \[ T_n f = \int_{R^m} K_n(x-y) f(y) \, dy \quad \text{for all } f \in L^p; \quad p \geq p_0; \quad K_n \text{ belongs to } L^{p_0} \text{ for all } n; \quad 1/p_0 + 1/p^{*0} = 1. \]

$F_2)$ If $\phi$ belongs to $H_\phi$ and $f \in L^2$, we have

\[ T_n f = \int_{R^m} K_n(x-y) f(y) \, dy \text{ where } K_n \in L^2 \text{ for all } n. \]

G) The kernels $K_n$ belong to $L^1$ and \[ \| K_n \|_1 \leq M \text{ if and only if } \phi \text{ reduces to a constant.} \]
H) If \( \mu \) is a regular measure with bounded variation, defined on the Borel subsets of \( \mathbb{R}^m \) then the following multipliers
\[
\bar{k}_n(u) = \int_{\mathbb{R}^m} \bar{k}_n(u-x)(u-x) \, d\mu (x) = T_{1, \ldots, m}^n
\]
verify \( A, B \) and also \( F \) with
\[
\bar{T}_nf = \int_{\mathbb{R}^m} K_n(x-y) \, g(x-y) \, f(y) \, dy
\]
where the \( K_n(x) \) are those associated to \( \bar{k}_n \) and \( g(x) \) is the Fourier Transform of \( \mu \).

The limit operator is defined by the multiplier
\[
\bar{\phi}(u) = \int_{\mathbb{R}^m} \phi (u-x) \, d\mu (x).
\]

The key to the most important part of the proof of the above Theorem is the following lemma.

2) Lemma. Let \( F_n(x) = F_{n_1, \ldots, n_m}(x_1, \ldots, x_m) \) be a denumerable family of measurable functions defined on \( \mathbb{R}^m \), and let \( \lambda_n = (\lambda_{n_1}, \ldots, \lambda_{n_m}) \) be an \( m \)-dimensional sequence of real positive parameters such that the auxiliary functions
\[
\Psi_n = F_{n_1, \ldots, n_m}(\lambda_{n_1} x_1, \ldots, \lambda_{n_m} x_m) \prod_{j=1}^m (\lambda_{n_j}^2)
\]
verify the following compactness property
\[
(2.1) \quad \| \Psi_n \prod_{j=1}^m |x_j|^{s_j} \|_{p_0} \leq M, \quad p_0 > 1
\]
where \( M \) does not depend on \( n \) or \( s = (s_1, \ldots, s_m) \) and each \( s_j \) can take the values 0 or 1 only. Then, if \( f^* = \sup |F_n \ast f| \) we have

i) \( \| f^* \|_{p_0} < C(p) \| f \|_{p_0} \) for all \( p > p_0^* \); \( 1/p_0 + 1/p_0^* = 1 \).

ii) If \( |f| \in L^{p_0^*} \) and \( |f|^{p_0^*} (\log^+ |f|)^{m-1} \) belongs locally to \( L^1 \); then, for all measurable set \( E \) such that \( |E| < \infty \) we have
\[
\int_E f^{p_0^*} \, dx \leq A(r, |E|) + B(r, |E|) \int_E |f|^{p_0^*} (\log^+ |f|)^{m-1} \, dx
\]
where \( 1 \leq r < p_0^* \).

iii) If \( f \in L^{p_0^*} \) and \( |f|^{p_0^*} (\log^+ |f|)^{m} \) belongs locally to \( L^1 \); then, for all measurable set \( E \) such that \( |E| < \infty \)
\[
\int_E f^{p_0^*} \, dx \leq A'(|E|) + B'(|E|) \int_E |f|^{p_0^*} (\log^+ |f|)^{m} \, dx
\]

iv) If the \( \lambda_{n_j} \) are strictly decreasing to 0 for each \( j \), we have
\[
|E(f^* > \lambda)| < (C_0/\lambda^{p_0^*}) \int_E |f|^{p_0^*} \, dx
\]

Proof. Without loss of generality we may assume that the \( F_n \) are zero on the complement of the set \( \{ x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0 \} \).
Let \( f \) be a continuous and compact supported function; then, by a change of variables we have

\[
\int_{\mathbb{R}^m} F_n(x - y) f(y) \, dy = \int_{\mathbb{R}^m} \Psi_n(y) f(x - \lambda_n y) \, dy =
\]

\[
= \int_{\mathbb{R}^m} \Psi_n(y_1, \ldots, y_m) f(x_1 - \lambda_n y_1, \ldots, x_m - \lambda_n y_m) \, dy_1 \ldots dy_m
\]

The modulus of the last integral is dominated by

\[
\sum_{\nu = 0}^{\infty} \int_{S_{k_1} \ldots S_{k_m}} |\Psi_n(y)| \, |f(x - \lambda_n y)| \, dy
\]

The sets \( S_{k_1}, \ldots, S_{k_m} \) are defined in the following way

\[
S_{k_1}, \ldots, S_{k_m} = [2^k_1, 2^k_1 + 1] \times \ldots \times [2^k_m, 2^k_m + 1] \text{ for } k_j > 0,
\]

\( j = 1, \ldots, m \). If \( k_j = 0 \) the corresponding interval is \([0, 2]\).

By an application of Hölder's inequality, the series (2.3) is dominated by

\[
\sum_{\nu = 0}^{\infty} \int_{S_{k_1} \ldots S_{k_m}} (|\Psi_n| \, dy)^{1/\nu} (\int_{S_{k_1} \ldots S_{k_m}} |f(x - \lambda_n y)\nu^* dy)^{1/\nu^*}
\]

Estimates for the terms \( t_{k_1}, \ldots, t_m = (\int_{S_{k_1} \ldots S_{k_m}} |\Psi_n| \, dy)^{1/\nu} \)

If \( k_j > 0, j = 1, \ldots, m \); then

\[
2^{k_1 + \ldots + k_m} t_{k_1}, \ldots, t_m \leq \left\{ \int_{S_{k_1} \ldots S_{k_m}} (|\Psi_n| \prod_{j=1}^{m} |x_j|)^{\nu} \, dx \right\}^{1/\nu} \leq M_0
\]

The last inequality holds from Hypothesis (2.1). Then we must have

\[
\sum_{\nu = 0}^{\infty} \int_{S_{k_1} \ldots S_{k_m}} (|\Psi_n| \prod_{j=1}^{m} |x_j|)^{s_j} \, dx \right\}^{1/\nu} \leq M_0
\]

If the \( m \)-tuple \((k_1, \ldots, k_m)\) contains zeros in its coordinates, according to Hypothesis (2.1) we have

\[
2^{k_1 + \ldots + k_m} t_{k_1}, \ldots, t_m \leq M_0 2^{k_1} - k_2 - \ldots - k_m
\]

\( k_j \) must be a finite number. Then we can conclude that in every case

\[
\sum_{\nu = 0}^{\infty} \int_{S_{k_1} \ldots S_{k_m}} (|\Psi_n| \prod_{j=1}^{m} |x_j|^{s_j} \, dx \right\}^{1/\nu} \leq M_0 \text{ where } s_j = 1 \text{ if } k_j \neq 0, s_j = 0 \text{ if } k_j = 0.
\]

Now we conclude that in every case
(2.9) \[ t_{k_1}, \ldots, t_{k_m} \leq M_0 2^{-k_1} \ldots 2^{-k_m}. \]

Estimates for the terms \( \left( \int_{s_{k_1} \ldots s_{k_m}} |f(x - \lambda_n y)| \, d\nu^* \, dy \right)^{1/\nu^*} \)

(2.10) \( \left( \int_{s_{k_1} \ldots s_{k_m}} |f(x - \lambda_n y)| \, d\nu^* \, dy \right)^{1/\nu^*} \leq \left( \int_{s_{k_1} \ldots s_{k_m}} \ldots \int_{s_{k_1} \ldots s_{k_m}} |f(x - \lambda_n y)| \, d\nu^* \, dy \right)^{1/\nu^*} \)

The second (right hand) member of (2.10), by a change of variables, is readily seen to be equal to

(2.11) \[
\left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right) \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right) \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right) \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right)
\]

If \( M(f) \) is the maximal operator of the strong differentiation (see [4], pp. 306-307) we have

(2.12) \[
\left( \int_{s_{k_1} \ldots s_{k_m}} |f(x - \lambda_n y)| \, d\nu^* \, dy \right)^{1/\nu^*} \leq 2^{2m/\nu^*} 2^{(1/\nu^*) (k_1 + \ldots + k_m)}
\]

(2.13) \[
M \left( \left| f \right| \right)^{1/\nu^*} \leq \left( \sum_{0=0}^{\infty} \right) 2^{(1/\nu^*) (k_1 + \ldots + k_m)}
\]

Consequently

(2.14) \[
f^* \leq M_0 2^{2m/\nu^*} \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right) \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right) \left( \frac{\lambda_1}{2}, \ldots, \frac{\lambda_k}{2}, \ldots, \frac{\lambda_m}{2} \right)
\]

and \( f^* \leq C(p_0) \left( \left| f \right| \right)^{1/\nu^*} \)

If \( p > p_0 \) then \( p = rp_0 \) with \( r > 1 \), and from the Jessen-Marcinkiewicz-Zygmund Theorem (see [4], pp. 306-307) we have

(2.15) \[
||f^*||_p^p = \int_{\mathbb{R}_m} \left( \left| f^* \right| \right)^r \, dx \leq C(p_0) \int_{\mathbb{R}_m} \left( \left| M \left( \left| f \right| \right) \right| \right)^r \, dx \leq C(p_0) C(r) \int_{\mathbb{R}_m} \left( \left| f \right| \right)^r \, dx = C(r, p_0) \int_{\mathbb{R}_m} \left| f \right|^r \, dx.
\]

Now the inequality (2.15) can be extended by continuity to all \( L^p, p > p_0 \) (see [5], p. 165) and part i of the Thesis follows. Taking \( f \in L^\nu^* \), an application of the Jessen-Marcinkiewicz-Zygmund Theorem to the function \( \left| f \right| \nu^* \) also gives ii) and iii).
Part iv) requires a different technique. Let us return to (2.11) and consider

\[ (2.16) \quad f^{(n)}_1, \ldots, f^{(n)}_m = \left( \frac{\lambda^{m/2} + \cdots + \lambda^{m/2}}{\lambda^{m/2} + \cdots + \lambda^{m/2}} \right) \int f(x-y) |p_0^* dy | \]

Putting \( f^{(n)}_1, \ldots, f^{(n)}_m = \sup_n f^{(n)}_1, \ldots, f^{(n)}_m \), we shall show that

\[ (2.17) \quad E(f^{(n)}_1, \ldots, f^{(n)}_m) > \lambda \bigg| < (C/\lambda) \int f |p_0^* dx \]

where \( C \) does not depend on \( k_1, \ldots, k_m \) or on the function \( f \).

The \( \lambda_{ij} \) are strictly decreasing to zero for each \( j, j = 1, \ldots, m \).

Therefore we can construct a family of continuous and strictly decreasing to zero functions \( h_j(t) \), \( j = 1, \ldots, m \), such that there exists a sequence \( t_n \) of real numbers, strictly decreasing to zero, verifying

\[ (2.18) \quad \lambda_{ij} = h_j(t_n) \quad (j = 1, \ldots, m) \]

Now, the differentiation operators of (2.16) are in the same conditions as those of [4], p. 310, (3.5); therefore (2.17) follows. Let us rewrite (2.5) taking into account (2.16); namely

\[ (2.19) \quad |F_n f| \leq M_0 2^{-m/n_0^*} \sum_{k_1, \ldots, k_m} \int f^{(n)}_1, \ldots, f^{(n)}_m ]^{1/n_0^*} \]

Calling \( a_k = 2^{(-1/n_0^*) (k_1 + \cdots + k_m)} \), we have

\[ (2.20) \quad |F_n f| \leq M_0 2^{-m/n_0^*} \sum_{k_1, \ldots, k_m} a_k 2^{(-1/n_0^*) (k_1 + \cdots + k_m)} \]

and also
After a new normalization of the coefficients $a_k, \ldots, k_m$ such that $\sum a^{1/2} k_1, \ldots, k_m = 1$, the inequality (2.21) holds for such coefficients, but with a modification on the value of $C$. Now

$$E\left( \sum_{0 \leq k_1 \leq m} a_k \ldots f^{*} k_1, \ldots, k_m > \lambda \right) \subseteq \bigcup_{k_1, \ldots, k_m} E(a^{1/2} k_1, \ldots, k_m f^{*} k_1, \ldots, k_m > \lambda)$$

Therefore

$$|E\left( \sum_{0 \leq k_1 \leq m} a_k \ldots f^{*} k_1, \ldots, k_m > \lambda \right)| \leq \sum_{k_1, \ldots, k_m} |E(a^{1/2} k_1, \ldots, k_m f^{*} k_1, \ldots, k_m > \lambda)| \leq \sum_{k_1, \ldots, k_m} a^{1/2} k_1, \ldots, k_m$$

$$(A/\lambda) \int_{R} |f| \int \phi^{*} dx = (A/\lambda) \int_{R} |f| \phi^{*} dx$$

The last inequality of (2.23) follows from (2.17). Now, from (2.21) and (2.23) we have

$$|E(f > \lambda)| = E(|f| \phi^{*} > \lambda \phi^{*}) < (A'/\lambda \phi^{*}) \int_{R} |f| \phi^{*} dx.$$  

This ends the proof of part iv).

3) Remark. The Hypothesis (2.1) can be replaced by a weaker one, namely

$$||\Psi_{n} \Pi_{j=1}^{m} x_{j}^{\beta} ||_{p_0} < M_0; p_0 > 1; \beta > (p_0 - 1)/p_0$$

$M_0$ does not depend on $n$ or on $s = (s_1, \ldots, s_m)$ and $s_j$ can take the values 0 or 1 only: $j = 1, \ldots, m$.

The condition $\beta > (p_0 - 1)/p_0$ ensures the convergence of

$$\sum a^{1/2} k_1, \ldots, k_m$$

4) Proof of Theorem 1. Since $k_n \phi^{-1} \in L^p_0$ for some $p_0$ such that $1 < p_0 \leq 2$, there exists a function $F_n \in L^p_0$ such that
its Fourier Transform in the distributions sense coincides with $\hat{k}_n \phi^{-1}$. Furthermore, from the Hausdorff-Young-Titchmarsh Theorem it follows

\begin{equation}
\| F_n \|_{p_0^*} \leq A(p_0) \| \hat{k}_n \phi^{-1} \|_{p_0}
\end{equation}

Calling $\varphi_n = (1/ \prod_{j=1}^m \epsilon_{n_j}) F_{n_1}, \ldots, n_m (x_1 \epsilon^{-1}_{n_1}, \ldots, x_m \epsilon^{-1}_{n_m})$ we have that the Fourier Transform in the distributions sense of $\varphi_n$ is $\Psi_n$ and also

\begin{equation}
\| \varphi_n \|_{p_j^*} \leq A(p_0) \| \Psi_n \|_{p_0}
\end{equation}

By a similar procedure we obtain

\begin{equation}
\| \varphi_n \|_{p_j^*} \leq A(p_0) \| \Psi_n \|_{p_0} < M_0
\end{equation}

$M_0$ does not depend on $n$ or $s = (s_1, \ldots, s_m)$ and each $s_j$ can take the values 0 or 1 only.

Now it is clear that the functions $F_n$ with auxiliary functions $\varphi_n$ are in the conditions of Lemma 2. If $f$ belongs to the space $C^\infty$ of rapidly decreasing functions, then

\begin{equation}
T_n \varphi_n (f) = \hat{k}_n, \ldots, n_m f = \hat{k}_n \phi^{-1} \phi f
\end{equation}

The Fourier Antitransform in the distributions sense gives

\begin{equation}
T_n \varphi_n (f) = F_n^*, K(f)
\end{equation}

Let us observe that $F_n \in L^p$ for all $p$ such that $1 \leq p \leq p_0^*$; this follows from the fact that $\| F_n \prod_{j=1}^m |x_j|^{s_j} \|_{n^*} < \infty$ for all $s = (s_1, \ldots, s_m)$ such that each $s_j$ takes the values 0 or 1 only.

On the other hand, we shall show that $\| F_n \|_1 \leq N_0$ where $N_0$ does not depend on $n$. In fact, from (2.5)

\begin{equation}
\int_{R_m} |F_n(x)| \, dx = \int_{R_m} |\varphi_n(x)| \, dx \leq \int_{R_m} |\varphi_n(x)| \, dx \leq
\end{equation}

\begin{equation}
\leq \sum_{t_1, \ldots, t_m} \epsilon_{t_1, \ldots, t_m} 2^{x_m/p_0} 2^{(1/p_0)} (k_1 + \ldots + k_m) \leq
\end{equation}

\begin{equation}
\leq M_0 2^{x_m/p_0} \sum_{t_1, \ldots, t_m} 2^{(-1/p_0)} (k_1 + \ldots + k_m) = N_0.
\end{equation}

If for some $p > 1, K(f)$ maps continuously $L^p$ into itself, by using the Young inequality we can extend by continuity the representation (4.5) for all $f \in L^p$ and furthermore, we have
(4.7) \[ \| T_n, \ldots, n_n(f) \|_p = \| F_n \ast K(f) \|_p \leq \| F_n \|_p \cdot \| K(f) \|_p \leq N_0 \cdot C \cdot \| f \|_p. \]

(4.8) As we have already shown, the \( F_n \) are on the conditions of Lemma 2 and, therefore, if \( K(f) \) maps continuously \( L^p \) into itself, \( \sup_{n} T_n f = \sup_{n} | F_n \ast K(f) | \) verifies similar inequalities to those proved for \( \sup_{n} | F_n \ast f | \).

(4.9) On the other hand, \( T_n f \) converges pointwise in a dense subset. In fact, if \( f \in \mathcal{S} \) we have

\[
(4.10) T_n f = (1/(2 \Pi)^n) \int_{R_n} e^{i < \zeta, \nu >} \hat{k}_n(y) \cdot \hat{f}(y) \, dy \quad \text{and}
\]

\[
(4.11) | T_n f - K(f) | \leq \int_{R_n} | \hat{k}_n - \phi | \cdot | \hat{f} | \, dy \leq (\int_{R_n} | \hat{k}_n - \phi |^2 \cdot | \hat{f} | \, dy)^{1/2}.
\]

Since \( f \in \mathcal{S} \), then \( | f |^{1/2} \in L^2 \); therefore, the last term of the inequality tends to zero. (condition 1).

A suitable combination of the two preceding arguments (4.8) and (4.9) gives parts A, B, C, D and E. (see [4], p. 307 and [5] p. 160).

(4.12) If \( f \in L^p \) and \( p > p_0 \) we shall show that \( T_n(f) = K_n \ast f \), where \( K \in L^{p_0} \).

Let us observe that \( \hat{F}_n = \hat{k}_n \phi^{-1} \), that is \( \hat{F}_n = \hat{k}_n \), and since \( K \) maps continuously \( L^p \) into itself for all \( p > 1 \) when \( \phi \) belongs to \( H \) or \( H_x \), then in both cases we have

(4.13) \( K(F_n) = K_n \in L^p \) for all \( p \) such that \( 1 < p < p_0 \) since \( F_n \in L^p \) for the same values of \( p \).

On the other hand, if \( f \in \mathcal{S} \), taking (4.5) into account, we have (4.14) \( T_n f = F_n \ast K(f) = K_n \ast f \).

Now we can extend (4.14) by continuity to all \( L^p, p \geq p_0 \), since both sides of the equality verify respectively

\[
(4.15) \| F_n \ast K(f) \|_p \leq \| F_n \|_p^{(p-1)} \cdot \| K(f) \|_p
\]

\[
| K_n \ast f | \leq | K_n |_p^{(p-1)} \cdot \| f \|_p
\]

Therefore part \( F_i \) is proved.
Part $F_2$ is also valid since this case requires
\[ ||K(f)||_2 \leq A ||f||_2 \text{ only.} \]

(4.16) **Proof of Part G.** If $\phi = C_0$ then $F_n = C_0^{-1} K_n$; therefore
\[ ||K_n||_1 < M' \text{ where } M' \text{ does not depend on } n. \]

Let us suppose that $||K_n||_1 < M'$; consequently $\hat{k}_n$ must be
continuous for each $n$, and furthermore

(4.18) $\phi = \hat{k}_n/F_n$

(4.19) If $F_n(0) \neq 0$ for some $n = n_0$; then $\phi$ must be continuous at
$x = 0$; and since $\phi$ is a bounded homogeneous of degree zero func-
tion it must reduce to a constant.

(4.20) If $F_n(0) = 0$ for all $n$ we shall show that $K(f) = 0$ for
all $f \in L^2$.

In fact, let us consider a subsequence $\{n'\}$ of $\{n\}$ such that $F_{n'}$ and $\varphi_{n'}$ are in
the conditions of iv) of Lemma 2. Then

(4.21) $|E(\sup_{\varphi_{n'}} |F_{n'} * f| > \lambda )| \leq (C_0/\lambda^2) \int_{\mathbb{R}^m} ||f||^2 dx$

On the other hand, from the compacity conditions on the $\varphi_{n'}$ for
each $\epsilon > 0$ the $\varphi_{n'}$ admit the decomposition

(4.22) $\varphi_n = \varphi_n^{(1)} + \varphi_n^{(2)}$; $||\varphi_n^{(1)}||_1 < \epsilon$

$\varphi_n^{(1)} = 0$ if $x \in Q$, where $Q$ is a cube centered at the origin
and depends on $\epsilon > 0$ only. If $x \in Q$, $|\varphi_n^{(1)}(x)| < M_2$, where $M_2$
depends on $\epsilon > 0$ only.

Thus, if $f \in D$, by changing variables and taking into
account that $F_{n'}(0) = 0$, we obtain

(4.23) $| (F_{n'} * f)(x) | = | \int_{\mathbb{R}^m} \varphi_{n'}(y) \cdot f(x - \epsilon^{-1} y) \, dy |$

\[ = | \int_{\mathbb{R}^m} \varphi_{n'}(y) [f(x) - f(x - \epsilon^{-1} y)] \, dy | \leq \]

\[ \leq \int_{\mathbb{R}^m} \varphi_{n'}^{(1)}(y) |f(x) - f(x - \epsilon^{-1} y)| \, dy + 2\epsilon ||f||_\infty. \]

where $f(x - \epsilon^{-1} y) = f(x_1 - \epsilon^{-1} y_1, \ldots, x_m - \epsilon^{-1} y_m)$.
Now if $Q_{\epsilon/n'}$ denotes the cube obtained from $Q$ by dividing its edges by $\epsilon/n'$ respectively, then the right term of the inequality (4.23) is readily seen to be less or equal than

$$(4.24) \quad |Q| M_2 (1/|Q_{\epsilon/n'}|) \int_{Q_{\epsilon/n'}} |f(x) - f(x - y)| \, dy + 2 \epsilon \|f\|_\infty$$

Therefore

$$(4.25) \quad \lim \frac{|F_{\epsilon/n'} f|}{\epsilon} \leq 2 \epsilon \|f\|_\infty.$$ 

Consequently, for all $f \in D$ it is valid

$$(4.26) \quad \lim F_{\epsilon/n'} f = 0$$

Thus, (4.26) together with (4.21) shows that $\lim F_{\epsilon/n'} f = 0$ a.e. for all $f$ belonging to $L^2$, whence $\lim F_{\epsilon/n'} K(f) = 0$ a.e. for all $f$ belonging to $L^2$. This completes the proof of Part G, since $F_{\epsilon/n'} K(f) = K_{\epsilon/n'}(f)$ and by Hypothesis i) $\|K_{\epsilon/n'}(f) - K(f)\|_2 \rightarrow 0$.

(4.27) Proof of Part H. The $K_n$ and $\phi$ are supposed to be Borel measurable. Let us consider the functions

$$K_n(x) = \int_{R^m} K_n(x - y) \, d\mu(y).$$

By an application of the Minkowski Integral Inequality we have

$$(4.28) \quad \|K_n\|_{L^1} \leq \left( \int_{R^m} dw(y) \right)^{1/2} \|K_n\|_{L^1} \leq V(\mu) \cdot M'$$

$dw(y)$ denotes the variation of $\mu$ and $V(\mu)$ the variation in the whole $R^m$. Now if $g(x) = 1/(2\pi)^m \int_{R^m} \phi(x, y) \, d\mu(y)$, from (4.12),

(4.13) and (4.28) we conclude that

$$(4.29) \quad K_n = (K_n \cdot g)^*$$

On the other hand if $f \in <\phi$, we have

$$(4.30) \quad T_n f = (K_n \cdot g)^* f$$

Since $\|g\|_\infty < V(\mu)$, $(K_n \cdot g)^* f$ verifies the same type of inequality as that of (4.15). Therefore, by the same density argument we conclude that

$$(4.31) \quad \overline{T_n f} = \int_{R^m} K_n(x - y) \, g(x - y) \, f(y) \, dy$$

for all $f \in L^p$ with $p > p_0$ and $\phi$ belonging to $H$ or $H_0$. By a similar method, one obtains the representation for $f \in L^p$ and $\phi \in H_0$. 

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(4.32) Now we are going to show that \( T_n f \) converges pointwise for all \( f \in \mathcal{F} \).

(4.33) \( T_n(f)(x) = (1/2\pi)^m \int_{\mathbb{R}^m} e^{i<x,y>} f(y) (\int_{\mathbb{R}^m} k_n(y-s) \, d\mu(s)) \, dy \)

Interchanging the order of integration (which we may) we have

(4.34) \( \int_{\mathbb{R}^m} d\mu(s) (1/2\pi)^m \int_{\mathbb{R}^m} e^{i<x,y>} \hat{k}_n(y-s) \hat{f}(y) \, dy \)

By a similar procedure than that employed in (4.9), (4.10) and (4.11), it is easy to show that the inner integral of (4.34) converges pointwise and uniformly to

(4.35) \( (1/2\pi)^m \int_{\mathbb{R}^m} e^{i<x,y>} \hat{\phi}(y-s) \hat{f}(y) \, dy \)

On the other hand, since \( V(\mu) < \infty \), \( T_n(f) \) converges pointwise for all \( f \in \mathcal{F} \) to the operator \( T(f) \), whose multiplier is \( \hat{k}_n \cdot \hat{\mu} \).

(4.36) Estimates for the maximal operator associated to \( T_n \). In this part we shall use a technique introduced by A. P. Calderón and A. Zygmund in [2]. Let \( \phi \) belong to \( H \) or \( H_\alpha \). Then, if \( f \in L^p \), \( p > p_0 \), we have

(4.37) \( T_n f = \int_{\mathbb{R}^m} K_n(x-y) g(x-y) f(y) \, dy = \)

\( = (1/2\pi)^m \int_{\mathbb{R}^m} K_n(x-y) (\int_{\mathbb{R}^m} e^{i<x-y,s>} d\mu(s)) f(y) \, dy \)

Interchanging the order of integration we have

(4.38) \( (1/2\pi)^m \int_{\mathbb{R}^m} d\mu(s) e^{i<x,s>} \int_{\mathbb{R}^m} K_n(x-y) f(y) e^{i<y,v>} d\mu(v) \, dy \).

The modulus of the integral (4.38) is dominated by

(4.39) \( 1/(2\pi)^m \int_{\mathbb{R}^m} d\mu(s) [\hat{T}(f e^{-i<x,v>})(x)] \)

Taking the \( L^p \) norm of (4.39) with respect to \( x \) we obtain

(4.40) \( (1/2\pi)^m \int_{\mathbb{R}^m} \left[ \int_{\mathbb{R}^m} T(f e^{-i<x,v>})(x) \, d\mu(s) \right]^p \, dx \right)^{1/p} \leq \)

\( \leq (1/2\pi)^m \int_{\mathbb{R}^m} d\mu(s) \left[ \int_{\mathbb{R}^m} T^*(f e^{-i<x,v>})(x) \, d\mu(s) \right]^{1/p} \leq \)

\( \leq (V(\mu))/(2\pi)^m C \| f \|_p \).
Thus

\[(4.41) \quad \| \sup_n \overline{T}_n f \|_p \leq (V(\mu)/(2\pi)^m) \cdot C(p) \cdot \| f \|_p.\]

A similar estimate holds for \( \phi \in H_0 \) and \( f \in L^p \). (4.41) and (4.32) give the corresponding convergence results. This completes the proof of Part H.

5) Examples.

(5.1) Let us consider the single Hilbert-Transform, that is

\[ \tilde{f} = \lim_{n \to \infty} \int_{|x-y| \leq |x|+\varepsilon} f(y)(x-y)^{-1} dy \]

We know that \( \int_{|y| \leq |x|/n} e^{-iyx} y^{-1} dy = -2i \cdot \text{sg}(u) \cdot \int_{|y| \leq |x|/n} (\sin t) \cdot t^{-1} dt \). Here, the role of \( \phi(u) \) is played by \(-i \cdot \text{sg}(u) \cdot \pi\); the role of \( k_n \) is played by \(-2i \cdot \int_{|w|/n}^{+\infty} (\sin t) \cdot t^{-1} dt \cdot \text{sg}(u)\); the role of \( \varepsilon_n \) are played by the natural numbers \( \{n\} \).

Finally, the function \( \int_{|w|/n}^{+\infty} (\sin t) \cdot t^{-1} dt \) and its derivative in the distributions sense belong to \( L^p \) for all \( p_0 \) such that \( 1 < p_0 \leq 2 \). Now an application of Theorem 1 will give the well known results concerning pointwise convergence of the Hilbert Singular Integral in \( L^p, p > 1 \).

(5.2) If \( K_n \) and \( \tilde{K}_n \) denote the Féjer Kernel and its conjugate, respectively, then

\[ \hat{K}_n = (1 - |u/n|)^{+} \quad \text{and} \quad \tilde{K}_n = (1 - |u/n|)^{+} \quad (\text{sg}(u) \cdot \pi).\]

Here the roles of \( \hat{k}_n, \phi \) and \( \varepsilon_n \), are played by \( \hat{K}_n, \int \{n\} \) and \( \tilde{K}_n, (-i \pi \cdot \text{sg}(u) \cdot \int \{n\} \) respectively. Finally, since \( (1 - |u|)^{+} \) and its derivative in the distributions sense belong to \( L^p \) for all \( p_0 \) such that \( 1 < p_0 \leq 2 \), then the same conclusions as in (5.1) hold.

Remark. Analog considerations are valid for the Poisson Kernel and its conjugate.

6) Theorem 2. Let \( k(x) \) be a function belonging to \( L^1(R^m) \) and submitted to the following two conditions
i) $\int_{\mathbb{R}^m} k(x) \, dx = 1.$

ii) There exists $p_0 > 0$ such that $k(x) \prod_{j=1}^{m} (1 + |x_j|) \in L^{p_0}.$

If $K$ is a singular integral operator with symbol belonging to $H,$ we shall denote by $\tilde{f} = K(f).$ By $k(nx)$ we denote $k(n x_1, \ldots, n x_m).$

Under the two preceding assumptions we have

a) $\int_{\mathbb{R}^m} n^m k(nx) f(y - x) \, dx \to f(y) \text{ a.e. for all } f \in L^p; \ p \geq p_0^*; 1/p_0 + 1/p_0^* = 1.$

b) $\int_{\mathbb{R}^m} n^m \tilde{k}(nx) f(y - x) \, dx \to \tilde{f}(y) \text{ a.e. for all } f \in L^p; \ p \geq p_0^*.$

Calling $f^* = \sup_n \int_{\mathbb{R}^m} n^m k(nx) f(y - Tx) \, dx$ and $\tilde{f} = \sup_n \int_{\mathbb{R}^m} n^m \tilde{k}(nx) f(y - x) \, dx$ we have the inequalities

c) $|E(f^* > \lambda)| \leq (C_0/\lambda^{p_0^*}) \int_{\mathbb{R}^m} |f|^\lambda^* \, dx; \ |E(\tilde{f} > \lambda)| \leq (C_0^1/\lambda^{p_0^*}) \int_{\mathbb{R}^m} |f|^\lambda^* \, dx.$

d) If $p > p_0^*,$ then $\|f^*\|_p < C(p) \|f\|_p; \ \|\tilde{f}\|_p < C'(p) \|f\|_p$ and therefore the convergence in mean of order $p$ of a) and b) is valid.

Proof. If $f \in \mathcal{S},$ then

(6.1) $\int_{\mathbb{R}^m} n^m k(nx) f(y - x) \, dx = (2\pi)^{-m} \int_{\mathbb{R}^m} e^{i \langle y, u \rangle} \tilde{k}(u/n) \hat{f}(u) \, du.$

Since $\|k(u/n)\|_{\infty} \leq \|k\|_1$ and the fact that $k(u/n) \to 1$ for each $u,$ it follows that, for $f \in \mathcal{S},$

(6.2) $\int_{\mathbb{R}^m} n^m k(nx) f(y - x) \, dx \to f(y).$

Now, $n^m k(nx)$ and $k(x)$ are respectively under the conditions of the $F_n$ and $\Psi_n$ of Lemma 2; the condition ii) implies the condition (2.1) of Lemma 2. Finally, since $(n, \ldots, n)$ are in the conditions of iv) Lemma 2, the maximal inequalities c) and d) with respect to $\hat{f}$ follow. A combination of (6.2) and the maximal inequalities gives a) and also the convergence in the mean of order $p$ for all $p \geq p_0^*.$ Now, if $f \in \mathcal{S}$
(6.3) \[ \int_{n^m} k(nx) f(y-x) \, dx = (2\pi)^{-m} \int e^{i\langle u, x \rangle} \hat{f}(u) \phi(u) \, du \]

\[ \hat{k}(s/n) \hat{f}(y-x) \, dx \]

The representation (6.3) may be extended by continuity to all \( f \in L^p \) with \( p \geq p_0 \), since \( n^m k(nx) \) belongs to \( L^q \) for all \( q \) such that \( q \leq p_0 \). (A similar argument was given in (4.12 - 13 - 14 - 15).) Now the representation proved above gives the results concerning \( n^m \int_{n^m} \hat{k}(nx) f(y-x) \, dx \).

7) **Remark.** A large family of Féjer-like kernels are particular cases of Theorem 2, namely: the \( m \)-dimensional Poisson kernel and its conjugates by the Marcel Riesz Transform, the multiple Féjer kernel, the multiple Weierstrass kernel, etc.

8) **Remark.** Example 1 shows that there exists a kernel \( k \) under the conditions of Theorem 2 such that, for all \( f \in L^p \), \( p > 1 \),

\[ \int_{[|x-y| > 1/n]} \frac{f(y)}{n} \, dy = \int_{-\infty}^{+\infty} n[k(n(x-y))] \hat{f}(y) \, dy \]

The kernel is precisely the function whose Fourier Transform is \( (2/\pi) \int_{[|x|]} (\sin t) t^{-1} \, dt \).

9) **Remark.** Another type of Féjer-like kernels is studied in [8] (see Lemma (1.5), Part I) and also in [1].

10) **Singular Integrals of Odd Non-homogeneous Kernel.** Let \( k(x) \) be a measurable and odd function defined on the real line, belonging to \( L^2 \), submitted to the following conditions

i) \( k(0+) \) and \( k(0-) \) exist and are different from zero.

ii) \( k(\lvert u \rvert) \) and its derivative in the distributions sense belong to \( L^{p_0} \) for some \( p_0 \) such that \( 1 < p_0 \leq 2 \). Now let \( S(x) \) be an odd homogeneous function of degree \( (m-1) \), defined on \( \mathbb{R}^m \) such that

\[ \int_{\mathbb{R}^m} |S(x)| \, dx < \infty \]

If \( K(x) = S(x) \cdot k(|x|) \); then we call Old Singular Integral of nonhomogeneous kernel to
10.2) \[ \int_{-\infty}^{\infty} K(ny) f(x - y) \, dy \]

11) **Lemma.** Let \( k(x) \) be under the conditions of 10), then the operators

(11.1) \[ k_n(f) = \int_{-\infty}^{\infty} \frac{1}{n} \, k(n(x - y)) f(y) \, dy \]

have the properties

i) If \( f \in L^p, \frac{1}{p} \geq \frac{1}{p_0} \), then

\[ k_n(f) \to T(f) \text{ a.e. where } T(f) \text{ is a multiple of the single Hilbert Transform.} \]

ii) If \( f \in L_{p_0} \) then

\[ |E(\sup_n |k_n(f)| > \lambda)| \leq \lambda^{-p_0} \int |f|^{p_0} \, dx \]

iii) If \( p > p_0 \) then

\[ \| k_n(f) \|_p < C(p) \| f \|_p \]

Proof. Let us consider the function \( \phi(u) = k(0+) \) if \( u > 0 \) and \( \phi(u) = k(0-) \) if \( u < 0 \). Since \( k(u) \) is odd we have

(11.2) a) \( \phi^{-1} \hat{k}(u) = c_0 \hat{k}(u) \)

b) \( \phi = c' I \) (where \( I \) is the symbol of the single Hilbert Transform).

Now the corresponding multipliers of the operators (11.2) are

(11.3) \[ c \phi(u) \hat{k}(|u/n|) \]

Thus, taking into account (10,ii) the multipliers (11.3) are in the conditions of Theorem 1, with \( \varepsilon_n = n \), since the condition (10 ii) also shows that \( \hat{k}(|x|) \) is the Fourier Transform of a function belonging to \( L^{p_0*} \cap L^1 \) and therefore

(11.4) \[ \int_{-\infty}^{\infty} \hat{k}(u/n) - \phi \hat{f}(u) \, du = \int_{-\infty}^{\infty} c \hat{k}(|u/n|) \]

\[ -1 \| \phi \|_2 \| f \|_2 \to 0 \] from the boundedness of \( \hat{k}(|u|) \) and the continuity at \( u = 0 \). Now, an application of Theorem 1 gives i), ii) and iii).

12) **Theorem 3.** The operators \( K_n(f) \) defined in (10.2) converge pointwise, almost everywhere and in mean of order \( p \) to a limit operator \( K(f) \), for all \( f \) belonging to \( L^p; p_0 < p < \infty \). Furthermore

i) \[ \sup_n \| K_n(f) \|_p < C(p) \| f \|_p; p_0 < p < \infty. \]
Proof. We shall use the "method of rotation" introduced in [2].

\begin{equation}
\int n^m K(nx) f(y - x) \, dx
\end{equation}

(12.1)

Taking polar coördinates and using the fact that \( K \) is odd, (12.1) is readily seen to be equal to

\begin{equation}
\int \frac{1}{2} | S(a) | \, d\sigma \int_{-\infty}^{\infty} n \, k(n \rho) \, f(y - \rho a) \, d\rho
\end{equation}

(12.2)

If \( \sup_n | k_n(f) | = \hat{k}(f) \), then the inner integral of (12.2) is dominated in modulus by

\begin{equation}
\sup_n \left| \int_{-\infty}^{\infty} n \, k(n \rho) \, f(y - \rho a) \, d\rho \right| = \hat{k}(f(a, s, u)) \, (R)
\end{equation}

(12.3)

where \((s, R)\) are the coördinates of the point \( y \) in the system defined by the direction of \( a \) and a hyperplane \((m - 1)\) dimensional orthogonal to the same direction. Now

\begin{equation}
\int s^n \left( \sup_n \left| \int_{-\infty}^{\infty} n \, k(n \rho) \, f(y - \rho a) \, d\rho \right| \right)^p \, dy =
\end{equation}

(12.4)

\begin{equation}
= \int ds \int_{-\infty}^{\infty} k(f(s, a)) \, (p(R)) \, dR \leq \int_{-\infty}^{\infty} ds \, C(p) \int_{-\infty}^{\infty} dy \left( f(s + aR) \right)^{1/p} \, dR = C(p) \left\| f \right\|_p^p
\end{equation}

Taking into account that

\begin{equation}
\sup_n | K_n(f) | \leq \int \frac{1}{2} | S(a) | \, d\sigma.
\end{equation}

(12.5)

From (12.4) and using the Minkowski Integral Inequality we have

\begin{equation}
\left\| \sup_n | K_n(f) | \right\|_p \leq \left( (C(p))^{1/p} \right) \int \left( (1/2) | S(a) | \, d\sigma \right) \left\| f \right\|_p
\end{equation}

(12.6)

(12.6) shows that the integrals (12.1) always exist a.e. and also proves part i) of the Thesis. Now, we are going to prove the pointwise convergence in a dense subset.

Let us observe that for \( f \in D \)

\begin{equation}
k_n(f) = \int_{-\infty}^{\infty} n \, k(ny) \, f(x - y) \, dy = \int_{-\infty}^{\infty} n \, \hat{k}(ny) \, k(f)(x - y) \, dy
\end{equation}

(12.7)

where \( k \in L^1 \) and is precisely the function whose Fourier Trans-
A form is \( c_0 k(\left| u \right|) \), see (11.3), and \( k(f) \) is a multiple of a single Hilbert Transform. Therefore

\[
(12.8) \quad \| k_n(f) \|_\infty < A \| \tilde{f} \|_\infty
\]

Now if \( f \in D \) in \( \mathbb{R}^n \) we have

\[
(12.9) \quad K_n(f) = \int_{x} (1/2) \left| S(a) \right| \, d\sigma \left[ \int_{-\infty}^{\infty} n k(n\rho) f(y - \rho a) \, d\rho \right]
\]

According to (12.8) the inner integral is uniformly bounded by

\[
(12.10) \quad A \sup_{a} \left| \int_{-\infty}^{\infty} f(y - \rho a) \cdot \rho^{-1} \, d\rho \right|
\]

Since the inner integral converges pointwise, the bound (12.10) gives the pointwise convergence of \( K_n(f) \). The above argument together with the maximal inequalities already shown complete the proof of Theorem 3.

**Remark.** If we take \( k(x) = 1/x \) if \( |x| > 1 \) and zero otherwise, the integrals of (10.2) become truncated singular integrals of odd kernel. See [2] and also [6].

**REFERENCES**


