AN ALGEBRAIC MODEL OF THE *m*-VALUED PROPOSITIONAL CALCULUS WITH VARIABLE FUNCTORS

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Algebraic models, in terms of elements of a Boolean algebra, have been given for the m - valued propositional calculus without variable functors (1) and for the 2 - valued propositional calculus with variable functors (2). We shall now combine the two methods to give an algebraic model of the m - valued propositoinal calculus with variable functors ($m = 2, 3, \dots$).

Since, with the notation of Lukasiewicz (3),

$$\delta P = {}_{T}DP\delta 1 \dots \delta m \qquad \text{(A)}$$

we may, as before, take as the algebrais representative of the propositional variable p the $(m-1)$ - tuple $(p_1, \dots, p_{m-1})$ of elements $p_1, \dots, p_{m-1}$ of a Boolean algebra, where

$$p_i \leqslant p_{i+1} \quad (i = 1, \dots, m-2)$$

(other propositional variables being, of course, treated similarry) and we may take as the algebraic representative of the variable functor δ the m $(m-1)$ - tuples $(\delta_{11}, \dots, \delta_{1, m-1}), \dots, (\delta_{m1}, \dots, \delta_{m, m-1})$ where

$$\delta_{ij} \leqslant \delta_{i, j+1} \; (i = 1, \dots, m; \; j = 1, \dots, m-2)$$

(other variable functors being, of course, treated similaryl).

By (A), if we regard $(\delta_{i1}, \dots, \delta_{i, m-1})$ as the algebraic representative of δ_i $(i = 1, \dots, m)$ and the algebraic representative of δp is $(a_1, \dots, a_{m-1})$, then

$$a_i = (p_1 \cap \delta_{1i}) \cup (p'_1 \cap p_2 \cap \delta_{2i}) \cup \dots \cup (p'_{m-2} \cap p_{m-1} \cap \\ \cap \delta_{m-1i}) \cup (p'_{m-1} \cap \delta_{mi}) \; (i = 1, \dots, m-1).$$

A formula of the propositional calculus with s designated truth-values takes desginated truth-values exclusively if and only if its algebraic representative is $(\beta_1, \ldots, \beta_{m-1})$ where $\beta_s = I$ $(s = 1, \ldots, m-1)$.

For example, if $m = 3$, the algebraic representative of δp is

$$((p_1 \cap \delta_{11}) \cup (p'_1 \cap p_2 \cap \delta_{21}) \cup (p'_2 \cap \delta_{31}),\ (p_1 \cap \delta_{12}) \cup (p'_1 \cap \cap p_2 \cap \delta_{22}) \cup (p'_2 \cap \delta_{32})).$$

A formula takes designated truth-values exclusively if and only if its algebraic representative is (β_1, β_2) where $\beta_1 = I$ is $s = 1$ and $\beta_2 = \mathrm{I}$ if $s = 2$.

Let A, K denote the functors of Lukasiewicz (4) and let C, I denote the implication functors whose truth-tables are given below. Clearly $C(I)$ satisfies the standard conditions of Rosser and Turquette (5) when $s = 1(2)$. As examples we shall show

CPQ	1	2	3	Q
1	1	2	3	
2	1	1	1	
3	1	1	1	
P				

IPQ	1	2	3	Q
1	1	2	3	
2	1	2	3	
3	1	1	1	
P				

that the formulae $C\delta KpqA\delta p\delta q$, $IKK\delta p\delta Rp\delta RRp\delta q$ take designated truth-values exclusively in the cases $s = 1$, $s = 2$ respectively, the symbol R denoting the primitive negation functor of Post (6).

We note first that the representative of Cpq, Kpq, Apq are $(p'_1 \cup q_1, p'_2 \cup q_2)$, $(p_1 \cap q_1, p_2 \cap q_2)$, $(p_1 \cup q_1, p_2 \cup q_2)$ respectively. Thus, with the above notation,

$$\beta_1 = ((p_1 \cap q_1 \cap \delta_{11}) \cup ((p'_1 \cup q'_1) \cap p_2 \cap q_2 \cap \delta_{21}) \cup \cup ((p'_2 \cup q'_2) \cap \delta_{31}))' \cup (p_1 \cap \delta_{11}) \cup (p'_1 \cap p_2 \cap \delta_{21}) \cup \cup (p'_2 \cap \delta_{31}) \cup (q_1 \cap \delta_{11}) \cup (q'_1 \cap q_2 \cap \delta_{21}) \cup (q'_2 \cap \delta_{31}).$$

But $(p_1 \cap \delta_{11}) \cup (q_1 \cap \delta_{11}) = (p_1 \cup q_1) \cap \delta_{11} \geqslant p_1 \cap q_1 \cap \delta_{11}$,
$(p'_1 \cap p_2 \cap \delta_{21}) \cup (q'_1 \cap q_2 \cap \delta_{21}) = ((p'_1 \cap p_2) \cup (q'_1 \cap q_2)) \cup \delta_{21} \geqslant \geqslant ((p'_1 \cap p_2 \cap q_2) \cup (q'_1 \cap p_2 \cap q_2)) \cap \delta_{21} = (p'_1 \cup q'_1) \cap \cap p_2 \cap q_2 \cap \delta_{21}$'
$(p'_2 \cap \delta_{31}) \cup (q'_2 \cap \delta_{31}) = (p'_2 \cup (q'_2 \cap \delta_{31}$.

Thus $\beta_1 \geqslant (p_1 \cap q_1 \cap \delta_{11}) \cup ((p'_1 \cup q'_1) \cap p_2 \cap q_2 \cap \delta_{21}) \cup$
$\cup ((p'_2 \cup q'_2 \cap \delta_{31}))' \cup (p_1 \cap q_1 \cap \delta_{11}) \cup ((p'_1 \cup q'_1) \cap$
$\cap p_2 \cap q_2 \cap \delta_{21}) \cup (p'_2 \cup q'_2) \cap \delta_{31})) = I.$

In the second example we note first that the representatives of *Ipq, Rp, RRp* are $(p'_2 \cup q_1, p'_2, \cup q_2)$, $(p'_2, p_1 \cup p'_2)$, $(p'_1 \cap p_2, p'_1)$ respectively. Thus

$$\beta_2 = (((p_1 \cap \delta_{12}) \cup (p'_1 \cap p_2 \cap \delta_{22}) \cup (p'_2 \cap \delta_{32})) \cap$$
$$\cap ((p'_2 \cap \delta_{12}) \cup (p_2 \cap (p_1 \cup p'_2) \cap \delta_{22}) \cup (p'_1 \cap p_2 \cap \delta_{32})) \cap$$
$$\cap ((p'_1 \cap p_2 \cap \delta_{12}) \cup ((p_1 \cup p'_2) \cap p'_1 \cap \delta_{22}) \cup (p_1 \cap \delta_{32})))' \cup$$
$$\cup (q_1 \cap \delta_{12}) \cup (q'_1 \cap q_2 \cap \delta_{22}) \cup (q'_2 \cap \delta_{32})$$
$$= (((p_1 \cap \delta_{12}) \cup (p'_1 \cap p_2 \cap \delta_{22}) \cup (p'_2 \cap \delta_{32})) \cap ((p'_2 \cap \delta_{12}) \cup$$
$$\cup (p_1 \cap \delta_{22}) \cup (p'_1 \cap p_2 \cap \delta_{32})) \cap ((p'_1 \cap p_2 \cap \delta_{12}) \cup (p'_2 \cap \delta_{22}) \cup$$
$$\cup (p_1 \cap \delta_{32})))', \cup (q_1 \cap \delta_{12}) \cup (q'_1 \cap q_2 \cap \delta_{22}) \cup (q'_2 \cap \delta_{32})$$
$$= ((p_1 \cap \delta_{12} \cap \delta_{22} \cap \delta_{32}) \cup (p'_1 \cap p_2 \cap \delta_{22} \cap \delta_{32} \cap \delta_{12}) \cup$$
$$(p'_2 \cap \delta_{32} \cap \delta_{12} \cap \delta_{22}))' \cup (q_1 \cap \delta_{12}) \cup (q'_1 \cap q_2 \cap \delta_{22}) \cup (q'_2 \cap \delta_{32})$$
$$= \delta'_{12} \cup \delta'_{22} \cup \delta'_{32} \cup (q_1 \cap \delta_{12}) \cup (q'_1 \cap q_2 \cap \delta_{22}) \cup (q'_2 \cap \delta_{32})$$
$$= \delta'_{12} \cup q_1 \cup \delta'_{22} \cup (q'_1 \cap q_2) \cup \delta'_{32} \cup q'_2 = I.$$

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3. See, for example, Jan Lukasiewicz, *On variable functors of propositional arguments*, Proceedings of the Royal Irish Academy, Section A, 54 (1951), 25 - 34. The symbol D derives the m - valued conditioned disjunction functor such that that $DQP_1 \ldots P_m$ takes the same truth-value as P_i when Q takes the truth-value i $(i = 1, \ldots, m)$. Conf. ALAN ROSE, *Conditioned disjunction as a primitive connective for the m-valued propositional calculus*, Mathematische Annalen, 123 (1951), 76 - 78.
4. See, for example, J. B. Rosser and A. R. Turquette, *Many-valued logics* (Amsterdam, 1952), p. 15.
5. See, for example, pp. 25 - 26 of the book referred to in footnote 4.
6. POST, Emil L., *Introduction to a general theory of elementary propositions*, American Journal of Mathematics, 43 (1921), 163 - 185.