ANOTHER NOTE ON SIMPLE STABLE POINTS IN TOPOLOGICAL LINEAR SPACES

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1. In our recent paper [2], we established some results concerning the simple stable points of games defined on separated, convex, compact, real topological linear spaces. We derived these results by using a generalization of a result given by Fan in [1], which is concerned with the intersection of sets with convex sections.

The object of this note is to prove some existence theorems for simple stable points of games given on convex, compact, real topological linear spaces, by using the same idea employed by Nikaido-Isoda [3] in order to prove the existence of an equilibrium point.

There is certain similarity between the results expressed in this paper and the respective results obtained by the mentioned technique. However, neither the results obtained here include the other, nor are included in them.

2. For our purpose, we need the basic result introduced in [3]. Its application will give the principal results.

THEOREM 1 (*Nikaido-Isoda*): Let ϕ be a real function defined on $\Sigma \times \Sigma$, where Σ is non-empty, convex and compact in a real topological linear space, such that the following two conditions are fulfilled:

(i) For each $\sigma \epsilon \Sigma$, the functions $\phi(\sigma, \tau)$ and $\phi(\tau, \tau)$ are continuous in $\tau \epsilon \Sigma$.

(ii) For each $\tau \epsilon \Sigma$, the function $\phi(\sigma, \tau)$ is concave in $\sigma \epsilon \Sigma$.

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Then there exists a point $\bar{\tau} \in \Sigma$ such that

$$\phi(\bar{\tau}, \bar{\tau}) = \max_{s \in \Sigma} \phi(s, \bar{\tau})$$

PROOF: Assume that there is not a point having the property just mentioned.

Then, for each $\tau \in \Sigma$, there is a $\sigma \in \Sigma$ such that

$$\phi(\tau, \tau) < \phi(\sigma, \tau).$$

Let

$$\theta_\sigma = \{\tau \in \Sigma : \phi(\tau, \tau) < \phi(\sigma, \tau)\}$$

be a set in Σ.

By the continuity of $\phi(\sigma, \tau)$ and $\phi(\tau, \tau)$ in $\tau \in \Sigma$ for each $\sigma \in \Sigma$ there exists a finite number of $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that

$$\bigcup_{i=1}^{n} \theta_{\sigma_i} = \Sigma.$$

Consider the functions

$$\rho_i(\tau) = \max\,[\phi(\sigma_i, \tau) - \phi(\tau, \tau),\ 0] \qquad \text{for } i = 1, \ldots, n.$$

Therefore by definition

$$\rho(\tau) = \sum_{i=1}^{n} \rho_i(\tau) > 0 \qquad \text{for all } \tau \in \Sigma.$$

Let

$$\psi(\tau) = \sum_{i=1} \frac{\rho_i(\tau)}{\rho(\tau)} \sigma_i$$

be a function

$$\psi : \Sigma \to \Sigma,$$

since Σ is convex.

The convex hull of $\sigma_1, \ldots, \sigma_n$ in Σ is homeomorphic to a simplex in an Euclidean space. Then, the application of Brower's fixed point to the function ψ guarantees the existence of a fixed point $\tilde{\tau}$:

$$\tilde{\tau} = \sum_{i=1}^{n} \frac{\rho_i(\tilde{\tau})}{\rho(\tilde{\tau})} \sigma_i$$

From the last conditions we obtain

$$\phi(\tilde{\tau}, \tilde{\tau}) > \phi(\tilde{\tau}, \tilde{\tau})$$

which is impossible. Q.E.D.

3. Let

$$\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$$

be a n-person game where, for each $i \in N = \{1, \ldots, n\}$, the set Σ_i is non-empty, compact, and convex in a real topological linear space and the payoff function A is defined on $\Sigma = \underset{i \in N}{X} \Sigma_i$ with values in the real numbers.

Let

$$e(i) \subseteq N - \{i\} \quad \text{and} \quad f(i) = N - (e(i) \cup \{i\})$$

be sets of players for each $i \in N$, and consider $\Sigma_R = \underset{j \in R}{X} \Sigma_j$ with R : $e(i)$ or $f(i)$.

A point $\bar{\sigma} \in \Sigma$ is said to be an $\underline{e_m}$ simple stable point of the game Γ if, for all $i \in N$,

$$\min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) =$$

$$= \max_{s_i \in \Sigma_i} \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(s_i, s_{e(i)}, \bar{\sigma}_{f(i)}).$$

Such a point can easily be characterized by the function

$$\Phi_1(\sigma, \tau) = \sum_{i \in N} F_i(\sigma_i, \tau_{f(i)}),$$

where, for each $i \in N$, the function F_i is defined by

$$F_i(\sigma_i, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}).$$

Lemma 2: A point $\bar{\sigma} \in \Sigma$ is a e_m simple stable point of the game Γ if and only if

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) = \max_{s \in \Sigma} \Phi_1(s, \bar{\sigma}).$$

Proof: Let $\bar{\sigma} \in \Sigma$ be a e_m simple stable point of the game Γ. Then, for each $i \in N$,

$$F_i(\sigma_i, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}),$$

and therefore,

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) = \max_{s \epsilon \Sigma} \Phi_1(s, \bar{\sigma}).$$

Now, examine the sufficiency. Let $\bar{\sigma} \epsilon \Sigma$ be such a point which fulfills, for each $\tau \epsilon \Sigma$,

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) \geqslant \Phi_1(\tau, \bar{\sigma}).$$

Suppose that there is a $\tau \epsilon \Sigma$ and a non-empty subset $I \subseteq N$ such that, for each $i \epsilon I$,

$$F_i(\bar{\sigma}_i, \bar{\sigma}_{f(i)}) < F_i(\tau_i, \bar{\sigma}_{f(i)}).$$

Consider the strategy $\bar{\tau} \epsilon \Sigma$ defined by

$$\bar{\tau} = \begin{cases} \tau_i & \textit{if} \quad i \epsilon I \\ \sigma_i & \textit{if} \quad i \epsilon N - I \end{cases},$$

Then the following is satisfied:

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) < \Phi_1(\bar{\tau}, \bar{\sigma})$$

which is absurd. Q.E.D.

A point $\bar{\sigma} \epsilon \Sigma$ is said to be an e^m simple stable point of the game Γ if, for all $i \epsilon N$,

$$\max_{s_i \epsilon \Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} \max_{s_i \epsilon \Sigma_i} A_i(s_i, s_{e(i)}, \bar{\sigma}_{f(i)}).$$

Introducing for each $i \epsilon N$ the function G_i defined by

$$G_i(\sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i \epsilon \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)}),$$

then, it is possible to characterize an e^m simple stable point by the function

$$\Phi_2(\sigma, \tau) = \sum_{i \epsilon N} [-G_i(\sigma_{e(i)}, \tau_{f(i)})],$$

as is illustrated in the following:

LEMMA 3: If for each $\sigma \epsilon \Sigma$ there is a $\tau \epsilon \Sigma$ such that for each $i \epsilon N$:

$$-G_i(\tau_{e(i)}, \sigma_{e(i)}) = \max_{s_{e(i)} \epsilon \Sigma_{e(i)}} [-G_i(s_{e(i)}, \sigma_{f(i)})],$$

Then a point $\bar{\sigma} \in \Sigma$ is an $\underline{e}^m$ simple stable point of the game Γ if and only if

$$\Phi_2(\bar{\sigma}, \bar{\sigma}) = \max_{s \in \Sigma} \Phi_2(s, \bar{\sigma}) .$$

Poof: Let $\bar{\sigma} \in \Sigma_N$ be an $\underline{e}^m$ simple stable point of the game Γ. Then, for each $i \in N$

$$-G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{e(i)} \in \Sigma_{e(i)}} [-G_i(s_{e(i)}, \bar{\sigma}_{f(i)})] .$$

Therefore

$$\sum_{i \in N} -G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = \sum_{i \in N} \max_{s_{e(i)} \in \Sigma_{e(i)}} [-G_i(s_{e(i)}, \sigma_{f(i)})] ,$$

which implies the validity of the equality

$$\Phi_2(\bar{\sigma}, \bar{\sigma}) = \max_{s \in \Sigma} \Phi_2(s, \bar{\sigma}) .$$

Now, consider a point $\bar{\sigma} \in \Sigma$ which satisfied

$$\Phi_2(\bar{\sigma}, \bar{\sigma}) = \max_{s \in \Sigma} \Phi_2(s, \bar{\sigma}) ,$$

and suppose that there is a non-empty subset $I \subseteq N$ such that, for each $i \in I$,

$$-G_i(\bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) < \max_{s_{e(i)} \in \Sigma_{e(i)}} [-G_i(s_{e(i)}, \bar{\sigma}_{f(i)})] .$$

By hypothesis, given the point $\bar{\sigma} \in \Sigma$, there exists a point $\bar{\tau} \in \Sigma$ such that, for each $i \in N$,

$$-G_i(\bar{\tau}_{e(i)}, \bar{\sigma}_{f(i)}) = \max_{s_{e(i)} \in \Sigma_{e(i)}} [-G_i(s_{e(i)}, \bar{\sigma}_{f(i)})] ,$$

and therefore, we obtain

$$\Phi_2(\bar{\sigma}, \bar{\sigma}) < \Phi_2(\bar{\tau}, \bar{\sigma}) \text{ which}$$

contradicts the hypothesis. Q.E.D.

An inmediate consequence of the preceding lemmas is given in the following result:

Corollary 4: If for each $\sigma\epsilon\Sigma$ there is a $\tau\epsilon\Sigma$ such that, for all $i\epsilon N$,

$$F_i(\tau_i, \sigma_{f(i)}) = \max_{s_i \epsilon \Sigma_i} F_i(s_i, \sigma_{f(i)})$$

and

$$-G_i(\tau_{e(i)}, \sigma_{f(i)}) = \max_{s_{e(i)} \epsilon \Sigma_{e(i)}} G_i(s_{e(i)}, \sigma_{f(i)}),$$

then, a point $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}_m$ and $\underline{e}^m$ simple stable point of the game Γ if and only if

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) = \max_{s \, \epsilon \, \Sigma} \Phi_1(s, \bar{\sigma})$$

and

$$\Phi_2(\bar{\sigma}, \bar{\sigma}) = \max_{s \, \epsilon \, \Sigma} \Phi_2(s, \bar{\sigma})$$

4. In this section, we will obtain some general theorems which are concerned with the existence of simple stable points of n-person games.

These theorems will be obtained as a direct application of the above results.

Theorem 5: Let $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be a game, where for each $i\epsilon N$, the set Σ_i is convex, compact in a real topological linear space, such that the following conditions are fulfilled:

(i) For each $i\epsilon\Sigma$ and each $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$, the function $F_i(\sigma_i, \sigma_{f(i)})$ is concave in $\bar{\sigma}_i \; \epsilon \; \Sigma_i$.

(ii) For each N and each $\sigma_i \epsilon \Sigma_i$, the function $F_i(\sigma_i, \sigma_f$ is continuous in $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$.

(iii) The function

$$\sum_{i \, \epsilon \, N} F_i(\sigma_i, \sigma_{f(i)})$$

is continuous in $\sigma\epsilon\Sigma$.

Then, there exists an $\underline{e}^m$ simple stable point of the grame Γ.

Poof: Consider the function

$$\Phi_1(\sigma, \tau) = \sum_{i\epsilon N} F_i(\sigma_i, \tau_{f(i)})$$

defined on the set $\Sigma \times \Sigma$. For each $\tau \epsilon \Sigma$, the function $\Phi_1(\sigma, \tau)$ is concave in $\sigma \epsilon \Sigma$. On the other hand, the function $\Phi_1(\sigma, \sigma)$ is continuous in $\sigma \epsilon \Sigma$; and for each $\sigma \epsilon \Sigma$, the function $\Phi_1(\sigma, \tau)$ is continuous in $\tau \epsilon \Sigma$.

Then, by direct application of Theorem 1 to the function Φ_1, the existence of a point $\bar{\sigma} \epsilon \Sigma$ such that

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) = \max_{s \epsilon \Sigma} \Phi_1(s, \bar{\sigma})$$

is guaranteed.

By Lemma 2, such a point is an $\underline{e_m}$ simple stable point of the game Σ. Q.E.D.

Theorem 6: Let $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be a game, where for each $i \epsilon N$, the set Σ_1 is convex, compact in a real topological linear space, such that the following conditions are fulfilled:

(i) For each $i \epsilon N$ and fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$, the function

$$G_i(\sigma_{e(i)}, \sigma_{f(i)})$$

is convex in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$.

(ii) Por each $i \epsilon N$ and each $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ the function $G_i(\sigma_{e(i)}, \sigma_{f(i)})$ is continuous in $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$.

(iii) The function

$$\sum_{i \epsilon N} G_i(\sigma_{e(i)}, \sigma_{f(i)})$$

is continuous in $\sigma \epsilon \Sigma$.

(iv) For each $\sigma \epsilon \Sigma$ there is a $\tau \epsilon \Sigma$ such that for each $i \epsilon N$:

$$-G_i(\tau_{e(i)}, \sigma_{f(i)}) = \max_{s_{e(i)} \epsilon \Sigma_{e(i)}} [-G_i(s_{e(i)}, \sigma_{f(i)})].$$

Then, there exists an $\underline{e^m}$ simple stable point of the game Γ.

Proo: Consider the function

$$\Phi_2(\sigma, \tau) = \sum_{i \epsilon N} G_i(\sigma_{e(i)}, \tau_{f(i)})$$

defined an $\Sigma \times \Sigma$. On one hand, for each $\tau \epsilon \Sigma$, the function $\Phi_2(\sigma,\tau)$ is concave in $\sigma \epsilon \Sigma$; and on the other hand, the function $\Phi_2(\sigma, \sigma)$ is continuous in $\sigma \epsilon \Sigma$. Furthermore, for each $\sigma \epsilon \Sigma$, the function $\Phi_2(\sigma, \tau)$ is continuous in $\tau \epsilon \Sigma$.

Then, Theorem 1 applied to the function Φ_2 guarentees the existence of a point $\bar{\sigma} \epsilon \Sigma$ such that

$$\Phi_2 (\bar{\sigma}, \bar{\sigma}) = \max_{s \epsilon \Sigma} \Phi_2 (s, \bar{\sigma}) .$$

By Lemma 3, such a point is an $\underline{e}^m$ simple stable point of the game Γ. Q.E.D.

THEOREM 7: Let $\Gamma = \{ \Sigma_1, \ldots, \Sigma_n ; A_1, \ldots, A_n \}$ be a game where for each $i \epsilon N$ the set Σ_i is convex, compact in a real topological linear space, such that the following conditions are fulfilled:

(i) For each $i \epsilon N$ and each $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$, the function $F_i (\sigma_i, \sigma_{f(i)})$ is concave in $\sigma_i \epsilon \Sigma_i$, and the function $G_i (\sigma_{e(i)}, \sigma_{f(i)})$ is convex in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$.

(ii) For each $i \epsilon N$ and every $\sigma_i \epsilon \Sigma_i$ and $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ the functions $F_i (\sigma_i, \sigma_{f(i)})$ and $G_i (\sigma_{e(i)}, \sigma_{f(i)}) \epsilon \Sigma_{e(i)}$ are continuous in $\sigma_{f(i)}$ e $\Sigma_{f(i)}$.

(iii) The functions

$$\sum_{i \epsilon N} F_i (\sigma_i, \sigma_{f(i)}) \text{ and } \sum_{i \epsilon N} G_i (\sigma_e, \sigma_{f(i)})$$

are continuous in $\sigma \epsilon \Sigma$.

(iv) For each $\sigma \epsilon \Sigma$ there is a $\tau \epsilon \Sigma$ such that for each $i \epsilon N$:

$$F_i (\tau_i, \sigma_{f(i)}) = \max_{s_i \epsilon \Sigma_i} F_i (s_i, \sigma_{f(i)})$$

and

$$- G_i (\tau_{e(i)}, \sigma_{f(i)}) = \max_{s_{e(i)} \epsilon \Sigma_{e(i)}} [- G_i (s_{e(i)}, \sigma_{f(i)})] .$$

Then, there exists an $\underline{e}_m$ and $\underline{e}^m$ simple stable point of the game Γ.

PROOF: Again, consider the function

$$\Phi (\sigma, \tau) = \Phi_1 (\sigma, \tau) + \Phi_2 (\sigma, \tau)$$

defined on $\Sigma \times \Sigma$. The function $\Phi (\sigma, \sigma)$ is continuous in $\sigma \epsilon \Sigma$ since the functions $\Phi_1 (\sigma, \sigma)$ and $\Phi_2 (\sigma, \sigma)$ are continuous in $\sigma \epsilon \Sigma$. Moreover, since the functions $\Phi_1(\sigma, \tau)$ and $\Phi_2(\sigma, \tau)$ are continuous in $\tau\epsilon\Sigma$ for each $\sigma\epsilon\Sigma$, then the function $\Phi(\sigma, \tau)$ is continuous in $\tau\epsilon\Sigma$ for each $\sigma\epsilon\Sigma$. Finally, for each $\tau\epsilon\Sigma$, the function $\Phi(\sigma, \tau)$ is concave in $\sigma\epsilon\Sigma$, since it is a sum of concave functions.

Then, Theorem 1 applied to the function $\Phi(\sigma, \tau)$ guarantees the existence of a point $\bar{\sigma} \epsilon \Sigma$ such that

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) + \Phi_2(\bar{\sigma}, \bar{\sigma}) = \max_{s \epsilon \Sigma} [\Phi_1(s, \bar{\sigma}) + \Phi_2(s, \bar{\sigma})].$$

By the last condition, there is a $\bar{\tau}\epsilon\Sigma$ such that

$$\Phi_1(\bar{\tau}, \bar{\sigma}) + \Phi_2(\bar{\tau}, \bar{\sigma}) = \max_{s \epsilon \Sigma} [\Phi_1(s, \bar{\sigma}) + \Phi_2(s, \bar{\sigma})] .$$

On the other hand, for each $\tau\epsilon\Sigma$

$$\Phi_1 (\tau, \bar{\sigma}) \leqslant \Phi_1(\bar{\tau}, \bar{\sigma}) = \sum_{i\epsilon N} \max_{s_i\epsilon\Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)})$$

and

$$\Phi_2(\tau, \bar{\sigma}) \leqslant \Phi_2(\bar{\tau}, \bar{\sigma}) = \sum_{i\epsilon N} \max_{s_{e(i)} \epsilon\Sigma_{e(i)}} [- G_i(s_{e(i)} , \bar{\sigma}_{f(i)})]$$

which implies that

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) + \Phi_2(\bar{\sigma}, \bar{\sigma}) = \max_{s\epsilon\Sigma} \Phi_1(s, \bar{\sigma}) + \max_{s\epsilon\Sigma} \Phi_2(s, \bar{\sigma}),$$

and therefore

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) = \max_{s\epsilon\Sigma} \Phi_1(s, \bar{\sigma})$$

and

$$\Phi_1(\bar{\sigma}, \bar{\sigma}) = \max_{s\epsilon\Sigma} \Phi_2(s, \bar{\sigma}) .$$

Then, by corollary 4, such a point is an $\underline{e_m}$ and $\underline{e^m}$ simple stable point of the game Γ. Q.E.D.

The above results are the principal of this paper. We note that in [3], since a $\underline{e_m}$ simple stable point of a game $\Sigma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ is an equilibrium point of the game $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; F_1 \ldots, F_n\}$ and conversely.

An immediate consequence of the last theorem is the following:

COROLLARY 8: Let $\Gamma = \{\Sigma_1, \ldots, \Sigma_n; A_1, \ldots, A_n\}$ be a game that satisfies all the conditions of the last theorem. If for each $\sigma \epsilon \Sigma$ and each $i \epsilon N$.

$$\max_{s_i \epsilon \Sigma_i} F_i(s_i, \sigma_f{}^{(i)}) = \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} G_i(s_c{}^{(i)}, \sigma_f{}^{(i)}),$$

then there exists an e_m e^m simple stable point $\sigma \epsilon \Sigma$ such that for each $i \epsilon N$,

$$\begin{aligned} A_i(\bar{\sigma}, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) &= \max_{s_i \epsilon \Sigma_i} F_i(s_i, \bar{\sigma}_{f(i)}) \\ &= \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} G_i(s_{e(i)}, \bar{\sigma}_{f(i)}) . \end{aligned}$$

Such a point is called an $\underline{e}$ - simple saddle point of the game Γ.

Indeed, there is a kind of games for which the additional condition in this last corollary can be in some sense weakened.

In fact, this is possible by using the Sion's minimax theorem [1] for games defined on separated, real topological linear spaces.

BIBLIOGRAFIA

[1] FAN, K.: *Sur un theorem minimax C.R.* Acad. Sci. Paris 259, 3925-3928 (1964).

[2] MARCHI, N.: *Simple Stable Points in Topological Linear Spaces.* J. Math. Anal. Appl. Vol. 20, Nº 1, 62-73 (1967).

[3] NIKAIDO, H., ISODA, K.: *Note on Non cooperative Convex Games.* Pac. J. Math. 5, 807-815, (1955).

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