SOME REMARKS ON THE POINTWISE CONVERGENCE OF SEQUENCES OF MULTIPLIER OPERATORS

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Introduction. The purpose of the present paper is to give some sufficient conditions for the pointwise convergence of operators of the form

$$\int_{R^m} K_n(x-y)\ f(y)\ dy$$

The kernels K_n need not belong to L^1. Some classes of singular integrals as well as a large family of Féjer-like kernels may be regarded as particular cases of theorems 1, 2, 3. The main results are contained in theorems 1, 2 and 3 and some corollaries and examples are given thereaffer.

Definitions and Notation.

1) $f * g$ will denote the convolution between f and g; namely

$$\int_{R^m} f(x-y)\ g(y)\ dy = \int_{R^m} f(x_1-y_1, \dots, x_m - y_m)\ g(y_1, \dots, y_m)\ dy_1 \dots dy_m$$

R^m denotes the euclidean m-dimensional space.

2) By L^p we shall denote the class of all measurable functions defined on R^m such that $\int_{R^m} |f|^p\, dx = ||f||_p^p < \infty; p \geqslant 1$.

3) By $E(f > \lambda)$; $f \geqslant 0$, we shall denote the set of points of R^m such that $f > \lambda$ and by $|E(f > \lambda)|$ its measure.

4) A multiplier operator (*) acting from L^2 to L^2 is a linear operator such that $\widehat{T(f)} = k.\hat{f}$ for all $f \epsilon L^2$.

$\hat{f}(u) = \int_{R^m} \exp(-i <u, x>)\ f(x)\ dx; <u, x> = \sum_{j=1}^{m} x_j u_j$

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(*) we shall be concerned with multiplier operators acting always from L^2 into L^2.

5) By H_∞, H, H_0 we shall denote respectively the classes of bounded functions ϕ defined on R^m homogeneous of degree zero, such that

a) $\phi \in C^\infty$ if $x \neq 0$; b) ϕ is the symbol of a singular integral operator K acting continuously from L^p into L^p for all $p > 1$; c) No further restriction is imposed on ϕ.

By $K(f)$ we shall always denote the operator defined by the multiplier ϕ. In cases a) and b) the corresponding K is a singular integral operator.

1) Theorem. Let $T_{n_1 \ldots n_m} = T_n$ be a sequence of multiplier operators and $\hat{k}_n(u) = \hat{k}_{n_1 \ldots n_m}$ their corresponding multipliers; suppose that

i) There exists a function ϕ, bounded and homogeneous of degree zero such that

$$\int_{R^m} | k_n(u) - \phi(u) |^2 | g(u) |^2 \, d\hat{u} \to 0 \text{ as } n \to \infty$$

[for all g belonging to L^2]

ii) There exists a sequence $\epsilon_n = (\epsilon^1{}_{n_m}, \ldots, \epsilon^m{}_{n_m})$ of real and positive parameters such that

a) $\epsilon_n \to \infty$; $\epsilon^j{}_{n_j} \neq 0$ for all n_j and for all j.

b) The functions $\Psi_n(u) = \hat{k}_{n_1 \ldots n_m}(\epsilon^1{}_{n_1} u_1, \ldots,$

$$\epsilon^m{}_{n_m} u_m) \, \phi^{-1}(\epsilon^1{}_{n_1} u_1, \ldots, \epsilon^m{}_{n_m} u_m).$$

have the following compacity property

$$\left\| \frac{\partial^{s_1 + \ldots + s_m} \Psi_n}{\partial^{s_1} u_1 \cdots \partial^{s_m} u_m} \right\|_{p_0} \leqslant M$$

for some p_0 such that $1 < p_0 \leqslant 2$ and for all $s = (s_1, \ldots, s_m)$ submitted to the conditions $0 \leqslant s_1 + \ldots + s_m \leqslant m$, where each s_j can take the values 0 or 1 only. The constant M does not depend on n and the derivates are taken in the distributions sense.

Under the above assumptions we have

A) If $p_0 < 2$ and ϕ belongs to the class H_∞ or H then

A$_1$) $\| T_n f - K(f) \|_p \to 0$ for all $f \in L^p$; $p > p_0$.

A$_2$) If $f \in L^p$, $p > p_0$ then $T_n f$ converges $a.e.$ to $K(f)$; furthermore the operator $\sup\limits_{n_1 \dots n_m} | T_{n_1 \dots n_m} f | = T^* f$ verifies $\| T^* f \|_p < C(p) \| f \|_p, p > p_0$ where $C(p)$ depends on p only.

B) If ϕ belongs to H_0 and $p_0 < 2$ then

B$_1$) $\| T_n f - K(f) \|_2 \to 0$ for all $f \in L^2$.

B$_2$) $T_n f$ converges a.e. to $K(f)$ and furthermore, if $\sup\limits_n | T_n f | = T^* f$ we have $\| T^* f \|_2 < C \| f \|_2$.

C) Limiting Cases. Let $p = p_0$ and ϕ belong to H or H_∞; if $f \in L^{p_0}$ and $| K(f) |^{p_0} \log^+ (K(f))^{m-1}$ belongs locally to L^1 then

C$_1$) $T_n f$ converges a.e. to $K(f)$.

C$_2$) *If* $| E | < \infty$ we have, for $1 \leqslant s < p_0$
$\int_E | T^* f |^s dx \leqslant 0_1 (s, | E |) + 0_2 (s, | E |) \int_E K(f)^{p_0} (\log^+ |K(f)|)^{m-1} dx.$

C$_3$) If $| K(f) |^{p_0} (log^+ | K(f) |)^m$ belongs locally to L^1 and $| E | < \infty$, then $\int_E (T^* f)^{p_0} dx < A_1(| E |) + A_2(| E |) \int_E | K(f) |^{p_0} (\log^+ | K(f) |)^m dx.$

D) If $p_0 = 2$ and ϕ belongs to H_0, the same conclusions of C are valid.

E) If the sequence $\epsilon_n = (\epsilon^1_{n_1}, \epsilon^2_{n_2}, \dots, \epsilon^m_{n_m})$ of ii) is such that each $\epsilon^j_{n_j}$ is strictly increassing to ∞, then

E$_1$) If ϕ belongs to H or H_∞, we have

$$| E \{ T^* f > \lambda \} | < (C / \lambda^{p_0}) \int_{R^m} | f |^{p_0} dx, f \in L^{p_0}$$

and consequently $T_n f$ converges a.e. to $K(f)$.

E$_2$) If ϕ belongs to H_0 and $p_0 = 2$ we have that the same conclusions of E_1 are valid.

F) For ϕ belonging to H or H_∞, $T_n f$ admits the representation

F$_1$) $T_n f = \int_{R^m} K_n(x - y) f(y)$ dy for all $f \in L^p$; $p \geqslant p_0$; K_n belongs to $L^{p^*_0}$ for all n; $1/p_0 + 1/p^*_0 = 1$.

F$_2$) If ϕ belongs to H_0 and $f \in L^2$, we have $T_n f = \int_{R^m} K_n(x - y) f(y)$ dy where $K_n \in L^2$ for all n.

G) The kernels K_n belong to L^1 and $\| K_n \|_1 \leqslant M$ if and only if ϕ reduces to a constant.

H) If μ is a regular measure with bounded variation, defined on the Borel subsets of R^m then the following multipliers

$$\bar{\hat{k}}_n(u) = \int_{R^m} \hat{k}_n(u-x)(u-x)\ d\mu(x) = \bar{\hat{T}}^{n_1,\ldots,n_m}$$

verify A, B and also F with

$\bar{T}_n f = \int_{R^m} K_n(x-y)\ g(x-y)\ f(y)\ dy$ where the $K_n(x)$ are those associated to $\hat{k}_n$ and $g(x)$ is the Fourier Transform of μ.

The limit operator is defined by the multiplier

$$\bar{\phi}(u) = \int_{R^m} \phi(u-x)\ d\mu(x).$$

The key to the most important part of the proof of the above Theorem is the following lemma.

2) *Lemma*. Let $F_n(x) = F_{n_1 \ldots n_m}(x_1, \ldots, x_m)$ be a denumerable family of measurable functions defined on R^m, and let $\lambda_n = (\lambda^1{}_{n_1}, \ldots, \lambda^m{}_{n_m})$ be an m-dimensional sequence of real positive parameters such that the auxiliary functions

$$\Psi_n = F_{n_1,\ldots,n_m}(\lambda^1{}_{n_1} x_1, \ldots, \lambda^m{}_{n_m} x_m) \prod_{j=1}^{m} (\lambda^j{}_{n_j})$$

verify the following compacity property

$$(2.1) \qquad \| \Psi_n \prod_{j=1}^{m} |x_j|^{s_j} \|_{p_0} \leqslant M\ ,\ p_0 > 1$$

where M does not depend on n or $s = (s_1, \ldots, s_m)$ and each s_j can take the values 0 or 1 only. Then, *if* $f^* = \sup_n |F_n * f|$ we have

i) $\| f^* \|_p < C(p) \| f \|_p$ for all $p > p_0^*$; $1/p_0 + 1/p_0^* = 1$.

ii) If $|f| \in L^{p_0^*}$ and $|f|^{p_0^*} (\log^+ |f|)^{m-1}$ belongs locally to L^1; then, for all measurable set E such that $|E| < \infty$ we have

$$\int_E f^{*r}\ dx \leqslant A(r, |E|) + B(r, |E|) \int_E |f|^{p_0^*} (\log^+ |f|)^{m-1}\ dx$$

where $1 \leqslant r < p_0^*$

iii) If $f \in L^{p_0^*}$ and $|f|^{p_0} (\log^+ |f|)^m$ belongs locally to L^1; then, for all measurable set E such that $|E| < \infty$

$$\int_E f^{*p_0^*}\ dx \leqslant A'(|E|) + B'(|E|) \int_E |f|^{p_0^*} (\log^+ |f|)^m\ dx$$

iv) If the $\lambda^j{}_{n_j}$ are strictly decreasing to 0 for each j, we have

$$|E(f^* > \lambda)| < (C_0/\lambda^{p_0^*}) \int_{R^m} |f|^{p_0^*}\ dx$$

Proof. Without loss of generality we may assume that the F_n are zero on the complement of the set $\{ x_1 \geqslant 0, x_2 \geqslant 0, \ldots, x_m \geqslant 0 \}$.

Let f be a continuous and compact supported function; then, by a change of variables we have

$$(2.2)\qquad \int_{R^m} F_n(x-y)\, f(y)\, dy = \int_{R^m} \Psi_n(y)\, f(x-\lambda_n y)\, dy =$$

$$= \int_{R^m} \Psi_n(y_1, \ldots, y_m) f(x_1 - \lambda_{n_1} y_1, \ldots, x_m - \lambda_{n_m} y_m)\, dy_1 \ldots dy_m$$

The modulus of the last integral is dominated by

$$(2.3)\qquad \sum_{0\cdots 0}^{\infty\ldots\infty} \int \cdots \int_{S_{k_1 \ldots k_m}} |\Psi_n(y)|\, |f(x-\lambda_n y)|\, dy$$

The sets $S_{k_1, \ldots, k_m}$ are defined in the following way

$$(2.4)\qquad S_{k_1, \ldots, k_m} = [2^{k_1}, 2^{k_1+1}] \times \cdots \times [2^{k_m}, 2^{k_m+1}] \text{ for } k_j > 0,$$

$j = 1, \ldots, m$. If $k_j = 0$ the corresponding interval is $[0, 2]$.

By an application of Hölder's inequality, the series (2.3) is dominated by

$$(2.5)\qquad \sum_{0\cdots 0}^{\infty\ldots\infty} \Big(\int_{S_{k_1 \ldots k_m}} |\Psi_n|^{p_0}\, dy\Big)^{1/p_0} \Big(\int_{S_{k_1 \ldots k_m}} |f(x-\lambda_n y)|^{p_0^*}\, dy\Big)^{1/p_0^*}$$

Estimates for the terms $t_{k_1, \ldots, k_m} = \Big(\int_{S_{k \ldots k_m}} |\Psi_n|^{p_0}\, dy\Big)^{1/p_0}$

If $k_j > 0$, $j = 1, \ldots, m$; then

$$(2.6)\qquad 2^{k_1 + \ldots + k_m}\, t_{k_1, \ldots, k_m} \leqslant \Big\{ \int_{S_{k_1 \ldots k_m}} \Big(|\Psi_n| \prod_{j=1}^{m} |x_j| \Big)^{p_0} dx \Big\}^{1/p_0} \leqslant M_0.$$

The last inequality holds from Hypothesis (2.1). Then we must have

$$(2.7)\qquad t_{k_1, \ldots, k_m} \leqslant M_0\, 2^{-k_1 - k_2 - \ldots - k_m}$$

If the m-tuple $(k_1, \ldots, k_m)$ contains zeros in its coordinates, according to Hypothesis (2.1) we have

$$(2.8)\qquad 2^{s_1 k_1 + \ldots + s_m k_m}\, t_{k_1, \ldots, k_m} \leqslant \Big(\int_{S_{k_1 \ldots k_m}} \Big(|\Psi_n| \cdot \prod_{j=1}^{m} |x_j|^{s_j} \Big)^{p_0} dx \Big)^{1/p_0} \leqslant$$

$\leqslant M_0$ where $s_j = 1$ if $k_j \neq 0$, $s_j = 0$ if $k_j = 0$.

Now we conclude that in every case

$$(2.9)\qquad t_{k_1,\ldots,k_m} \leqslant M_0\, 2^{-k_1-\ldots-k_m}$$

Estimates for the terms $\displaystyle \Big(\int_{S_{k_1\ldots k_m}} | f(x-\lambda_n y) |^{p_0^*} dy \Big)^{1/p_0^*}$

$$(2.10)\qquad \Big(\int_{S_{k_1\ldots k_m}} | f(x-\lambda_n y) |^{p_0^*} dy \Big)^{1/p_0^*} \leqslant \Big(\int_{-2^{k_1+1}}^{2^{k_1+1}} \cdots \int_{-2^{k_m+1}}^{2^{k_m+1}} | f(x- \lambda_n y) |^{p_0^*} dy \Big)^{1/p_0^*}$$

The second (right hand) member of (2.10), by a change of variables, is readily seen to be equal to

$$(2.11)\qquad \left[\frac{\big(\prod_{j=1}^{m} \lambda^{n}_{j} \big)^{-1}}{2^{2m} \prod_{j=1}^{m} 2^{k_j}} \int_{-\lambda^{n}_{1} . 2^{k_1}}^{\lambda^{n}_{1} . 2^{k_1}} \cdots \int_{-\lambda^{m}_{n_m} . 2^{k_m}}^{\lambda^{m}_{n_m} . 2^{k_m}} | f(x-y) |^{p_0^*} dy \right]^{1/p_0^*} . \, 2^{2m/p_0^*} . \, 2^{(1/p_0^*)(k_1+\ldots+k_m)}$$

If $M(f)$ is the maximal operator of the strong differentiation (see [4], pp. 306 - 307) we have

$$(2.12)\qquad \Big(\int_{S_{k_1\ldots k_m}} | f(x-\lambda_n y) |^{p_0^*} dy \Big)^{1/p_0^*} \leqslant 2^{2m/p_0^*}\, 2^{(1/p_0^*)(k_1+\ldots+k_m)} . \{ M[| f |^{p_0^*}](x) \}^{1/p_0^*}$$

Now from (2.9) and (2.12) we have

$$(2.13)\qquad | F_n * f | \leqslant M_0\, 2^{2m/p_0^*} \Big(\sum_{0\cdots 0}^{\infty\ldots\infty} 2^{(-1/p_0)(k_1+\ldots+k_m)} \Big) . \{ M(| f |^{p_0^*}) \}^{1/p_0^*}$$

Consequently

$$(2.14)\qquad f^* \leqslant M_0 2^{2m/p_0^*} \Big(\sum 2^{(1/p_0)(k_1+\ldots+k_m)} \Big) \{ M(| f |^{p_0^*}) \}^{1/p_0^*}$$

and $\qquad f^{*p_0^*} \leqslant C(p_0)\, M\{ | f |^{p_0^*} \}$

If $p > p_0^*$ then $p = r p_0^*$ with $r > 1$, and from the Jessen-Marcinkiewicz-Zygmund Theorem (see [4], pp. 306 - 307) we have

$$(2.15)\qquad || f^* ||_p^p = \int_{R_m} (| f^* |^{p_0^*})^r dx \leqslant C(p_0) \int_{R_m} \{ M(| f |^{p_0^*}) \}^r dx \leqslant$$
$$\leqslant C(p_0)\, C'(r) \int_{R_m} | f |^{p_0^* r} dx = C(r, p_0) \int_{R_m} | f |^p dx = C(p, p_0)\, || f ||_p^p .$$

Now the inequality (2.15) can be extended by continuity to all $L^p, p > p_0^*$ (see [5], p. 165) and part i of the Thesis follows. Taking $f \in L^{p_0^*}$, an application of the Jessen-Marcinkiewicz-Zygmund Theorem to the function $| f |^{p_0^*}$ also gives ii) and iii).

Part iv) requieres a different technique. Let us return to (2.11) and consider

$$(2.16)\qquad f^{(n)}{}_{k_1,\dots,k_m} = \int_{-\lambda^1{}_{n_1}2^{k_1+1}}^{\lambda^1{}_{n_1}2^{k_1+1}} \cdots \int_{-\lambda^m{}_{n_m}2^{k_m+1}}^{\lambda^m{}_{n_m}2^{k_m+1}} |f(x-y)|^{p_0{}^*}\,dy\}$$

$$2^{-2m}\left(\prod_{j=1}^{m}\lambda_{n_j}2^{k_j}\right)^{-1}$$

Putting $f^*{}_{k_1,\dots,k_m} \underset{def}{=} \sup_n f^{(n)}{}_{k_1,\dots,k_m}$, we shall show that

$$(2.17)\qquad |E(f^*{}_{k_1,\dots,k_m} > \lambda)| < (C/\lambda)\int_{R_m} |f|^{p_0{}^*}\,dx$$

where C does not depend on $k_1,\dots,k_m$ or on the function f.

The λ_{n_j} are strictly decreasing to zero for each j, $j=1,\dots,m$.

Therefore we can construct a family of continuous and strictly decreasing to zero functions $h_j(t)$, $j=1,\dots,m$, such that there exists a sequence t_n of real numbers, strictly decreasing to zero, verifying

$$(2.18)\qquad \lambda_{n_j} = h_j(t_n) \qquad (j=1,\dots,m)$$

Now, the differentiation operators of (2.16) are in the same conditions as those of [4], p. 310, (3.5); therefore (2.17) follows. Let us rewrite (2.5) taking into account (2.16); namely

$$(2.19)\qquad |F_n * f| \leqslant M_0\,2^{2m/p_0{}^*} \sum_{0\cdots0}^{\infty\cdots\infty} 2^{(-1/p_0)\,(k_1+\dots+k_m)}$$

$$\cdot\,[f^{(n)}{}_{k_1,\dots,k_m}]^{1/p_0{}^*} \leqslant M_0\,2^{2m/p_0{}^*} \sum_{0\cdots0}^{\infty\cdots\infty} 2^{(-1/p_0)\,(k_1+\dots+k_m)}$$

$$[f^*{}_{k_1,\dots,k_m}]^{1/p_0{}^*}$$

Calling $a_{k_1,\dots,k_m} = 2^{(-1/p_0)\,(k_1+\dots+k_n)}\,[\sum_{0\cdots0}^{\infty\cdots\infty}$

$$2^{(-1/p_1)\,(k_1+\dots+k_m)}\,]^{-1}$$

and using the convexity of $u^{p_0{}^*}$ we have

$$(2.20)\qquad |F_n * f|^{p_0{}^*} \leqslant M_0{}^{p_0{}^*}2^{2m}\left[\sum_{0\cdots0}^{\infty\cdots\infty} 2^{(-1/p_0)\,(k_1+\dots+k_m)}\right]^{p_0{}^*}$$

$$\sum_{0\cdots0}^{\infty\cdots\infty} a_{k_1,\dots,k_m} f^*{}_{k_1,\dots,k_m}$$

and also

$$(2.21) \qquad f^{*p_0^*} \leqslant C \sum_{0\cdots 0}^{\infty\cdots\infty} a_{k_1,\ldots,k_m} f^*_{k_1,\ldots,k_m}$$

After a new normalization of the coefficients $a_{k_1,\ldots,k_m}$ such that $\sum_{0\cdots 0}^{\infty\cdots\infty} a^{1/2}_{k_1,\ldots,k_m} = 1$, the inequality (2.21) holds for such coefficients, but with a modification on the value of C. Now

$$(2.22) \qquad E\left(\sum_{0\cdots 0}^{\infty\cdots\infty} a_{k_1,\ldots,k_m} f^*_{k_1,\ldots,k_m} > \lambda\right) \subset \bigcup_{k_1\cdots k_m} E(a^{1/2}_{k_1,\ldots,k_m} f^*_{k_1,\ldots,k_m} > \lambda)$$

Therefore

$$(2.23) \qquad \left| E\left(\sum_{0\cdots 0}^{\infty\cdots\infty} a_{k_1,\ldots,k_m} f^*_{k_1,\ldots,k_m} > \lambda\right)\right| \leqslant \sum_{k_1\ldots k_m} \left| E(a^{1/2}_{k_1,\ldots,k_m} f^*_{k_1,\ldots,k_m} > \lambda)\right| \leqslant \sum_{k_1\ldots k_m} a^{1/2}_{k_1,\ldots,k_m} (A/\lambda) \int_{R_m} |f|^{p_0^*} dx = (A/\lambda) \int_{R_m} |f|^{p^*_0} dx$$

The last inequality of (2.23) follows from (2.17). Now, from (2.21) and (2.23) we have

$$(2.24) \qquad |E(\overset{*}{f} > \lambda)| = E(|\overset{*}{f}|^{p_0^*} > \lambda^{p_0^*}) < (A'/\lambda^{p_0^*}) . \int_{R_m} |f|^{p_0^*} dx.$$

This ends the proof of part iv).

3) *Remark.* The Hypothesis (2.1) can be replaced by a weaker one, namely

$$(3.1) \qquad \left\| \Psi_n \prod_{j=1}^{m} | x_j^{\beta} |^{s_j} \right\|_{p_0} < M_0;\ p_0 > 1;\ \beta > (p_0 - 1)/p_0$$

M_0 does not depend on n or on $s = (s_1, \ldots, s_m)$ and s_j can take the values 0 or 1 only; $j = 1, \ldots, m$.

The condition $\beta > (p_0 - 1)/p_0$ ensures the convergence of $\sum_{k_1\ldots k_m} a^{1/2}_{k,\ldots,k_m}$

4) *Proof of Theorem* 1. Since $\hat{k}_n \phi^{-1} \in L^{p_0}$ for some p_0 such that $1 < p_0 \leqslant 2$, there exists a function $F_n \in L^{p_0^*}$ such that

its Fourier Transform in the distributions sense coincides with $\hat{k}_n \cdot \phi^{-1}$. Furthermore, from the Hausdorff-Young-Titchmarsch Theorem it follows

$$(4.1) \qquad \| F_n \|_{p_0^*} \leqslant A(p_0) \, \| \hat{k}_n \, \phi^{-1} \|_{p_0}$$

Calling $\varphi_n = (1/ \prod_{j=1}^{m} \epsilon_{n_j}) \, F_{n_1, \ldots, n_m} \, (x_1 \epsilon^{-1}_{n_1}, \ldots, x_m \epsilon^{-1}_{n_m})$ we have that the Fourier Transform in the distributions sense of φ_n is Ψ_n and also

$$(4.2) \qquad \| \varphi_n \|_{p_0^*} \leqslant A(p_0) \, \| \Psi_n \|_{p_0}$$

By a similar procedure we obtain

$$(4.3) \qquad \| \varphi_n \prod_{j=1}^{m} | x_j |^{s_j} \|_{p_0^*} < A(p_0) . \| D^{s_1 + \cdots + s_m}_{s_1, \ldots, s_m} \Psi_n \|_{p_0} < M_0$$

M_0 does not depend on n or $s = (s_1, \ldots, s_m)$ and each s_j can take the values 0 or 1 only.

Now it is clear that the functions F_n with auxiliary functions φ_n are in the conditions of Lemma 2. If f belongs to the space $\mathcal{S}$ of C^∞ rapidly decreasing functions, then

$$(4.4) \qquad T^{n_1, \cdots, n_m}(f)^\wedge = \hat{k}_{n_1, \ldots, n_m} \cdot \hat{f} = \hat{k}_n \, \phi^{-1} . \phi \, \hat{f}$$

The Fourier Antitransform in the distributions sense gives

$$(4.5) \qquad T_{n_1, \ldots, n_m}(f) - F_n^* \, K(f)$$

Let us observe that $F_n \in L^p$ for all p such that $1 \leqslant p \leqslant p_0^*$; this follows from the fact that $\| F_n \prod_{j=1}^{m} | x_j |^{s_j} \|_{p_0^*} < \infty$ for all $s = (s_1, \ldots, s_m)$ such that each s_j takes the values 0 or 1 only.

On the other hand, we shall show that $\| F_n \|_1 \leqslant N_0$ where N_0 does not depend on n. In fact, from (2.5)

$$(4.6) \qquad \int_{R_m} | F_n(x) | \, dx = \int_{R_m} | \varphi_n(x) | \, dx \leqslant \int_{k_1 \ldots k_m} | \varphi_n(x) | \, dx \leqslant$$

$$\leqslant \sum_{k_1 \ldots k_m} t_{k_1, \ldots, k_m} \cdot 2^{2m/p_0} \, 2^{(1/p_0)(k_1 + \ldots + k_m)} \leqslant$$

$$\leqslant M_0 \, 2^{2m/p_0} \sum_{k_1 \ldots k_m} 2^{(-1/p_0^*)(k_1 + \ldots + k_m)} = N_0 .$$

If for some $p > 1$, $K(f)$ maps continuously L^p into itself, by using the Young inequality we can extend by continuity the representation (4.5) for all $f \in L^p$ and furthermore, we have

$$(4.7)\qquad \| T_{n_1, \dots, n_m}(f) \|_p = \| F_n * K(f) \|_p \leqslant \| F_n \|_1 \| K(f) \|_p \leqslant N_0 . C . \| f \|_p .$$

(4.8) As we have already shown, the F_n are on the conditions of Lemma 2 and, therefore, if $K(f)$ maps continuously L^p into itself, $\sup_n T_n f = \sup_n | F_n * K(f) |$ verifies similar inequalties to those proved for $\sup_n | F_n * f |$.

(4.9) On the other hand, $T_n f$ converges pointwise in a dense subset. In fact, if $f \epsilon \mathcal{S}$ we have

$$(4.10)\ T_n f = (1/(2\Pi)^m) \int_{R_m} e^{i<x,y>} \hat{k}_n(y) . \hat{f}(y)\, dy \text{ and}$$

$$(4.11)\ | T_n f - K(f) | \leqslant \int_{R_m} | \hat{k}_n - \phi | \, | \hat{f} | \, dy \leqslant (\int_{R_m} | \hat{f} | \, dy)^{1/2} . (\int_{R_m} | \hat{k}_n - \phi |^2 | \hat{f} | \, dy)^{1/2}.$$

Since $\hat{f} \epsilon \mathcal{S}$, then $| \hat{f} |^{1/2} \epsilon L^2$; therefore, the last term of the inequality tends to zero. (condition i)).

A suitable combination of the two preceding arguments (4.8) and (4.9) gives parts A, B, C, D and E. (see [4], p. 307 and [5] p. 160).

(4.12) If $f \epsilon L^p$ and $p > p_0$ we shall show that $T_n(f) = K_n * f$, where $K \epsilon L^{p_0 *}$.

Let us observe that $\hat{F}_n = \hat{k}_n \phi^{-1}$, that is $\phi \hat{F}_n =, \hat{k}_n$, and since K maps continuously L^p into itself for all $p > 1$ when ϕ belongs to H or H_∞, then in both cases we have

(4.13) $K(F_n) = K_n \epsilon L^p$ for all p such that $1 < p < p_0^*$ since $F_n \epsilon L^p$ for the same values of p.

On the other hand, if $f \epsilon \mathcal{S}$, taking (4.5) into account, we have (4.14) $T_n f = F_n * K(f) = K_n * f$.

Now we can extend (4.14) by continuity to all $L^p, p \geqslant p_0$, since both sides of the equality verify respectively

$$(4.15)\qquad | F_n * K(f) | \leqslant \| F_n \|_{p/(p-1)} . \| K(f) \|_p$$
$$| K_n * f | \leqslant \| K_n \|_{p/(p-1)} . \| f \|_p$$

Therefore part F_1 is proved.

Part F_2 is also valid since this case requires

$$\| K(f) \|_2 \leqslant A \|f\|_2 \text{ only.}$$

(4.16) *Proof. of Part G.* If $\phi = C_0$ then $F_n = C_0^{-1} K_n$; therefore

(4.17) $\| K_n \|_1 < M'$ where M' does not depend on n.

Let us suppose that $\| K_n \|_1 < M'$; consequently $\hat{k}_n$ must be continuous for each n, and furthermore

(4.18) $\phi = \hat{k}_n / \hat{F}_n$

(4.19) If $\hat{F}_n(0) \neq 0$ for some $n = n_0$; then ϕ must be continuous at $x = 0$; and since ϕ is a bounded homogeneous of degree zero function it must reduce to a constant.

(4.20) If $\hat{F}_n(0) = 0$ for all n we shall show that $K(f) = 0$ for all $f \epsilon L^2$.

In fact, let us consider a subsequence $\{ n' \}$ *of* $\{ n \}$ such that $F_{n'}$ *and* $\varphi_{n'}$ are in the conditions of iv) of Lemma 2. Then

(4.21) $| E(\sup_{n'} |F_{n'} * f| > \lambda) | \leqslant (C_0/\lambda^2) \int_{R^m} |f|^2 dx$

On the other hand, from the compacity conditions on the $\varphi_{n'}$, for each $\epsilon > 0$ the φ_n admit the decomposition

(4.22) $\varphi_n = \varphi_n^{(1)} + \varphi_n^{(2)} \quad ; \quad \| \varphi_n^{(2)} \|_1 < \epsilon$

$\varphi_n^{(1)} = 0$ if $x \bar{\epsilon} Q$, where Q is a cube centered at the origin and depends on $\epsilon > 0$ only. If $x \epsilon Q$, $| \varphi_n^{(1)}(x) | < M_2$, where M_2 depends on $\epsilon > 0$ only.

Thus, if $f \epsilon D$, by changing variables and taking into account that $\hat{F}_{n'}(0) = 0$, we obtain

$$(4.23) \quad | (F_{n'} * f)(x) | = | \int \varphi_{n'}(y) \, . f(x - \epsilon_{n'}^{-1} y) \, dy | =$$

$$= | \int_{R^m} \varphi_{n'}(y) [f(x) - f(x - \epsilon_{n'}^{-1} y)] \, dy | \leqslant$$

$$\leqslant \int_{Rm} | \varphi_{n'}^{(1)}(y) | \, . | f(x) - f(x - \epsilon_{n'}^{-1} y) | \, . \, dy + 2\epsilon \| f \|_\infty$$

where $f(x - \epsilon_{n'}^{-1} y) = f(x_1 - \epsilon_{n'_1}^{-1} y_1, \ldots, x_m - \epsilon_{n'_m}^{-1} y_m)$

Now if $Q_{\epsilon_{n'}}$ denotes the cube obtained from Q by dividing its edges by $\epsilon_{n'j}$ respectively, then the right and term of the inequality (4.23) is readily seen to be less or equal than

(4.24) $|Q|\, M_2 . (1/|Q_{\epsilon_{n'}}|) \int_{Q_{\epsilon_{n'}}} |f(x) - f(x-y)|\, dy + 2\epsilon\, \|f\|_\infty$

Therefore

(4.25) $\overline{\lim}\ |F_{n'} * f| \leqslant 2\epsilon\, \|f\|_\infty .$

Consequently, for all $f \in D$ it is valid

(4.26) $$\lim\ F_{n'} * f = 0$$

Thus, (4.26) together with (4.21) shows that $\lim F_{n'} * f = 0$ a.e. for all f belonging to L^2, whence $\lim\ F_{n'} * K(f) = 0$ a.e. for all f belonging to L^2. This completes the proof of Part G, since $F_{n'} * K(f) = K_{n'}(f)$ and by Hypothesis i) $\|\, K_{n'}(f) - K(f)\, \|_2 \rightarrow 0$.

(4.27) *Proof of Part H.* The $\hat{k}_n$ and ϕ are supposed to be Borel measurable. Let us consider the functions

$$\overline{k_n}(x) = \int_{R^m} \hat{k}_n(x-y) . d\mu(y) .$$

By an application of the Minkowski Integral Inequality we have

(4.28) $\|\overline{k_n}\|_{p_0} \leqslant (\int_{R^m} dw(y))\ \|\hat{k}_n\|_{p_0} \leqslant \mathrm{V}(\mu) . M'$

$dw(y)$ denotes the variation of μ and $\mathrm{V}(\mu)$ the variation in the whole R^m. Now if $g(x) = 1/(2\pi)^m \int_{R^m} e^{i<x,y>}\, d\mu(y)$, from (4.12), (4.13) and (4.28) we conclude that

(4.29) $\overline{k_n} = (K_n . g)^\wedge$

On the other hand if $f \in \mathcal{S}$, we have

(4.30) $\overline{T}_n f = (K_n . g) * f$

Since $\|g\|_\infty < V(\mu)$, $(K_n . g) * f$ verifies the same type of inequality as that of (4.15). Therefore, by the same density argument we conclude that

(4.31) $$\overline{T}_n f = \int_{R^m} K_n(x-y)\ g(x-y)\ f(y)\ dy$$

for all $f \in L^p$ with $p > p_0$ and ϕ belonging to H or H_∞. By a similar method, one obtains the representation for $f \in L^2$ and $\phi \in H_0$.

(4.32) Now we are going to show that $\overline{T}_n f$ converges pointwise for all $f \in \mathcal{S}$.

(4.33) $\overline{T}_n(f)(x) = (1/2\pi)^m \int_{R^m} e^{i<x,y>}$.

$\hat{f}(y) \left(\int_{R^m} \hat{k}_n(y-s)\, d\mu(s) \right) dy$

Interchanging the order of integration (which we may) we have

(4.34) $\int_{R^m} d\mu(s) . (1/2\pi)^m . \int_{R^m} e^{i<x,y>} \hat{k}_n(y-s)\, \hat{f}(y)\, dy$

By a similar procedure than that employed in (4.9), (4.10) and (4.11), it is easy to show that the inner integral of (4.34) converges pointwise and uniformly to

(4.35) $(1/2\pi)^m \int_{R^m} e^{i<x,y>} \hat{\phi}(y-s)\, \hat{f}(y)\, dy$

On the other hand, since $V(\mu) < \infty$, $\overline{T}_n(f)$ converges pointwise for all $f \in \mathcal{S}$ to the operator $T(f)$, whose multiplier is $\hat{k}_n * \mu$.

(4.36) Estimates for the maximal operator associated to $\overline{T}_n$. In this part we shall use a technique introduced by A. P. Calderón and A. Zygmund in [2]. Let ϕ belong to H or H_∞. Then, if $f \in L^p$, $p > p_0$, we have

(4.37) $\overline{T}_n f = \int_{R^m} K_n(x-y)\; g(x-y)\; f(y)\; dy =$

$= (1/2\pi)^m \int K_n(x-y) \left(\int e^{i<(x-y),s>} d\mu(s) \right) f(y)\; dy$

Interchanging the order of integration we have

(4.38) $(1/2\pi)^m \int_{R^m} d\mu(s)\; e^{i<x,s>} \int_{R^m} K_n(x-y)\; f(y)\; e^{i<s,y>} dy$.

The modulus of the integral (4.38) is dominated by

(4.39) $1/(2\pi)^m \int_{R^m} dw(s)\; [\overset{*}{T}(f\, e^{-i<s,y>})(x)]$

Taking the L^p norm of (4.39) with respect to x we obtain

(4.40) $(1/2\pi)^m . \left(\int [\int \overset{*}{T}(f\, e^{-i<s,y>})(x) \, . \, dw(s)]^p \, dx \right)^{1/p} \leqslant$

$\leqslant (1/2\pi)^m \int_{R^m} dw(s) \; [\int_{R^m} \overset{*}{T}(f\, e^{-i<s,y>})(x) \,|^p \, dx]^{1/p} \leqslant$

$\leqslant (V(\mu)/(2\pi)^m . C_p \,||\, f \,||_p$.

Thus

(4.41) $\qquad || \sup_n | \overline{T}_n f | \, ||_p \leqslant (V(\mu)/(2\pi)^m \cdot C(p) \cdot || f ||_p.$

A similar estimate holds for $\phi \in H_0$ and $f \in L^2$. (4.41) and (4.32) give the corresponding convergence results. This completes the proof of Part H.

5) Examples.

(5.1) Let us consider the single Hilbert - Transform, that is

$$\tilde{f} = \lim_{n \to \infty} \int_{|x-y|>1/n} f(y) \cdot (x-y)^{-1} \, dy$$

We know that $\int_{|y|>1/n} e^{-iuy} y^{-1} \, dy = -2i \; sg(u). \int_{|u/n|} (\sin t) \cdot t^{-1} \, dt$

Here, the role of $\phi(u)$ is played by $-i \; sg(u) \; \pi$; the role of $\hat{k}_n$ is played by $-2i \; (\int_{|u/n|}^{+\infty} (\sin t) \cdot t^{-1} dt) \; sg(u)$; the role of ϵ_n are played by the natural numbers $\{ n \}$.

Finally, the function $\int_{|u|}^{+\infty} (\sin t) \cdot t^{-1} \, dt$ and its derivate in the distributions sense belong to L^p for all p_0 such that $1 < p_0 \leqslant 2$. Now an application of Theorem 1 will give the well known results concerning pointwise convergence of the Hilbert Singular Integral in $L^p, p > 1$.

(5.2) If K_n and $\tilde{K}_n$ denote the Féjer Kernel and its conjugate, respectively, then $\hat{K}_n = (1 - | u/n |)_+$ and $\hat{\tilde{K}}_n = (1 - | u/n | |)_+ (-i \; sg(u) \cdot \pi.)$

Here the roles of $\hat{k}_n$, ϕ and ϵ_n, are played by $\hat{K}_n$, 1, $\{ n \}$ and $\hat{\tilde{K}}_n$, $(-i \, \pi \; sg(u))$, $\{ n \}$ respectively. Finally, since $(1 - |u|)_+$ and its derivate in the distributions sense belong to L^p for all p_0 such that $1 < p_0 \leqslant 2$, then the same conclusions as in (5.1) hold.

Remark. Analog considerations are valid for the Poisson Kernel and its conjugate.

6) *Theorem 2.* Let $k(x)$ be a function belonging to $L^1(R^m)$ submitted to the following two conditions

i) $\int_{R^m} k(x)\, dx = 1.$

ii) There exists $p_0 >$ such thet $k(x) \prod_{j=1}^{m} (1 + |x_j|) \in L^{p_0}$

If K is a singular integral operator with symbol belonging to H, we shall denote by $\tilde{f} = K(f)$. By $k(nx)$ we denote $k(nx_1, \ldots, nx_m)$. Nnder the two preceding assumptions we have

a) $\int_{R^m} n^m k(nx)\, f(y - x)\, dx \to f(y)$ a.e. for all $f \in L^p$; $p \geqslant p_0^*$; $1/p_0 + 1/p_0^* = 1$.

b) $\int n^m \tilde{k}(nx) f(y - x)\, dx \to \tilde{f}(y)$ a.e. for all $f \in L^p$; $p \geqslant p_0^*$ *Calling* $f^* = \sup_n |\int_{R^m} n^m k(nx)\, f(y - Tx)\, dx|$; $\overset{*}{\tilde{f}} = \sup_n |\int_{R^m} n^m \tilde{k}(nx)\, f(y - x)\, dx|$ we have the inequalities

c) $|E(f^* > \lambda)| \leqslant (C_0/\lambda^{p_0^*}) \int_{R^m} |f|^{p_0^*}\, dx$; $|E(\overset{*}{\tilde{f}} > \lambda)| \leqslant \leqq (C_0^1/\lambda^{p_0^*}) \int_{R^m} |f|^{p_0^*}\, dx$

d) If $p > p_0^*$, then $\|f^*\|_p < C(p)\, \|f\|_p$; $\|\overset{*}{\tilde{f}}\|_p < C'(p) \|f\|_p$ and therefore the convergence in mean of order p of a) and b) is valid.

Proof. If $f \in \mathcal{S}$, then

(6.1) $\int_{R^m} n^m k(nx)\, f(y - x)\, dx = (2\pi)^{-m} \int_{R^m} e^{i\langle y, u\rangle}\, \hat{k}(u/n)\, \hat{f}(u)\, du$

Since $\|\hat{k}(u/n)\|_\infty \leq \|k\|_1$ and the fact that $\hat{k}(n/u) \to 1$ for each u, it follows that, for $f \in \mathcal{S}$,

(6.2) $\int_{R^m} n^m k(nx)\, f(y - x)\, dx \to f(y)$.

Now, $n^m k(nx)$ and $k(x)$ are respectively under the conditions of *the* F_n and Ψ_n of Lemma 2; the condition ii) implies the condition (2.1) of Lemma 2. Finally, since $(n, \ldots, n)$ are in the conditions of iv) Lemma 2, the maximal inequalities c) and d) with respect to $\overset{*}{f}$ follow. A combination of (6.2) and the maximal inequalities gives a) and also the convergence in the mean of order p for all $p \geqslant p_0^*$. Now, if $f \in \mathcal{S}$

$$(6.3) \int_{R^m} n^m \tilde{k}(nx)\, f(y-x)\, dx = (2\pi)^{-m} \int_{R^m} e^{i<y,s>} \hat{f}(s)\, \phi(s)$$

$$\hat{k}(s/n)\, ds = \int n^m\, k(nx)\, \tilde{f}(y-x)\, dx$$

The representation (6.3) may be extended by continuity to all $f \in L^p$ with $p \geqslant p_0{}^*$, since $n^m\, k(nx)$ belongs to L^q for all q such that $q \leqslant p_0$. (A similar argument was given in (4.12 - 13 - 14 - 15)). Now the representation proved above gives the results concerning $n^m \int_{R^m} \tilde{k}(nx)\, f(y-x)\, dx$.

7) *Remark.* A large family of Féjer-like kernels are particular cases of Theorem 2, namely: the m - dimensional Poisson kernel and its conjugates by the Marcel Riesz Transform, the multiple Féjer kernel, the multiple Weierstrass kernel, etc.

8) *Remark.* Example 1 shows that there exists a kernel k under the conditions of Theorem 2 such that, for all $f \in L^p$, $p > 1$,

$$(8.1) \int_{|x-y|>1/n} [(f(y)/(x-y)]\, dy = \int_{-\infty}^{+\infty} n\, k[n(x-y)]\, \tilde{f}(y)\, dy$$

The kernel is precisely the function whose Fourier Transform is $(2/\pi) \int_{|u|}^{+\infty} (\sin t)\, t^{-1}\, dt$.

9) *Remark.* Another type of Féjer-like kernels is studied in [8] (see Lemma (1.5), Part I) and also in [1].

10) *Singular Integrals of Odd Non-homogeneous Kernel.* Let $k(x)$ be a measurable and odd function defined on the real line, belonging to L^2, submitted to the following conditions

i) *$\hat{k}(0+)$ and $\hat{k}(0-)$ exist and are different from zero.*

ii) *$\hat{k}(|u|)$ and its derivate in the distributions sense belong to L^{p_0} for some p_0 such that* $1 < p_0 \leqslant 2$. Now let $S(x)$ be an odd homogeneous function of degree $(m-1)$, defined on R^m such that

$$(10.1) \int_{|x|=1} |S(x)|\, d\sigma < \infty$$

If $K(x) = S(x)\,.\,k(|x|)$; then we call Old Singular Integral of nonhomogeneous kernel to

10.2) $\int_{R^m} n^m K(ny)\, f(x-y)\, dy$

11) *Lemma.* Let $k(x)$ be under the conditions of 10), then the operators

(11.1) $k_n(f) = \int_{-\infty}^{+\infty} n\, k[n(x-y)]\, f(y)\, dy$ have the properties

i) If $f \in L^p$, $p \geqslant p_0$, $1/p_0 + 1/p_0^* = 1$; then
$k_n(f) \rightarrow T(f)$ a.e. where $T(f)$ is a multiple of the single Hilbert Transform.

ii) If $f \in L^{p_0}$ then
$| E\, (\sup_n | k_n(f) | > \lambda) < (1/\lambda^{p_0}) \int |f|^{p_0} dx$

iii) If $p > p_0$ then $\| \sup_n | k_n(f) | \|_p < C(p) \| f \|_p$

Proof. Let us consider the function $\phi(u) = k(0+)$ if $u > 0$ and $\phi(u) = k(0-)$ if $u < 0$. Since $\hat{k}(u)$ is odd we have

(11.2) a) $\phi^{-1} \hat{k}(u) = c_0\, \hat{k}(u)$
b) $\phi = c'$ I (where I is the symbol of the single Hilbert Transform.

Now the corresponding multipliers of the operators (11.2) are

(11.3) $c\, \phi(u)\, \hat{k}(| u/n |)$

Thus, taking into account (10,ii) the multipliers (11.3) are in the conditions of Theorem 1, with $\epsilon_n = n$, since the condition (10 ii) also shows that $\hat{k}(| x |)$ is the Fourier Transform of a function belonging to $L^{p_0^*} \cap L^1$ and therefore

(11.4) $\int_{-\infty}^{+\infty} | \hat{k}(u/n) - \phi |^2 | \hat{f}(u) |^2 du = \int_{-\infty}^{+\infty} | c\, \hat{k}(| u/n |)$

$- 1 |^2 | \phi |^2 | \hat{f}(u) |^2 du \rightarrow 0$ from the boundedness of $\hat{k}(| u |)$ and the continuity at $u = 0$. Now, an application of Theorem 1 gives i), ii) and iii).

12) *Theorem 3.* The operators $K_n(f)$ defined in (10.2) converge pointwise, almost everywhere and in mean of order p to a limit operator $K(f)$, for all f belonging to L^p; $p_0 < p < \infty$. Furthermore

i) $\| \sup_n | K_n(f) | \|_p < C(p) \| f \|_p$; $p_0 < p < \infty$.

Proof. We shall use the "method of rotation" introduced in [2].

(12.1) $\int n^m K(nx) f(y-x)\, dx$

Taking polar coördinates and using the fact that K is odd,

(12.1) is readily seen to be equal to

(12.2) $\int_\Sigma (1/2) \mid S(\alpha) \mid d\sigma \int_{-\infty}^{+\infty} n\, k(n \rho) f(y - \rho\, \alpha)\, d\rho$

If $\sup_n |k_n(f)| = \overset{*}{k}(f)$, then the inner integral of (12.2) is dominated in modulus by

(12.3) $\sup_n \mid \int_{-\infty}^{+\infty} n\ k(n\rho)\ f(s + (R - \rho)\ \alpha)\ d\rho \mid = \overset{*}{k}\ (f(\rho, s, \alpha))\ (R)$

where (s, R) are the coördinates of the point y in the system defined by the direction of α and a hyperplane $(m-1)$ dimensional orthogonal to the same direction. Now

(12.4) $\int_{R^m} (\sup_n \mid \int_{-\infty}^{+\infty} n\ k(n\rho)\ f(y - \rho\alpha)\ d\rho\ |)^p\ dy =$

$= \int_{R^{m-1}} ds \int_{-\infty}^{+\infty} k(f(\rho, s, \alpha))^p(R)\ dR \leqslant \int_{R^{m-1}}^{+\infty} ds\ C(p) \int_{-\infty}$

$| f(s + \alpha R) |^p dR = C(p) \, \| f \|_p{}^p$

Taking into account that

(12.5) $\sup_n | K_n(f) | \leqslant \int_\Sigma (1/2 \mid S(\alpha) \mid d\sigma.$

$.\sup_n | \int_{-\infty}^{+\infty} n\ k(n\rho)\ f(y - \rho\alpha)\ d\rho \mid$

From (12.4) and using the Minkowski Integral Inequality we have

(12.6) $\| \sup_n | K_n(f) | \|_p \leqslant [(C(p)^{1/p} \int_\Sigma (1/2) \mid S(\alpha) \mid d\sigma] . \| f \|_p$

(12.6) shows that the integrals (12.1) always exist a.e. and also proves part i) of the Thesis. Now, we are going to prove the pointwise convergence in a dense subset.

Let us observe that for $f \in D$

(12.7) $k_n(f) = \int_{-\infty}^{+\infty} n\ k(ny)\ f(x-y)\ dy = \int_{-\infty}^{+\infty}$

$n\ \tilde{k}(ny)\ k(f)\ (x-y)\ dy$

where $k \in L^1$ and is precisely the function whose Fourier Transform is $c_0 \hat{k}(|u|)$, see (11.3), and $k(f)$ is a multiple of a single Hilbert Transform. Therefore

$$(12.8) \qquad \| k_n(f) \|_\infty < A \| \tilde{f} \|_\infty$$

Now if $f \in D$ in R^m we have

$$(12.9) \qquad K_n(f) = \int_\Sigma (1/2) \, | S(\alpha) | \, d\sigma \, [\int_{-\infty}^{+\infty} n \, k(n\rho) \, f(y - \rho\alpha) \, d\rho]$$

According to (12.8) the inner integral is uniformly bounded by

$$(12.10) \qquad A . \sup_{y,\alpha} | \int_{-\infty}^{+\infty} f(y - \rho\alpha) . \rho^{-1} \, d\rho |$$

Since the inner integral converges pointwise, the bound (12.10) gives the pointwise convergence of $K_n(f)$. The above argument together with the maximal inequalities already shown complete the proof of Theorem 3.

Remark. If we take $k(x) = 1/x$ if $|x| > 1$ and zero otherwise, the integrals of (10.2) become truncated singular integrals of odd kernel. See [2] and also [6].

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