ON THE EXISTENCE AND CONSTRUCTION OF POLYNOMIAL SOLUTIONS OF CERTAIN TYPES OF DIFERENTIAL EQUATIONS

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§ 1. *Introduction*

In this paper we consider the problem of the existence and construction oy polynomial solutions of equations of the form

$$D\ y(x) = 0 \tag{1}$$

defined by linear differential operators with polynomial coefficients.

The problem of detecting these solutions is related to that of finding solutions in integers of the algebraic equations defined by (3) and (5).

The effective solution of this problem has special interest in connection with some methods for the numerical approximation of the solution of (1).

§ 2. *Definitions and notation*

Let be D the operator

$$D = \sum_{i=0}^{k} q_i(x)\, D^{(i)} \tag{2}$$

where the coefficients $q_i(x)$ are polynomial with real coefficients and $D^{(i)}$ indicates the operator "derivate of order i" when $i > 0$ and the identity when $i = 0$; k is the order of D. With N_0 we shall indicate the set formed by zero and the positive integers and with I_n the set $\{ i/i \leqslant n,\ i \in N_0 \}$. For every $i \in I_k$, n_i and a_i are respectively the degree and the leading coefficient of $q_i(x)$.

If the operator D is applied to x^n $(n \in N_0)$ we obtain a polynomial

$$P_n(x) = D\, x^n$$

the degree of which shall be called h_n.

For every $i \in I_k$ we define an element $m_i \in N_0$ such that

$$m_i = n_i + k - i \qquad \text{when } a_i \neq 0$$

and

$$m_i = 0 \qquad \text{otherwise.}$$

H shall be the set of elements $M_r \in N_0$ for which there is a subset $J_r \subset I_k$ such that:

i) $\mathrm{Card}(J_r) > 1$
ii) $j \in J_r \leftrightarrow m_j = M_r$
iii) $i \in I_k$ and $i < j$ for some $j \in J_r \rightarrow m_i < M_r$

§ 3. *Existence and detection of polynomial solutions*

We shall examine separately the cases where the degree of the polynomial solution is $\geqslant k$ (case (i) and where it is $< k$ (case (ii)).

(i) Let be $M = \max\limits_i (m_i)$. When $M \bar{\in} H$, (1) has no polynomial solutions of degree greater or equal to k. In fact, in this case it follows that if s is the only element of I_k such that $m_s = M = \max\limits_i (m_i)$, then (1) has no polynomial solutions of degree $\geqslant s$.

We shall assume then, that $M \in H$. Let $J = \{ j_1, j_2, \ldots, j_h \}$ be the subset of I_k associated with M (i.e.: $m_{j_1} = m_{j_2} = \ldots = m_{j_h} = M$), where $j_1 > j_2 > \ldots > j_h$.

Let us introduce the polynomial

$$\Psi(n) = \sum_{j \in J} \frac{(n - j_h)!}{(n - j)!}\, a_j \qquad (3)$$

If the equation $\Psi(n) = 0$ has no integral roots $\geqslant k$, then, (1) has no polynomial solutions of degree $\geqslant k$.

Let be $W = \{ w_1, w_2, \ldots, w_p \}$ the set of integral solutions of $\Psi(n) = 0$, greater than or equal to k. This set is a finite one and we shall assume that it is not empty.

For every $w \in W$ we consider the sets

$$T'_w = \{ t' \in I_w \ , \ h_{t'} = h_w \}$$
$$T_w = \{ t \in I_w \ , \ h_t \leqslant h_w \} .$$

It follows then that $D\, y(x) \equiv 0$ may have a polynomial solution of degree w only when Card$(T'_w) > 1$. In this case there exists a polynomial solution of the general form

$$\sum_{i \in T_W} c_i \, x^i$$

if and only if there are real numbers c_i $(i \in T_w,\ c_w \neq 0)$ such that

$$\sum_{i \in T_W} c_i \, P_i(x) = 0 \qquad (4)$$

Identity (4) leads to an homogeneous system of Card(T_w) linear algebraic equations with Card(T_w) unknowns. The equation $D\, y(x) = 0$ has a polynomial solution of degree w if and only if such system has a non - trivial solution.

(ii) Let us point out firstly that if $m_i = 0$ for all $i \in I_z (z < k)$, then every polynomial of the general form

$$a_z \, x^z + a_{z-1} x^{z-1} + \ldots + a_0 ,$$

where $a_j (j = 0, 1, \ldots, z)$ are real numbers, is a solution of (1).

We shall say that D is a *proper* operator if $q_0 \neq 0$. Throughout the paper we shall assume that (2) is a proper operator (*)

Let be

$$H = \{ M_0,\ M_1,\ \ldots,\ M_d \}$$

where

$$M_0 < M_1 < \ldots < M_d$$

and for every $M_r \in H$ let be

$$J_r = \{ j_{r_1},\ j_{r_2},\ \ldots,\ j_{r_n} \}$$

where

$$j_{r_1} > j_{r_2} > \ldots > j_{r_n}$$

is the subset of I_k associated with M_r.

(*) If (2) is not a proper operator, it admits the polynomial solutions considered above. Once these solutions are eliminated, a proper operator remains.

With every $M_r \in H$, $M_r > 0$ we also associate the following sequence of subsets, every one with at least two elements

$$F_{r_1} = \{ j_{r_1}, j_{r_2}, \ldots, j_{r_n} \}$$
$$F_{r_2} = \{ j_{r_2}, j_{r_3}, \ldots, j_{r_n} \}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F_{r_{n-1}} = \{ j_{r_{n-1}}, {}^{r_n} \}$$

and for every $F_r^{\,i}$ we define

$$s_{r_i} = \min \{ s \in I_k - F_{r_i}, \quad m_s \geqslant M_r \} \quad \text{if} \quad r_i \neq d_i$$

and

$$s_{r_1} = k \qquad \text{otherwise.}$$

We finally form the sets W_r, the elements of which are the integral roots w of the equation

$$\Psi_{r_i}(n) = \sum_{j \in F_{r_i}} \frac{(n - j_{r_n})!}{(n - j)!} a_j \tag{5}$$

such that

$$j_{r_i} \leqslant w < s_{r_i}$$

It follows that if for every r_i is $W_r = \phi$, then (1) has no polynomial solutions of degree $< k$. For every $w \in W_r$ and some r_i we define, as we did in case (i), sets T'_w and T_w and use the same argument.

It is interesting to point out that if D is a proper operator, (1) has no polynomial solutions provided that $H = \phi$. Necessary and suficient conditions for the existence of polynomial solutions are given in [1].

§ 4. *Applications of the method*

We shall consider now some applications of the method given in § 3 to decide whether a given differential equation with polynomial coefficients has polynomial solutions and, if so, we shall compute them.

Example 1: Let be the differential equation

$$x^{10}y^{viii}(x) + 2x^5y^{vi}(x) + x^7y^v(x) -$$
$$- x^2y^{iv}(x) + 5x^2y^{iii}(x) - 7y'(x) = 0 \qquad (6)$$

defined by an operator D which is not proper.

In this case

$k = 8$

$a_0 = 0$	$n_0 = 0$	$m_0 = 0$
$a_1 = -7$	$n_1 = 0$	$m_1 = 7$
$a_2 = 0$	$n_2 = 0$	$m_2 = 0$
$a_3 = 5$	$n_3 = 2$	$m_3 = 7$
$a_4 = -1$	$n_4 = 2$	$m_4 = 6$
$a_5 = 1$	$n_5 = 7$	$m_5 = 10$
$a_6 = 2$	$n_6 = 5$	$m_6 = 7$
$a_7 = 0$	$n_7 = 0$	$m_7 = 0$
$a_8 = 1$	$n_8 = 10$	$m_8 = 10$

Hence

$$H = \{10,7\}\,; \quad M = M_d = 10 \quad ; \quad M_o = 7$$

and

$$J = J_d = \{8,5\}\,; \quad J_o = \{3,1\}$$

Therefore

$$\Psi(n) = \frac{(n-5)!}{(n-8)!}\cdot 1 + \frac{(n-5)!}{(n-5)!}\cdot 1 = (n-5)\,(n-6)\,(n-7) + 1$$

As no root of $\Psi(n) = 0$ belongs to N_0, $W \equiv \phi$. Then, there is no polynomial solution of degree $\geqslant 8$. On the other hand, as $m_0 = 0$ and $m_1 \neq 0$, any constant is a solution of (6).

As

$$F_{d_1} = \{8,5\}\,; \quad s_{d_1} = 8 \quad ; \quad F_{o_1} = \{3,1\}\,; \quad s_{o_1} = 5$$

we have

$$\Psi_{d_1}(n) = \Psi(n)\ , \text{ i.e. } \ W_{d_1} = \phi$$

and

$$\Psi_{0_1}(n) = \frac{(n-1)!}{(n-3)!} \cdot 5 + \frac{(n-1)!}{(n-1)!} \cdot (-7) =$$

$$\equiv (n-1\ (n-2)\ 5-7.$$

Again, $W_{0_1} = \phi$, as both, $\Psi_{d_1}(3)$ and $\Psi_{0_1}(4)$ are different from zero. We conclude that (6) has no other polynomial solution than a constant.

Example 2: Let us consider the differential equation

$$(x^4 - \tfrac{1}{2}x^2 - .3x/5 + 3/40)\,y'''(x) + 2xy''(x) +$$

$$+ (-12x^2 - x - 2)\,y'(x) + 5y(x) = 0 \qquad (7)$$

Here is $k = 3$

$a_0 = 5$	$n_0 = 0$	$m_0 = 3$
$a_1 = -12$	$n_1 = 2$	$m_1 = 4$
$a_2 = 2$	$n_2 = 1$	$m_2 = 2$
$a_3 = 1$	$n_3 = 4$	$m_3 = 4$

Hence

$$H = \{\ 4\ \};\quad M \equiv M_d = 4 \text{ and } J = \{3,1\}$$

$$\Psi(n) = \frac{(n-1)!}{(n-3)!} \cdot 1 + \frac{(n-1)!}{(n-1)!} \cdot (-12) =$$

$$= (n-1)\ (n-2) - 12 = n^2 - 3n - 10.$$

The roots of $\Psi(n) = 0$ are the integers $n = 5$ and $n = -2$. Therefore $W = \{\ 5\ \}$ and $W_{d_1} = \phi$.

As $m_0 \neq 0$, we can only expect polynomial solutions of degree five.

Then, as

$$P_0(x) = 5$$
$$P_1(x) = -12x^2 + 4x - 2$$
$$P_2(x) = -24x^3 + 3x^2$$
$$P_3(x) = -30x^4 + 2x^3 + 3x^2 - 18x/5 + 9/20$$
$$P_4(x) = -24x^5 + x^4 + 4x^3 - 72x^2/5 + 9x/10$$
$$P_5(x) = -36x^3 + 9x^2/2$$

it follows that

$$h_5 = 3 \ ; \ T'_5 = \{ 5{,}2 \} ; \quad T_5 = \{ 5, 2, 1, 0 \} \ .$$

If there is a polynomial solution to (7) it will have the general form

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_5 x^5 .$$

The condition $D\, y(x) = 0$ leads to an homogeneous system of linear algebraic equations which has a nontrivial solution. From it we deduce that

$$y(x) = 2x^5 - 3x^2 \qquad (8)$$

is a polynomial solution of (7). We can also state that there is no other polynomial solution of (7) linearly independent of (8).

REFERENCES

[1] ORTIZ, E. L., *Polynomlösungen von Differentialgleichungen,* Zeitschrift für Angewandte Mathematik und Mechanik, V (1966).