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ON MEASURABLE SUBALGEBRAS ASSOCIATED TO COMMUTING CONDITIONAL EXPECTATION OPERATORS, II,

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SUMMARY. The objective we pursued was the same as in {M P_2},i.e., to give necessary and sufficient conditions to make sure that two conditional expectation operators $E^{\mathcal{B}}$ and $E^{\mathcal{C}}$ commute. We restricted ourselves to seek for conditions on the σ-algebras $\mathcal{B}$ and $\mathcal{C}$. We found that $E^{\mathcal{B}}E^{\mathcal{C}}f = E^{\mathcal{C}}E^{\mathcal{B}}f$, $\forall\ f \in L^2$, essentially when and only when in a partition of sets: $\mathcal{B}\subset\mathcal{C}$ or $\mathcal{C}\subset\mathcal{B}$, $\mathcal{B}$ is independent of $\mathcal{C}$, or $\mathcal{B}$ and $\mathcal{C}$ behave as the algebras of Borel measurable sets of R^3 independents of z and x respectively. It is noted also that when the commutation is asked not for one but several probabilities equivalent among them, inclusion is the only possible relation between $\mathcal{B}$ and $\mathcal{C}$. That is, it is not only the most natural relation assuring commutation but also the most stable under variation of the probability measure. This paper is a self-contained continuation of {M P_2}.

1. INTRODUCTION. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\mathcal{B}$,$\mathcal{C}$, σ-subalgebras of $\mathcal{A}$ containing all the sets of measure zero,or as we shall say $\mathcal{A}$-complete. This type of completeness will be supposed of any σ-algebra appearing in this paper even when not mentioned explicitly. Call $\mathcal{D} = \mathcal{B}\wedge\mathcal{C}$, the greatest σ-algebra contained in $\mathcal{B}$ and $\mathcal{C}$. Then, the trivial subalgebra τ will contain all the sets of measure zero. $\mathcal{B}$ and $\mathcal{C}$ are conditionally independent with respect to the σ-algebra $\mathcal{M}$ if $P(BC/\mathcal{M}) = P(B/\mathcal{M})P(C/\mathcal{M})$ for $B\in\mathcal{B}$, $C\in\mathcal{C}$. This concept presents two extreme cases: $\mathcal{M}=\mathcal{A}$ and $=\tau$. In the first case conditionally independence does not establish any tie between the algebras. In the other one it is equivalent to independence. When $\tau\subset\mathcal{M}\subset\mathcal{A}$ the conditionally independence is also an intermediate case between the independence and the absence of conditioning between the algebras. Call $E = E(.\ /\mathcal{B})$, $F = E(.\ /\mathcal{C})$, $G = E(.\ /\mathcal{D})$ the conditional expectation operators associated to the mentioned algebras and e(f) = expectation of f.
Let us prove now a useful lemma.

LEMMA 1. *i)* E *is a projector on* $L^2(\Omega,\mathcal{A},P)$ *with range* $L^2(\Omega,\mathcal{B},P)$.
ii) EF *is a projector iff* E *and* F *commute.*
iii) E *and* F *commute iff* EF = G.
iv) E *and* F *commute iff* E:$\mathcal{C}$*-measurable positive bounded functions*→

→ *C-measurable functions.*

Proof: i) and ii) follow from the definitions and the theory of Hilbert spaces. If the commutator of F and E, [E,F] , is zero then EFf is $\mathcal{D}$-measurable, and conversely, any $\mathcal{D}$-measurable function is invariant under EF, this proves iii). iv) follows from: a subspace with projector F reduces an operator E iff E and F commute, and the self adjointness of E.

A theorem of Burkholder and Chow asserts that $(EF)^n f \to Gf$ a.e. and in L^2 if $f \in L^2(\Omega, A, P)$, (cf. {BC}) . What conditions must be imposed on the associated algebras as to have $(EF)^m f = Gf$ for every f? In particular, how are $\mathcal{B}$ and $\mathcal{C}$ related in these cases?

There is a formal parallelism between this situation and the preceding one where conditional independence was considered. Moreover, if $\mathcal{B}$ is independent of $\mathcal{C}$, then the commutator of F and E, [E,F] , equals 0. Is there some relation between the concept of independence and the property of commutation of the associated conditional expectation operators? Since the commutation is present whenever one algebra contains the other the question must be properly posed as follows: When inclusion ($\mathcal{B} \subset \mathcal{C}$ or $\mathcal{C} \subset \mathcal{B}$) is not present and E and F commute, is $\mathcal{B}$ independent of $\mathcal{C}$? In a sense the answer is yes and this paper is essentially devoted to prove it. Another clue is given next.

It is well-known that if F, G are (closed) subspaces of the subspace E constituted by functions of mean zero, square integrable and finite normal joint distributions and if $\mathcal{B} = g(F)$, $\mathcal{C} = g(G)$ are the σ-algebras generated by the functions of the mentioned subspaces, then the restriction of E to the subspace of $\mathcal{B}$-measurable L^2- functions of mean zero has range equal to F. Briefly, in this case projection and conditioning coincide. Let us prove now the following proposition.

a) G is orthogonal to F iff b) G is independent of F iff c) EF=FE==e=0 on (F times G) $\cap L^2$, whenever is satisfied the hypothesis explicited above.

a) implies b). If $g \in G$ and f to F then the most general functions $\mathcal{C}$ and $\mathcal{B}$-measurable are of the form g+c , f+d, where c and d are constants. Since e(gf)=0=e(g).e(f) it follows that e((g+c)(f+d))==e(g+c).e(f+d). b) implies c), as it is easy to see since independence of $\mathcal{B}$ from $\mathcal{C}$ implies EF=FE=e. It holds: EF(fg)=e(fg)=0 whenever c) holds. QED.

The preceding proposition supports the suspicion that commutation

and independence are related, if there is not inclusion. This paper will be devoted to prove this.
Expository reasons oblige us to include with a proof most of the results of {M} and those of {M P_2}. Whenever this happens it will be explicitely mentioned.
To begin with, let us say that this introduction was already contained in {M P_2}.

2. AUXILIARY RESULTS. A Boolean σ-algebra where can be defined a probability measure will be called a measure Boolean algebra (cf.{H_3}). It is complete in the sense that the supremum of any family of elements exists, that is, there exists a least upper bound. Examples are the quotients of the σ-algebras of probability spaces by its ideal of sets of measure zero. (And they are the sole examples as it is seen using Stone's representation theorem, and Caratheodory's theorem of extension of measures). A σ-basis of a σ-algebra A is a set of generators of A (i.e. the least σ-algebra containing the set is A) with minimal cardinality. Because of the well ordering of the cardinals every σ-algebra contains a σ-base. A principal ideal of A is generated by an element $a \neq 0$, and is defined as $\{x;\ x \leq a,\ x \in A\}$. This ideal defines the σ-algebra $A \wedge a$. An algebra is called homogeneous if for every $a \in A$, $\dim(A \wedge a)$ is constant $(a \neq 0)$; in other words, all the proper principal ideals have the same dimension.
Example: The algebra of Borelian sets in (0,1) is homogeneous of dimension $\aleph_0$. In relation with homogeneous algebras cf. {M}.
Next we prove some results necessary for what follows.

THEOREM 1: *i) If* A *and* B *are Boolean σ-algebras and* h *is a surjective σ-homomorphism then if* $K \subset A$ *and* g(K) *designates the least σ-algebra containing* K, $h(g(K)) = g(h(K))$. *Also* $\dim B \leq \dim A$.
ii) $g_a(K \wedge a) = g(K) \wedge a$, *where* g_a *means "generated in the principal ideal* I_a*" and* a *is an element of* A.
iii) If $L \subset A$, $\dim L \leq \dim A$.

Proof: i) is easy and is left to the reader. ii) follows applying i) to $h: A \longrightarrow I_a$, $h(x) = x \wedge a$
iii) Let L' and A' be σ-basis of L and A respectively. We can suppose that they are ordered with the ordinals less than one which is the minimum among those of same cardinality. We can suppose more, as it is easy to see:
(*) if $A_i = g(a_s\ ;\ a_s \in A'\ ,\ s<i)$ then $i<j$ implies $A_i \subset A_j \neq A_i$.
Same for L and L'. Then $A = \bigcup_i A_i$. Call $K_i = g(L' \wedge A_i)$. Obviously $L = \bigcup_i K_i$. Besides:

(1) $$\dim L \leq \operatorname{card} \bigcup_i (L' \wedge A_i)$$

(Observe that (*) assures that no element of L' can be generated by the preceding ones and consequently that $\operatorname{card}(L' \wedge A_i) = \dim g(L' \wedge A_i)$). To finish the proof it would be enough to have $\operatorname{card}(L' \wedge A_i) \leq \operatorname{card} i$, since from (1) we had $\dim L \leq (\dim A)^2 = \dim A$. (The finite case is trivial). For this it suffices to have: any σ-subalgebra of A with dimension less than dim A verifies iii). If this is verified the theorem is proved, if not, there exists a σ-subalgebra B of A not verifying iii) and with minimal dimension with respect to this property. The preceding argument applied to B shows that iii) is verified, contradiction. QED.

THEOREM 2. *Given a measure Boolean algebra* A *without atoms there exists a partition of* A *in homogeneous ideals with different homogeneity,* (cf.{M}).

Proof: Let $\bar{N} = \{\dim a \ ; a \in A, a \neq 0\}$ and $a_\xi = \vee \{a; \dim a \leq \xi\}$, $\xi \in \bar{N}$. $\{a_\xi\}$ is well ordered and isomorphic to $\bar{N}$. In fact it will suffice to show that $\dim a_\xi = \xi$. Since in a measure Boolean algebra the sup of any family of elements coincides with the sup of a denumerable subfamily, we have $a_\xi = \vee b_n$. Therefore, $\dim a_\xi \leq \xi \cdot \aleph_0 = \xi$. By definition of a_ξ, $\dim a_\xi \geq \xi$. Putting now $x_{\xi_0} = a_{\xi_0}$, $\xi_0 = \inf \bar{N}$, and $x_\xi = a_{\xi+1} - a_\xi$ for $\xi \neq \xi_0$ we have the decomposition $\{x_\eta \ ; \eta \in \bar{N}\}$ we were looking for. QED.

COROLLARY. *A Boolean measure algebra without atoms of dimension* $\aleph_0$ *is homogeneous.*

EXAMPLE. $B((0,1)^{(n)})$.

Call B_Λ the Boolean measure algebra quotient of the Borel sets of $\Pi(I_\alpha \ ; \alpha<\Lambda)$, Λ an ordinal number, $I_\alpha = (0,1)$, with the set of sets of measure zero with respect to Lebesgue infinite product measure. If $\operatorname{card}\Lambda = \operatorname{card} \Phi$ then B_Λ and B_Φ are isomorphic. It is well-known for $\operatorname{card} \Lambda = \aleph_0$ and it is easy to see for other cardinals. What Maharam's theorem says is that those are the only homogeneous algebras. The isomorphism when the cardinals are equal follows then from her theorem, which will be proved later.

THEOREM 3. *Let* A *be a Boolean measure algebra with a probability* P *and* B *and* C *two* σ*-subalgebras such that if* $b \in B$ *and* $c \in C$ *then* $b \wedge c \neq 0$

whenever b,c≠0. *Assume* A = g(B,C) *and* P(b∧c) = P(b) P(c). *Then, there exist two probability spaces* (S,B,p), (T,C,q) *such that* B/p *is isomorphic to* B, C/q *is isomorphic to* C, *under the applications (resp.)* i,j *and these applications can be extended simultaneously to* k *that defines an isomorphism from* BxC/pxq *onto* A.

Proof: Let S and T be the Stone spaces associated to B and C respectively. R that associated to A. Let B and C be the σ-algebras generated by the clopens of S and T, and A that generated by those of R. Consider the family of finite unions of intersections of clopens of the form .x T, S x., in the product SxT; it is exactly the algebra of all the clopens of SxT. From the hypothesis it follows that this algebra is isomorphic to that generated by B and C. Therefore, it exists a continuous application f from R onto SxT that induces the isomorphism. Moreover, f pulls back BxC onto a σ-subalgebra of A. Call i(j) the isomorphism from the clopens of S(T) to B(C). Define p(q) on the clopens of S(T) as the value of P on the image under the isomorphism just described.

Calling again P, p, q, the extensions of P, p, q, from the clopens to A,B and C respectively we shall see that f^{-1} induces the promised isomorphism k. We shall only sketch the proof.

Define P' on f^{-1}(BxC) as P'(f^{-1}(H)) = (p x q)(H). We must see that 1) the probability P' coincides with P on the σ-algebra where the first is defined, 2) every element of A is equivalent [P] to a certain element of f^{-1}(BxC). This would prove the theorem.

1) The clopens of S generate B and the restriction of P to the inverse image of them by f coincides with p on them. Therefore P and P' coincide on f^{-1}(B). Same for f^{-1}(C).

Every element of B is equivalent [p] to a certain clopen as it is easy to see using nonotone classes (cf. {My}).

Let M be an element of B and N its clopen associated, let U∈C and V the equivalent clopen [q] . Then MxU is equivalent to NxV[pxq]. Since f^{-1}(NxV)has measure P equal to P(f^{-1}(N)).P(f^{-1}(V)) coinciding with (pxq)(NxV) and since f^{-1}U ∧ f^{-1}M is P-equivalent to f^{-1}V ∧ f^{-1}N, it follows: P(f^{-1}(MxU)) = (pxq)(MxU). Therefore, the same holds for any element of BxC , which proves 1).

2) Let us consider the set of elements of A such that the corresponding clopens are P-equivalent to a set of f^{-1}(BxC). This set contains the algebra of finite unions of sets of the form b∧c and it is a monotone family. An application of the theorem of monotone families for Boolean σ-algebras proves that every clopen of A is equivalent [P] to a set of f^{-1}(BxC). Since every element of A is equivalent to a clopen, the thesis is proved. QED.

We return now to the situation that will be the setting of what follows: a complete probability space $(\Omega,\mathcal{A},P)$; two $\mathcal{A}$-complete σ-algebras, $\mathcal{B}$ and $\mathcal{C}$; $\mathcal{D}$ the intersection of them; and we shall suppose from now on $\mathcal{A}$ is the least complete σ-algebra containing $\mathcal{B}\cup\mathcal{C}$. This situation will be written as $\mathcal{A} = g(\mathcal{B},\mathcal{C})$. Let Q be a finite measure on $\mathcal{A}$, absolutely continuous with respect to P and f = dQ/dP. From $E'(\cdot\ /\mathcal{B}) = E'(\ .\ ) = E_Q(\ .\ )$, the conditional expectation operator associated to Q and $\mathcal{B}$, and

(1) $$\int_B E'(h)\, dQ = \int_B h\, dQ = \int_B hf\, dP = \int_B E(hf)\, dP = \int_B E'(h)f\, dP$$

we obtain: $$\int_B E(hf)\, dP = \int_B E(E'(h)f)\, dP = \int_B E'(h)E(f)\, dP$$

From this:

(2) $$E'(h) = E(hf)/E(f), \quad [P]$$

Putting $f = 1_A$, the indicator of A, we get on A at least that

$$E'(h) = E(h\,1_A)\ /\ E(1_A)\ . \qquad \text{Therefore,}$$

(3) $$E'(h) = E(h\,1_A)\,1_A\ /\ E(1_A)\ , \quad [P]$$

That is, (3) defines the conditional expectations of the restrictions to A, (cf. {HN}).

If R and Q are equivalent probability measures with Radon-Nikodym derivatives r and q, from (2) it is easy to obtain:

$$E_R(h\, q/r)\ /\ E_Q(h) = E(q)\ /\ E(r)$$

independent of h. This formula will not be used in the paper.

PROPOSITION. *If* Q *is a probability equivalent to* P *and* f *is* $\mathcal{B}$ *or* $\mathcal{C}$*-measurable then the commutation of* E *and* F *implies that of* E_Q, F_Q.

Proof: Assume f is $\mathcal{C}$-measurable. From (2), $E_Q(h)$ is a $\mathcal{C}$-measurable function, if $h\in\mathcal{C}$-measurable. It follows from iv), lemma 1 and the hypothesis that E_Q and F_Q commute.

This proposition also follows immediatly from Proposition 2 of § 10. This alternative proof is left to the reader.

3. EXAMPLES. We shall introduce here some examples to avoid later interferences. The next three examples were contained in $\{MP_2\}$.

I. Let X = Y x Z and card Y = card Z = $\aleph_1$. Call $\mathcal{B}(\mathcal{C})$ the σ-algebra generated by the sets contained in a denumerable family of vertical (horizontal) lines. Let $\mathcal{A}$ = $\mathcal{P}$(X). Since card X = $\aleph_1$, a theorem of Ulam asserts that every measure on X is discrete and with respect to them $\mathcal{B}$ and $\mathcal{C}$ are equivalent. Same thing if the two algebras contain the points as measurable sets and together generate $\mathcal{A}$. Therefore, the associated conditional expectation operators commute. This example shows, for example, that what really matters is the Boolean structure which can be unexpectedly "different" from the set theoretic setting.

II. We have seen that commutation holds whenever $\mathcal{B}$ and $\mathcal{C}$ are independent. The following situation, which will be involved in what will be called g-independence, generalizes the case of independence. What we are going to show in this paper is that g-independence and inclusion ($\mathcal{B}\subset\mathcal{C}$ or $\mathcal{C}\subset\mathcal{B}$) are (essentially) the fundamental stones on which commutation is based on, and when $\mathcal{A}$ is under the influence of several measures inclusion is the only stable situation under which commutation does appear.

Let Ω = X x Y x Z, X = Y = Z = (0,1), $\mathcal{B}$ = the algebra of Borel measurable sets independent of z and $\mathcal{C}$ that independent of x. Obviously E(f) = ∫ f(x,y,z) dz, F(f) = ∫ f(x,y,z) dx and EF = FE, as it follows from Fubini's theorem.

III. Let $\mathcal{D}$ = Borel measurable sets independent of y in Ω= (0,1)x(0,1). $\mathcal{B}$ = g($\mathcal{D}$,{(x,y); y ≤ (1+x)/4}), $\mathcal{C}$ = g($\mathcal{D}$,{ x/4 ≤ y ≤ x/4 + 1/(x+1)}). Then E and F commute. We shall not make the calculations since the details will be given in the next example. Let us observe only that in the set {y ≤ (1+x)/4}ε$\mathcal{B}$-$\mathcal{D}$, $\mathcal{B}$ and $\mathcal{D}$ induce the same σ-algebras, that is, they intersect it in the same algebra. This means that in spite of $\mathcal{B}\supset\mathcal{D}$ on that set both coincide. This situation essentially replaces the inclusion case or if one wants is a generalization of it. In {M P_2}it is associated with the concept of "atomic relation". In {HN} it is said that sets of the mentioned kind are "conditional atoms".

IV) Let Ω = (0,1)x(0,1); $\mathcal{D}$ the algebra of Borel measurable sets independent of y; B = {(x,y); 0 ≤ y ≤ f(x)} ; f a Borel measurable function such that 0 ≤ f(x) ≤ 1 for xε(0,1). p and q two non-negative Borel measurable functions of x verifying

$$0 \le f-p \le f+q \le 1 \quad ;$$

$$C = \{f-p \le y \le f+q\} \quad ; \quad \mathcal{B} = g(\mathcal{D},B) \; ; \; \mathcal{C} = g(\mathcal{D},C).$$

PROPOSITION. EF = FE *iff* f(p+q) = p.

Proof: It will suffice to prove $F(I_B) = f(x)\, I_{B'}$ for any set $B = \{0 \le y \le f(x),\ x \in B'\}$ where B' is Borel measurable in (0,1). (This equality is equivalent to FE = EF since $f\, I_{B'}$ is $\mathcal{D}$-measurable). It is easy to see that $G(I_B) = f\, I_{B'}$. Besides: call $Q = \{(x,y);\ x \in Q',\ (f-p)(x) \le y \le (f+q)(x)\}$. Then:

$$(1) \qquad \int_Q f\, I_{B'}\, dx\, dy = \int_{B'Q'} f(x) \left(\int_{(f-p)(x)}^{(f+q)(x)} dy\right) dx = \int_{B'Q'} f(p+q)\, dx.$$

$$(2) \qquad \int_Q I_B\, dx\, dy = \int_{BQ} dx\, dy = \int_{B'Q'} p(x)\, dx.$$

Therefore, from (1) and (2) it follows that $F(I_B) = f\, I_{B'}$ iff $f(p+q) = p$ a.e. QED.

V. Commutation is present also in the following situation, which is a mixture of examples II and III. $\Omega = (0,1)^3$; $\mathcal{B}_x = \mathcal{B}_y$ = Borel sets of (0,1); $\mathcal{B}_z = g((0,1/2)$, Borel sets of $[1/2,1))$; $T = g(\Phi,\Omega)$; $\mathcal{B} = T x \mathcal{B}_y x \mathcal{B}_z$; $\mathcal{C} = \mathcal{B}_x x \mathcal{B}_y x T$.

VI. $\Omega = (0,1)^2$, $1 = \int\int p(x,y)\, dx\, dy$, $p \ge 0$; $M(x) = \int p\, dy$, $N(y) = \int p\, dx$ (M and N are the marginals densities); $\mathcal{B}$ and $\mathcal{C}$ are respectively the Borel sets independent of y and x; $E(h) = \int h(p/M(x))dy$, $F(h) = \int h(p/N(y))\, dx$. If, to fix ideas, we assume $p > 0$ and symmetric, then $0 < M = N$, and supposing f symmetric, we get:

$$FE\, f = k(y)/N(y) \ , \ EF\, f = k(x)/N(x),\ k = \int\{\int (fp/N)dy\}p\,dx/N(y)$$

The commutation of E and F requires k = constant x N, and therefore, in general, it is not verified.

REMARKS. It will be shown that on the atoms of $\mathcal{D}$, $\mathcal{B}$ is independent of $\mathcal{C}$ whenever E and F commute. In the examples II and IV, it is possible to say that the independence still holds in the sets "infinitely small" of $\mathcal{D}$, as we show next.

Case IV. Let us consider a cylinder Z independent of y whith basis (x,x+dx). Then, $P(B.C|Z) = (f(x) - (f(x) - p(x)))\, dx / 1.dx = p(x)$. Analogously, $P(B|Z)\, P(C|Z) = f(p+q)$. Since $p = f(p+q)$ is necessary and sufficient for the commutation we get the conditional independence of B and C given Z, that is, given an infinitely small set of $\mathcal{D}$.

Case II. If $B = B' \times (0,1)$, $C = (0,1) \times C'$, and $Z = (0,1)x(y,y+dy)x(0,1)$, then again $P(BC|Z) = P(B|Z)\, P(C|Z)$, as it is easy to see.

4. MONADIC AND BIADIC ALGEBRAS. The results of this section will not be used in the sequel. They are the precursors of results that will follow. or if one wants, a generalization of them. They show in which extent product structures play a role in this problem when

there is no inclusion, (cf. Th. 1). Next theorem 2 gives a representation that recalls Maharam's representation theorem (cf.{M}). Let us remember some definitions and theorems. A Boolean algebra A is called monadic with respect to the subalgebra B if B is conditionally complete: $\forall$ $a\varepsilon A, \exists$ $\inf(x\varepsilon B;\ x \geq a)\varepsilon B$. This is equivalent to give an operator ∇ in A with: $\nabla 0 = 0$, $\nabla x \geq x$, $\nabla(a\wedge\nabla b) = \nabla a\wedge\nabla b$; here $B = (a\varepsilon A;\ \nabla a = a)$. (Those properties immediately imply that ∇ is a closure operator: $\nabla(a\vee b) = \nabla a\vee\nabla b$, $\nabla\nabla a = \nabla a$, $\nabla 0 = 0$, $\nabla a \geq a$).
Given a filter F, $A/F = (a^*;\ c\varepsilon a^*$ iff $((a-c)\vee(c-a))'\varepsilon F)$. F is said monadic in the monadic algebra (A,B) if $F\wedge B$ generates F. It can be proved that this is equivalent to the possibility of introducing in a canonic way a ∇-operation in the quotient algebra: $\nabla h(a) = h(\nabla a)$ where h is the canonical homomorphism $A \to A/F$. (A,B) is said to be simple if $B = T = \{0,1\}$. It can be proved that A/F is simple iff F is a maximal monadic filter. Also that (A,B) is a subalgebra of the product Π (A/F; F maximal monadic in A).
An algebra is called biadic relative to the subalgebras B and C if it is monadic with respect to them and the corresponding closure operators, ∇_1, ∇_2, commute. Then $\nabla = \nabla_1\nabla_2$ defines another closure operator and obviously associated to the algebra $D = B\wedge C$. Every monadic filter F with respect to D is monadic with respect to B and C and therefore in A/F the induced operators ∇_i, i=1,2, commute. If F is also maximal monadic, in A/F, $\nabla x = 0$ or 1, that is $D/F = \{0,1\} = T$. In this situation, when D = T, A is called simple biadic.
Given the algebras M,N and P, we shall say that P is the direct sum of M and N, $P = M \oplus N$, if $P \supset M\cup N$ is generated by them, $M\wedge N = \{0,1\}$, and if when $m\varepsilon M$, $n\varepsilon N$ are comparable, one of them is 0 or 1.
From now on we shall suppose that whenever we speak of a biadic algebra (A,B,C), A is generated by B and C. We shall denote by S(A) the Stone's space associated to the algebra A.

THEOREM 1. *A biadic algebra* (A,B,C) *is simple iff* $A = B\oplus C$.

Proof: From the construction of the direct sum we see that the associated ∇-operators commute since their product is the trivial ∇-operator. If A is simple, since by a general hypothesis we already know that B and C generate A, it will suffice to show that of two comparable elements b,c, one is 0 or 1 to have $A = B \oplus C$. Let $b \leq c$, then, $d = \nabla_2 b = \nabla b \leq c$. When $0 \neq c \neq 1$, $d = 0$ and therefore b=0.

QED.

THEOREM 2. *If* A *is biadic, then it is isomorfic to a subalgebra of a product* $\Pi(B_M \oplus C_M;$ M *is a maximal monadic filter*).

Proof: In fact, it is known that it is isomorphic to a subalgebra of Π(A/M; Mϵmaximal monadic filters). Since A/M = $B_M \vee C_M$, where B_M and C_M are the algebras B/M and C/M, we can apply Th. 1. QED.

COROLLARY. *Let* βX *be the Stone-Čech compactification of* X, *and* $T = \beta(\Sigma_M(S(B_M) \times S(C_M)))$, *where* M *runs on the set of maximal monadic filters. Then, there exists a continuous application of* T *onto* S(A) *that induces the injective homomorphism of* A *into* Π(A/M).

Proof: It is a trivial consequence of the functoriality of Stone's representation and the fact that two complementary clopen sets on the topological direct sum of the spaces $S(B_M \oplus C_M)$ has disjoint complementary closures on T (cf. {GJ}, chp.6)

THEOREM 3. *When* A *is generated by* B *and* C, *and* (A,B,C) *is biadic, it is possible to decompose* S(A) *in a union of disjoint closed sets defining an open and closed equivalence relation associated to* $\nabla = \nabla_1\nabla_2$ *and such that the induced topology on each of those sets defines on them the algebras of clopen sets coinciding with the direct sum of those induced by the clopen sets of* B *and* C. *There are as many maximal monadic filters as there are equivalent classes and the quotient space is isomorphic to* S(D).

Proof: Let us only sketch the proof. From Stone's representation we know that if F is a filter the canonic application i: S(A/F) $\to$ S(A) is a continuous injection which is open iff F is principal. (This follows immediately observing that Stone's space of A/F is isomorphic to the closed set associated to F with the induced topology of S(A)). Recall now that (A,∇) is a monadic algebra if on S(A) it is possible to introduce an open and closed relation such that $\forall$ aϵA , ∇a = sat a ; (cf. {$H_{1,2}$}). Consider the family of equivalence classes of the points of S(A) under the closed relation induced by $\nabla = \nabla_1\nabla_2$. These classes are disjoint and closed as sets of S(A). With the induced topology they are Stone's spaces, and exactly,those corresponding to the algebras A/M, M a maximal monadic filter.(Given M, an equivalence class is defined by all the ultrafilters containing M). QED.

The difference with the corollary to theorem 2 is in the fact that each equivalence class defines a closed set and not a clopen set, as in this corollary.

REMARK: Since we will not develop systematically the algebraic approach, we shall not care about most general statements concerning monadic operators.

5. BOOLEAN MEASURE ALGEBRAS AND CLOSURE OPERATORS. A Boolean probability measure algebra A and a σ-subalgebra B provide one of the oustanding examples of monadic algebras as it is easy to see since if P denotes the measure, $\inf\{P(b);\ b \geq a,\ b \in B\}$ defines a number h for which exists a $b_0 \in B$ with $b_0 \geq a$, and $P(b_0) = h$, i.e., $\nabla a = b_0$. We shall call ∇ a closure operator or a ∇-operator. If two subalgebras B and C, are considered, we get a biadic algebra whenever the associated ∇-operators commute. In the setting in which we are interested described after theorem 3 of section 2: $\{A,B,C\}$ with probability P, $\nabla_1 A$, $A \in A$, is the set defined by $\{P(A|B) > 0\}$. Naturally comes the first question: in which extent commutation of the conditional expectation operators and that of the ∇-operators are related? We answer this next and also provide a result similar to lemma 1 in the introduction, but for the closure operators.

THEOREM 1. *i) If $\nabla_1\nabla_2$ defines an operator ∇ with the same closure properties as the ∇_j's, then this is the operator associated to* $B \wedge C$.
ii) If $\nabla_2\nabla_1 = \nabla$ is a closure operator then the ∇_j's, $j=1,2$, commute, and conversely, if they commute their product defines a closure operator.
iii) If two conditional expectation operators commute, then the associated closure operators also commute.
iv) If $\nabla_1 C \subset C$ then $\nabla_2 B \subset B$ and ∇_1 commutes with ∇_2.

Proof. i) The following properties define a closure operator: $\nabla 0 = 0$, $\nabla x \geq x$, $\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$, and $\{x;\ \nabla x = x\}$ determines the associated subalgebra. Therefore if $\nabla_1\nabla_2$ defines a closure operator ∇ then $x = \nabla x$ implies $x = \nabla_2 x \in B$, and therefore, to $B \wedge C$. It proves i).
ii) If the ∇_i's commute their product verifies the properties defining a closure operator, as it is easy to see. Assume $\nabla_2\nabla_1 = \nabla$. By i) we know the algebra associated to ∇. If the thesis were false it would exist a $c \in C$ such that $b = \nabla_1 c \notin B \wedge C$; in fact, if it always belonged to $B \wedge C$ it wouldbe possible to verify for $\nabla_1\nabla_2$ the conditions defining a closure operator. By i) it woul imply $\nabla_1\nabla_2 = \nabla$, which by hypothesis and i) must coincide with $\nabla_2\nabla_1$.
Let $b^\circ = \nabla_2 b$. Then $b^\circ \in B \wedge C$ and $b^\circ - b \neq 0$. It holds: $(\nabla_2(b^\circ - b)) \wedge c = 0$ and also: $(\nabla_2(b^\circ - b)) \wedge b \neq 0$. Therefore, $b^\circ - \nabla_2(b^\circ - b) \in B \wedge C$, $\neq b$, and $\geqslant c$, a contradiction.
iii) For $A \in A$, it always holds $\nabla_1\nabla_2 A \leq \nabla A \in B \wedge C$. On the other hand we have $E(P(A|B)|C) = P(A|B \wedge C)$. The second member is greater than zero exactly in the set ∇A. Finally, where the first member is greater than zero we also have:

$$(\bullet) \qquad P(\nabla_2 A | B) > 0$$

But (•) is verified exactly on $\nabla_1\nabla_2 A$. In consequence $\nabla_1\nabla_2 A \geq \nabla A$.
iv) Let $b \in B$. Then $\Delta_1(\nabla_2 b) = (\nabla_1(\nabla_2 b)')' \in C$. From $b \leq \Delta_1(\nabla_2 b) \leq \nabla_2 b$, it follows $\Delta_1\nabla_2 b = \nabla_2 b$, i.e., $\nabla_2 b \in B$. Let us see that the operators commute. $\nabla_1 C \subset C$ implies $\nabla_2\nabla_1 x < \nabla_2\nabla_1(\nabla_2 x) = \nabla_2[\nabla_1(\nabla_2 x)] = \nabla_1(\nabla_2 x) = \nabla_1\nabla_2 x$. The opposite inequality follows from $\nabla_2 B \subset B$. QED.

There exists formal analogy between Lemma 1, §1, and the preceding theorem, that is, between the ∇'s and E's operators, partly justified because of iii) of theorem 1. The study of this analogy is pursued further in theorems that will follow, but results not concerning the subject of this paper will not be included.

6. CONDITIONAL ATOMS AND MAXIMAL FREE FILTERS. Given (Ω, A, P) and a σ-subalgebra L, an element $A \in A$ is said a *conditional atom* or an *L-atom* if $L \wedge A = A \wedge A$, (cf. {HN}). We can introduce the concept in (A,L), a monadic algebra A with a subalgebra L: a is an L-atom if $L \wedge a = A \wedge a$. In what follows of this section we relate the concept of L-atom with that of free filter and it will not be used in the rest of this paper. The homomorphism i: $L \ni b \rightarrow b \wedge a \in I_a$ = principal ideal generated by a, has kernel $F_{\nabla a}$ = principal filter generated by ∇a. If this map is onto a is an L-atom. When $\nabla a = 1$ it is said that a is a *free element* relative to L. The restriction is not serious since for $b \in L$ the intersection algebra $(A \wedge b, L \wedge b)$ is again a monadic algebra. The relevant fact is that the homomorphism i is a bijection whenever a is simultaneously a free element and an L-atom, as it is easy to see. A filter is called *free* if any of its elements is free. It is well-known that any free filter is contained in a maximal free one.
Let us see now a characterization of conditional atoms.

THEOREM 1. *Let* $\nabla a = 1$. a *is an* L-*atom iff the principal ideal generated by* a *in* A, F_a, *is a maximal free filter.*

Proof: The condition is necessary: F_a is free, if not maximal there exists $y \leq a$, $y \neq a$, which generates a free filter. But $y = x \wedge a$, $x \in L$, and $\nabla y = 1$. Therefore, $\nabla x = x = 1 \geq a$, and $y = a$, contradiction
Sufficiency: if a were not L-atom, it would exist $y < a$ such that $y \notin L \wedge a$. Therefore $z = y \vee (a - \nabla y) \neq a$, $\nabla z = \nabla a = 1$. Then F_z is free and contains properly to F_a . QED.

We are tempted to mention now the following theorem due to Halmos: if F is a maximal free filter of a monadic algebra (A,L) and L is complete then any equivalence class of A/F contains exactly one element of L.

7. CONDITIONAL ATOMS AND COMPLETELY DIMINISHABLE ELEMENTS. Let (Ω, A, P) be a probability space and L a σ-subalgebra. Equalities and inequalities are always a.e.. An element $N \in A$ will be called *diminishable* (relative to L) if there exists $Q \subset N$, $Q \in A$, $Q \neq \emptyset$, such that $\nabla(N - Q) = \nabla N$, where the closure operation is taken with respect to L. $M \in A$ will be called *completely diminishable* if it is nonvoid and every nonvoid subset of M is diminishable.

PROPOSITION 1. *If* M *is completely diminishable, then there exists* $Q \subset M$, $Q \neq \Phi$, $Q \in A$, *such that* $\nabla Q = \nabla(M - Q) = \nabla M$.

Proof: Define $Q_0 = \Phi$, and transfinitely, $Q_k \subset M - \sum_{j<k} \nabla Q_j$ such that $\nabla(M - Q_k - \sum_{j<k} \nabla Q_j) = \nabla(M - \sum_{j<k} \nabla Q_j)$. Call $Q = \Sigma Q_k$. Then M has the same closure as M - Q. By construction the same as Q. QED.
The converse is false: take $\Omega = M = (0,1)$, $L = T$, $A = g(A)$ where A is the set $(0,1/2)$, $Q = (0,1/2)$.

EXAMPLE: (D. Maharam, cf. {M}). Assume that A is a homogeneous σ-algebra, and that L is also homogeneous and Boolean σ-isomorphic to a B_Λ (see section 2). If $\dim L < \dim A$, then Ω is completely diminishable.
In fact, it follows from the definition of homogeneity and the next theorem, proposition ii).
Let us introduce another definition. We shall say that a set $M \in A$ is *not sectionable* (relative to L) or has not the *sectioning property* if $\forall\, N \subset M$, $\Phi \neq N \in A$, there exists $Q \subset N$, $\Phi \neq Q \in A$ such that $\nabla Q = \{P(Q|L) > 0\} \subset \{0 < P(Q|L) < P(N|L)$. The reasons why we have chosen that adverb will be explained later.
Finally a useful remark to be used in the following theorem. A set N is a conditional atom iff for any $A \in A$, $A \subset N$, it holds: $A = N \wedge \nabla A$. That is, the closure of any subset A of N could be greater than A but only in a subset contained in the complement of N. The proof is immediate.

THEOREM 1. *i) A set is not diminishable iff it is an* L*-atom.*
ii) A set is completely diminishable iff it does not contain L*-atoms.*
iii) A set is diminishable iff it is not sectionable.

Proof: i) Let N be an L-atom. Assume it is diminishable: there exists $Q \neq \Phi$ contained in N with $\nabla(N - Q) = \nabla N$. Therefore $N = N \wedge \nabla(N-Q) = N - Q$.
Suppose now that N is not an L-atom. Therefore, there exists $H \subset N$ with $H \neq N \wedge \nabla H$. Set $Q = H \wedge \nabla(N - H)$, then $Q \neq \Phi$. Now, $\nabla(N - Q) =$

$= \nabla\{(N - H) \vee (N - \nabla(N - H))\} = [\nabla(N - H)] \vee [\nabla N - \nabla(N - H)] = \nabla N$.
In other words, N is diminishable.
ii) follows from i) and the definitions. iii) follows easily from the definitions and $P(Q \vee (N - Q)|L) = P(Q|L) + P(N - Q|L)$. QED.

PROPOSITION 2. *If* N *is an L-atom and* $A \ni A \subset N$, *then:*
$$(°) \qquad P(A|L) = P(NA|L) = 1_{\nabla A} \cdot P(N|L) .$$
Conversely, if (°) *holds,* N *is a conditional atom.*

Proof: If N is an atom, $P(NA|L) = P(N.\nabla A|L)$. Conversely, if $N.\nabla A - A \neq \Phi$ then $P(NA|L) \neq P(N.\nabla A|L)$. QED.

The proposition says that $P(A|L)$ is obtained "sectioning" with ∇A to $P(N|L)$. This cutting cannot be made in a diminishable set and explains the nomenclature used above. The proposition "N sectionable implies (°)" was proved in $\{MP_2\}$. The proposition also appears in {HN} where other properties of atomicity are studied. This paper contains also a proof of next theorem. In $\{MP_2\}$it was observed that non-sectionability, now shown to be equivalent to non-atomicity, implies the same thesis as next theorem. There, it was said that the result is in its essence, nothing but a lemma used in {M} by Maharam, and in fact, it is an abstraction of that lemma whose proof can be used without changes. As a matter of fact, we repeat the proof for the sake of completeness. In {HN} the demonstration follows a shorter way.

THEOREM 2. *If the set* M *is completely diminishable and* f *is an L-measurable function such that* $0 \leq f \leq P(M|L)$ *then there exists* $N \subset M$ *such that* $f = P(N|L)$.

(Hanen and Neveu prove the following proposition: for any set C and function $f \in L$-measurable satisfying $0 \leq f \leq P(C|L)$ there exists two disjoint subsets of C, A and B, B a conditional atom, such that $P(A|L) \leq f \leq P(B + A|L)$. If C does not contain conditional atoms then $P(A|L) = f$).
As an application of theorem 2 we have lemma 2 of {M} :

PROPOSITION 3. *If* A *is a homogeneous σ-algebra, and* L *a σ-subalgebra Boolean σ-isomorphic to a* B_Λ, *then* dim A > dim L *implies that if* $0 \leq f \leq P(M|L)$ *there exists* $N \subset M$ *with* $P(N|L) = f$.

(It follows from theorem 2 and the example described above).

Theorem 2 has a converse:

THEOREM 3. *If* $\forall$ $N \subset M$ *it holds that* $\forall$ f; $0 \leq f \leq P(N|L)$ *there exists* $M' \subset N$ *such that* $f = P(M'|L)$, *then* M *is completely diminishable.*

Proof: Take $f = P(N|L)/2$. Then $P(M'|L) = 1/2 \cdot P(N|L) = P(N - M'|L)$ and $\nabla M' = \nabla(N - M') = \nabla N$. QED.

If we did not require $M' \subset N$, the implication would be false. Take $N = (0,1/2)$, $\Omega = (0,1) = M$, $L = \{\Omega,\Phi\}$, $A = g(N,B(1/2,1))$. N is an atom and every constant function not greater than 1/2 is the conditional expectation with respect to L of a subset of M.

Proof of theorem 2. It is sufficient to prove that there exists $N' \subset M$ such that $P(N'|L) \leq f$. In fact, defining recursively N_j as a set contained in $M - \sum_{i<j} N_i$ verifying $P(N_j|L) \leq P(M - \sum_{i<j} N_i|L)$, we get finally a set $N' = \sum_k N_k$ with the desired property. This involves an exhaustion procedure which will be often used. One way of substituting this method by another one is to use axiom of choice in its maximal-element form. Let us prove the existence of such an N'. The crucial point is to exhibit a $B \subset M$ with

$$(*) \qquad \{0 < P(B|L) < P(M|L)\} = \{0 < P(M|L)\}.$$

But (*) says that ∇M coincides with the intersection of ∇B and $\{P(M - B|L) > 0\} = \nabla(M - B)$. Such a B exists because of proposition 1. Call $C = \{0 < P(B|L) < 1/2 \cdot P(M|L)\}$, $D = \{1/2 \cdot P(M|L) \leq P(B|L)\}$. $B_1 = (C \wedge B) \vee (D \wedge (M - B))$. Then, $\nabla B_1 = \nabla M$ and $P(B_1|L) \leq 1/2 \cdot P(M|L)$. Repeating the process, we can prove that exists a sequence $\{B_n\}$:

$$(**) \qquad \nabla B_n = \nabla M,\ P(B_n|L) \leq P(M|L)/\ 2^n,\ n = 1,2,\ldots,\ B_n \subset M.$$

For a certain n, $P(f > P(M|L)/\ 2^n) > 0$ if $f \neq 0$. Define

$$N' = B_n \wedge \{P(B_n|L) \leq f\} = B_n \wedge H$$

$N' \neq \Phi$ because $\nabla N' = H \wedge \nabla B_n = H \wedge \nabla M = H$. Besides $N' \subset M$ and from (**) $P(N'|L) \leq f$. QED.

From now to the end of this section we shall generalize the preceding notions and theorems. The proofs are trivial or similar to those given before.

We shall not go into troubles adapting them since in this moment what only matters for us is to have a prospective view of the subject. Let A, L and S be σ-algebras of subsets of Ω and $L \subset A \supset S$. (Ω, A, P) a probability space. A set N will be called S-diminishable if $N \in A$ and there exists $Q \in S \wedge N$, $\Phi \neq Q \subset N$ such that $\nabla N = \nabla(N - Q)$ and where ∇

is taken with respect to L. A set $M\in A$ will be called *completely S-diminishable* if every subset A-measurable is S-diminishable. Given L and S we shall say that $M\in A$ is an *(L,S)-atom* if $S\wedge M = L\wedge M$. Therefore, L-atom coincides with (L, A)-atom.

THEOREM 4. *i) M is completely S-diminishable iff* $\forall\, N\subset M, \Phi \neq N\in A, \exists$ $Q\subset N,\ \Phi \neq Q\in S\wedge M\ ;\ \nabla Q\subset\{0 < P(Q|L) < P(N|L)\}$.
ii) Assume $L\subset S$. *Then: a set of A is S-diminishable iff it is not an (L,S)-atom.*
iii) Under the same condition a set of A is completely S-diminishable iff it does not contain an (L,S)-atom.
iv) Assume M is completely S-diminishable. If f is L-measurable and $0 \leq f \leq P(M|L)$ *then there exists* $M'\in g(L,S)\wedge M$ *such that* $f = P(M'|L)$.
v) Let $L\subset S$. *If M does not contain an (L,S)-atom then for every* $f\in L$*-measurable,* $0 \leq f \leq P(M|L)$ *, there exists* $M'\in S\wedge M$ *such that* $f = P(M'|L)$.

(iv) was mentioned in $\{MP_2\}$, but in its equivalent form shown in i).)

8. ATOMS OF THE INTERSECTION ALGEBRA. The setting is the one described after theorem 3, section 2.We are interested in discovering what happens in the atoms of D when E and F commute. We shall show that there B and C are independent (*) and since this answer is pleasant enough we shall go into the complement of the atomic part to see how the algebras are related in that part. This "local" study can be done because of the following proposition.

PROPOSITION 1. *If* $\{D_n,\ n=1,2,\ldots\}$ *defines a partition of* Ω *by sets of D then E and F commute on each* D_i *iff they commute on* Ω.

Proof: Given the set $A\in A$, we shall denote by $A_\circ$, $B_\circ$, $C_\circ$, $D_\circ$, the restrictions to A of the algebras A, B, C, D. Observe that $A_\circ = g(B_\circ, C_\circ)$ but only $D_\circ\subset B_\circ\wedge C_\circ$.
To say that E and F commute on A means: $E_\circ$ and $F_\circ$ commute on $(A, A_\circ, P/P(A))$ where $E_\circ(F_\circ)$is the conditional operator associated to $B_\circ(C_\circ)$ and, if $A\in D$, then $E_\circ = E(F_\circ = F)$. The corresponding closure operator will be designated by ∇_{01} (∇_{02}).
We have: $EF\, f = EF\, \Sigma\, 1_{D_n}.f = \Sigma\, EF(1_{D_n}.f) = \Sigma\, E_nF_n(1_{D_n}.f)$ and this implies that the commutation of $E_\circ$ and $F_\circ$ is equivalent to that of

(*) Independence of B and C on D means $B\wedge D$ independent of $C\wedge D$ with respect to the probability $P(.)/P(D)$.

E and F. This proves the proposition.

Elimination of the atoms of $\mathcal{D}$ implies the eradication of the atoms of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, as the following proposition shows (the proof is left to the reader).

PROPOSITION 2. *Any atom* H *of* $\mathcal{A}$, $\mathcal{B}$ *or* $\mathcal{C}$ *is contained in an atom of* $\mathcal{D}$, *precisely*, ∇H.

When $\mathcal{A} = g(\mathcal{B},\mathcal{C})$, we defined in section 4 direct sum of $\mathcal{B}$ and $\mathcal{C}$ which is equivalent to say that any $B\in\mathcal{B}$ intersects any $C\in\mathcal{C}: B\wedge C \neq \Phi$ (then $B\wedge C = T$), if $B \neq \Phi$, $C \neq \Phi$.

THEOREM 1.*i) If* P *is a probability on* $\mathcal{A}$ *which is equal to* $\mathcal{B} \oplus \mathcal{C}$ *and* EF=FE, *then* $\mathcal{B}$ *and* $\mathcal{C}$ *are independent.*
ii) If $\nabla_1\nabla_2 = \nabla_2\nabla_1$ *on* $\mathcal{A}$, *then* $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$ *whenever* $\mathcal{B}\wedge\mathcal{C} = T$
iii) If EF = FE, *then* $\mathcal{B}$ *and* $\mathcal{C}$ *are independent, whenever* $\mathcal{B}\wedge\mathcal{C} = T$.

Proof: i) Let $C\in\mathcal{C}$, $B\in\mathcal{B}$. $G\, 1_C = a.1_\Omega$ because of the triviality of the intersection algebra and lemma 1, section 1. Then P(C) = a. Analogously P(B) = b. From $E\, F\, 1_{BC} = G\, 1_{BC} = E\, 1_B\, F\, 1_C = ab\, 1_\Omega$, we obtain P(BC) = P(B)P(C).
ii) If $B.C = \Phi$ then $\Omega - C \supset \nabla B = \nabla_2 B$ and the intersection algebra would not be trivial.
iii) follows from Th. 1, iii), section 5 and i) and ii) of this theorem. QED.
iii) was proved in $\{MP_2\}$.

COROLLARY 1. *On the atoms of* $\mathcal{D}$, $\mathcal{B}$ *and* $\mathcal{C}$ *are independent iff* EF=FE. *In the complement of the* $\mathcal{D}$*-atomic part of* Ω *neither* $\mathcal{A}$ *nor* $\mathcal{B}$ *nor* $\mathcal{C}$ *have atoms.*

In fact, it follows from the preceding theorem and propositions 1 and 2.

COROLLARY 2. *If* $\mathcal{A}$ *is purely atomic and* EF = FE, *then* Ω *can be represented on* NxN $(N=\{0,1,2,\ldots\})$ *in such a way that the* σ*-algebras* $\mathcal{B}$ *and* $\mathcal{C}$ *correspond with the* σ*-algebras of sets parallel to the axes,* NxN *is decomposed into rectangles with disjoint projections where the masses are concentrated, and on each rectangle* P *is the product of its marginal distributions but for a constant.*

In fact, if $\mathcal{A}$ is atomic so it is any σ-subalgebra. The rectangles correspond to the atoms of $\mathcal{D}$ and the corollary is a direct application of the preceding theorem. The mentioned constant is the measure of the rectangle.

COROLLARY 3. *If* P *is a probability measure on* NxN *positive on each point and* $\mathcal{B}$ *and* $\mathcal{C}$ *are respectively the "vertical and horizontal lines" then* $P = P_1 x P_2$, *whenever* EF = FE.

It follows from the preceding corollary.

REMARKS. 1) In the proof of the preceding theorem we showed that in an atom of $\mathcal{D}$ two sets, B, C, intersect if they are not void. This can be generalized: if $\nabla_2 B \supset A$ and $C.\nabla_1 A \neq \Phi$ then $B.C \neq \Phi$. In fact, $\nabla_2(B.C) = (\nabla_2 B).C = C.\nabla_2\nabla_1 B \supset C.\nabla_1 A \neq \Phi$ implies $B.C \neq \Phi$. $\nabla_2 B \supset A$ whenever A is an atom of $\mathcal{A}$ and $B.\nabla_2 A \neq \Phi$. In fact, in this case, if there is not inclusion $A.\nabla_2 B = \Phi$ and then $\nabla_2(B.\nabla_2 A) = \Phi$, contradiction.

2) If D is an atom of $\mathcal{D}$ the filter F_D is maximal monadic with respect to ∇ and therefore $\mathcal{A}/F_D \approx \mathcal{A}_o = \mathcal{B}_o \oplus \mathcal{C}_o$ where the direct sum is understood in the sense of Boolean algebras. Since $\mathcal{A}_o = g(\mathcal{B}_o, \mathcal{C}_o)$ it can be interpreted in the sense of Boolean σ-algebras, (cf. §5,6).

3) If $\Omega_o \subset \Omega$, $\Omega_o \in \mathcal{A}$, and $\mathcal{L} \subset \mathcal{A}$, then if $\mathcal{L}_o = \mathcal{L} \wedge \Omega_o$ is trivial, $\nabla^{\mathcal{L}} \Omega_o$ is an atom of $\mathcal{L}$, as it is easy to see. This implies that if ∇_1 and ∇_2 commute and $\mathcal{A}_o = \mathcal{B}_o \oplus \mathcal{C}_o$ on Ω_o, it is contained in an atom of $\mathcal{D}$ because $\mathcal{D}_o \subset \mathcal{B}_o \wedge \mathcal{C}_o = \{\Phi, \Omega_o\}$. This means that if on a set of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are independent then this set is contained in an atom of $\mathcal{D}$. Therefore, after discarding the atoms of $\mathcal{D}$ no trace of independence will be found. Spurious forms of independence can appear anyhow. The typical example of this bastard type of independence is shown in example II of section 3.

4) We already said and can be easily verified that the concept of conditional atom generalizes that of atom. Next lemma characterizes some of them when $\mathcal{L} = \mathcal{D}$.

LEMMA 1. *If* ∇_1 *and* ∇_2 *commute and* $B \in \mathcal{B}$ *then* B *is a* $\mathcal{D}$*-atom relative to* $\mathcal{B}$ *iff* $\mathcal{B}_o \subset \mathcal{C}_o$.

Proof: Assume B is a $\mathcal{D}$-atom relative to $\mathcal{B}$, i.e. $\mathcal{D}_o = \mathcal{B}_o$, then if B' is $\mathcal{B}$-measurable and contained in B, we have: $B' = B.\nabla B' = B.\nabla_2 B \in \mathcal{C}_o$. Assume $\mathcal{B}_o \subset \mathcal{C}_o$, $B' \in \mathcal{B}_o$. Then, $B' = B.C$ for certain $C \in \mathcal{C}$. Applying ∇_1, we get: $B' = \nabla_1 B' = \nabla_1(B.C) = B.\nabla_1 C = B.\nabla C$, and this means that B is a $\mathcal{D}$-atom with respect to $\mathcal{B}$.

5) We have shown in remark 3) that if independence of $\mathcal{B}$ and $\mathcal{C}$ ap-

pears in a set A of $\mathcal{A}$, it is included in a set of $\mathcal{D}$ where still those algebras are independent. An analogous fact occurs for inclusion as is shown in next lemma. Lemma 1 says that if $A_\circ \in \mathcal{B}$, and $\mathcal{B}_\circ \subset \mathcal{C}_\circ$, then $A_\circ$ is a $(\mathcal{D},\mathcal{B})$-atom. But if $A_\circ \in \mathcal{C}$, and $\mathcal{B}_\circ \subset \mathcal{C}_\circ$ then $A_\circ \subset A_1 = \nabla_1 A_\circ \in \mathcal{D}$ which is a $(\mathcal{D},\mathcal{B})$-atom (lemma 2), and then because of lemma 1, $\mathcal{B}_1 \subset \mathcal{C}_1$.

LEMMA 2. *Assume* $\nabla_1\nabla_2 = \nabla_2\nabla_1$. *If* $A_\circ \in \mathcal{C}$ *and* $\mathcal{B}_\circ \subset \mathcal{C}_\circ$ *then* A_1 *is a* $\mathcal{D}$*-atom relative to* $\mathcal{B}$.

Proof: Let $B \in \mathcal{B}$, $B_1 = B.A_1 = B.\nabla_1 A_\circ = \nabla_1(B.A_\circ) = \nabla_1(C.A_\circ)$ with $C \in \mathcal{C}$. Therefore $B_1 \in \mathcal{D}$.

6) In relation with this remark, cf. {HN} . Assume our probability space is $(\Omega, \mathcal{M}, P)$ and $\mathcal{L} \subset \mathcal{M}$, a σ-subalgebra. By a conditional atom we shall understand one $\mathcal{L}$-atom relative to $\mathcal{M}$. This will be applied in the case $\mathcal{M} = \mathcal{B}$ and $\mathcal{L} = \mathcal{D}$.

LEMMA 3. *i) If* $\{A_\alpha\}$ *is a chain of conditional atoms, then* $A = \sup A_\alpha$ *is a conditional atom. Every subset of a conditional atom is an* $\mathcal{L}$*-atom.*

ii) Given a conditional atom there exists a maximal conditional atom containing it.

iii) If $\{A_n\}$, $n=1,2,\ldots$, *is a sequence of conditional atoms such that the* ∇A_n *are pairwise disjoint, then* ΣA_n *is a conditional atom.*

iv) If A *is a maximal conditional atom and* $Z = \sup\{B;\ B$ *is a conditional atom*$\}$ *then* $\nabla A = \nabla Z$.

v) If A *and* A' *are disjoint conditional atoms there exists a conditional atom containing* A *maximal with respect to the property of beeing disjoint to* A'.

vi) $Z = \sum_{n=1}^{\infty} A_n$, *where* A_n *is a conditional atom maximal with respect to the property of beeing disjoint to* $A_1 + \ldots + A_{n-1}$.

Proof: Sup A_α is essentially denumerable, i.e. $A = \sup A_n$ a.e. where A_n is increasing. From $\nabla \sup A_n = \sup \nabla A_n$ we get i), and ii). iii) follows easily. Since $\nabla Z \supset \nabla A$, if $Z - \nabla A \neq \Phi$, it would contain an $\mathcal{L}$-atom and from iii) it would follow a contradiction. v) is proved like i).

7) Denote with $T_\circ$ the union of the family of atoms of $\mathcal{D}$ and with $T_1(T_2)$ the maximal set of $\mathcal{D}$ where $\mathcal{B} \subset \mathcal{C}$ $(\mathcal{C} \subset \mathcal{B})$.

LEMMA 4. *If* A *is an atom of* $\mathcal{D}$, *then* $A \subset T_\circ \wedge T_1$ *when and only when* A *is an atom of* $\mathcal{B}$.

Proof: It is left to the reader.

LEMMA 5. *If* A *is an atom of* $\mathcal{D}$, $A \subset T_o \wedge T_1 \wedge T_2$ *is equivalent to* A *is an atom of* $\mathcal{A}$.

Proof: Trivial, after lemma 4.

9. COMMUTATIVITY. In this section we study some situations -in the general context we already admitted- which preserve the commutation of E and F. For example generalizing proposition 1 of section 8 we have:

LEMMA 1. *i) If* $A \subset \Omega$ *is* $\mathcal{B}$*-measurable then* $E_o F_o = F_o E_o$ *on* A. *Idem if* $A \varepsilon \mathcal{C}$.
ii) If $\mathcal{C}' = g(\mathcal{C}, B_1, B_2, \ldots)$ *where* $B_i \varepsilon \mathcal{B}$ *and* $\{B_i\}$ *is a partition of* Ω *then* $EF' = F'E$; $F' = E(./\mathcal{C}')$.

Proof: i) From (3), section 2, we have: $F_o(.) = F(.)\ 1_A/F(1_A)$. Applied to sets of $\mathcal{B}_o$, because of the commutation of E and F and that $A \varepsilon \mathcal{B}$, the right member is $\mathcal{B}_o$ measurable. Lemma 1 of section 1 implies then that F_o and E_o commute.
ii) Every element C' of $\mathcal{C}'$ is of the form $\Sigma\ C_j B_j$; then

$$E(1_{C'}) = E(\ \Sigma\ 1_{B_j C_j}\ ) = \Sigma\ 1_{B_j}\ E(1_{C_j})$$

and this function is clearly $\mathcal{B}$-measurable. QED.

If instead of the conditional expectation operators we consider the closure operators, i) of the preceding lemma is generalized by i) of next lemma.

LEMMA 2. *i) If* $A_o \varepsilon \mathcal{B}$ *then* ∇_{01} *commute with* ∇_{02} , *whenever* $\nabla_1 \nabla_2 = \nabla_2 \nabla_1$
ii) If $A_n \uparrow A_o$ *and the commutation of the closure operators holds on each* A_n, *it is also valid on* A_o, $(n=1,2,\ldots)$.
iii) If $A_n \uparrow A_o$ *and on each* A_n, $E_n F_n = F_n E_n$, *then* $E_o F_o = F_o E_o$.

Proof: i) It follows immediately that:

(1) $\nabla_{01} H = (\nabla_1 H).A_o$ whenever $H \subset A_\bullet$, whatever it be $A_\bullet$.

Then: $\nabla_{02}(\nabla_{01} H) = A_o \nabla_2 \nabla_1 H$ since $A_o \varepsilon \mathcal{B}$. Besides: $\nabla_{01} \nabla_{02} H = \nabla_1 (A_o \nabla_2 H) = A_o \nabla_1 \nabla_2 H$.
ii) $\nabla_{n1} \nabla_{n2} (X.A_n) = A_n . \nabla_1 \{A_n . \nabla_2 (X.A_n)\} \uparrow A_o . \nabla_1 \{A_o . \nabla_2 (X.A_o)\}$.

Then, $\nabla_{n1}\nabla_{n2}(X.A_n)\uparrow\nabla_{01}\ \nabla_{02}\ (X.A_o)$. This implies the commutation of the closure operators on A_o.

iii) follows easily applying formula (3) of section 2.

(°) $F_n E_n(f\ 1_{A_n}) = F\ \{\ 1_{A_n} E(f\ 1_{A_n})\ /\ E(1_{A_n})\ \}\ 1_{A_n}\ /\ F(1_{A_n})$.

Taking $0 \leq f \leq 1$, we see that (°) converges to $F_o E_o(f\ 1_{A_o})$.

iii) is a ready consequence of this. QED.

LEMMA 3. *Let* $(\mathcal{B}_n)$ *and* $(\mathcal{C}_n)$ *be increasing sequences of σ-algebras (completed in A). If* $\mathcal{B} = g(\mathcal{B}_1,\mathcal{B}_2,\dots)$, $\mathcal{C} = g(\mathcal{C}_1,\mathcal{C}_2,\dots)$ *and for each* $n, n = 1,2,\dots$, $E_n F_n = F_n E_n$, *then* $EF = FE$.

Proof: Call $\mathcal{B}_o = \bigcup_1^\infty \mathcal{B}_n$. From martingale theory, we have for $B \in \mathcal{B}_o$

$$F\ 1_B = \lim_n F_n\ 1_B \quad \text{a.e.}$$

If $B \in \mathcal{B}_{n_o}$, $n \geq n_o$, because of the commutation the function $F_n 1_B$ is $\mathcal{B}_n$-measurable and therefore $F\ 1_B$ is $\mathcal{B}$-measurable.

The set $\{\ B \in \mathcal{B}\ ;\ F\ 1_B$ is $\mathcal{B}$-measurable$\}$ contains $\mathcal{B}_o$ and is a monotone class, therefore, since $\mathcal{B} = g(\mathcal{B}_o)$ it coincides with $\mathcal{B}$.

LEMMA 4. *Let* A *and* A' *be* A*-measurable sets and* D *a set of* $\mathcal{D}$ *such that* $A \subset D$, $A' \subset \Omega - D$. *If* $\mathcal{B}$ *and* $\mathcal{C}$ *commute on* A *and* A' *then they commute on* $A \cup A'$.

Proof: Denote E_o, F_o, $(E_1, F_1, \bar{E}, \bar{F})$, the conditional expectation operators associated to the restriction of the algebras to A (A', A + A'). From:

$$\bar{E}\left(1_{C(A+A')}\right) = \frac{E\ 1_{C(A+A')}}{E\ 1_{A+A'}}\ 1_{A+A'} = \frac{E\ 1_{CA}}{E\ 1_A}\ 1_A + \frac{E\ 1_{CA'}}{E\ 1_{A'}}\ 1_{A'} = f + g,$$

taking into account that f is $\mathcal{C}_o$-measurable and g is $\mathcal{C}_1$-measurable, we see that $\bar{E}\ 1_{C(A+A')}$ is $\bar{\mathcal{C}}$-measurable, thanks to the fact that D separates A from A'. QED.

(The same result holds if instead of commutation of the conditional expectation operators we ask for commutation of the closure operators.)

Combining lemma 2 and a passage to the limit we obtain:

LEMMA 5. *If* $\mathcal{B}$ *and* $\mathcal{C}$ *commute on each set* A_i, $i=1,2,\dots$, *such that the* ∇A_i*'s form a partition of* Ω, *then they commute on* $\sum_{i=1}^{\infty} A_i$.

10. COMMUTATION UNDER SEVERAL MEASURES. We have said that commu-

tation of the closure operators is possible under several circumstances: independence, inclusion, g-independence, etc.. We prove here that if E_P, F_P are the conditional expectation operators associated to $\mathcal{B}$ and $\mathcal{C}$ in the space $(\Omega,\mathcal{A},P)$, all the algebras completed in $\mathcal{A}$, then $E_Q F_Q = F_Q E_Q$ for every $Q \sim P$, iff only inclusion occurs. This solves partially also the following question.
Problem: How must $\mathcal{B}$ and $\mathcal{C}$ be related as to have $E_P F_P = F_P E_P$ for every measure P?
Conjecture: Ω is decomposable into two sets Ω_1, Ω_2 belonging to $\mathcal{D}$, such that $\mathcal{B} \subset \mathcal{C}$ in the first, $\mathcal{C} \subset \mathcal{B}$ in the second one.
The problem, the conjecture and next proposition appeared in $\{MP_2\}$.

PROPOSITION 1. *Assume given a set Ω, and the σ-algebra of sets $\mathcal{A}$ with two σ-subalgebras, $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A} = g(\mathcal{B},\mathcal{C})$. If for every P; E_P and F_P commute, then if $G \in \mathcal{D} = \mathcal{B} \wedge \mathcal{C}$ is not void then or G is decomposable in $\mathcal{D}$, or it is indecomposable in $\mathcal{B}$ or in $\mathcal{C}$.*

Proof: Indecomposability of A in $\mathcal{A}$ means that every $A' \in \mathcal{A}$ includes A or is disjoint to it. Commutation of the conditional expectation operators for every P, means $\forall$ P, $E_P 1_C$ is $\mathcal{C}$-measurable $[P]$, $\forall C \in \mathcal{C}$. Assume $P(G) > 0$, and also that G is indecomposable in $\mathcal{D}$. Therefore $\mathcal{B}$ and $\mathcal{C}$ are independent on G. If G were decomposable with respect to $\mathcal{B}$ and $\mathcal{C}$ then it would exist $B \in \mathcal{B}$, $C \in \mathcal{C}$, such that $B-C \neq \phi \neq C-B$. Take $x \in B-C$, $y \in C-B$ and let δ_x, δ_y, the probabilities concentrated on x and y respectively. Consider $Q = (P+\delta_x+\delta_y)/3$. G is an atom with respect to P and Q. Since by hypothesis E and F commute, $\mathcal{B}$ and $\mathcal{C}$ are independent with respect to Q on G. Then:

$Q(B)Q(C) = Q(BC)Q(G)$ which implies $(P(B)+1)(P(C)+1) = P(BC)(P(G)+2)$
From this we obtain: $P(B\Delta C)+1 = 0$. QED.

THEOREM 1. *If E_Q and F_Q commute for every Q equivalent to P then Ω can be decomposed into two sets of $\mathcal{D}$, T_0, T_1, where, respectively, $\mathcal{B}_0 \subset \mathcal{C}_0$, $\mathcal{C}_1 \subset \mathcal{B}_1$.*

Proof: Let T_1 be the greatest set of $\mathcal{D}$ where $\mathcal{C}_1 \subset \mathcal{B}_1$. If in $T_0 = \Omega - T_1$ were not true that $\mathcal{B}_0 \subset \mathcal{C}_0$, it would exist a set $B \in \mathcal{B}$, included in T_0 such that $\nabla_2 B \neq B$ and $B \neq \Delta B = \Omega - \nabla_2(\Omega-B)$. $D = \nabla_2 B - \Delta B \in \mathcal{D}$.
Let $C \in \mathcal{C}$, and $f = a\, 1_{BC} + b\, 1_{CB'} + c\, 1_{BC'} + d\, 1_{B'C'} \geq 0$, $B' = \Omega - B$, $\int f\, dP = 1$. Eventually passing to complements we can reduce the situation to one of the following cases: 1) B and C intersect in a positive set but no of them is included in the other; 2) B is contained in C. From (2) of section 2, we get for $dQ = f\, dP$: $F_Q 1_B = F(f\, 1_B)/F(f)$ and

from this:

(*) $(a\, 1_C + c\, 1_{C'})F\, 1_B\, F_0\, 1_{B'} = (b\, 1_C + d\, 1_{C'})F\, 1_{B'}.F_Q\, 1_B$.

Therefore, the function $h = 1_D(a\, 1_C + c\, 1_{C'})/(b\, 1_C + d\, 1_{C'})$ is $\mathcal{B}$-measurable.

Take b = d = 1, which is possible in the mentioned cases, and choose a and c in such a way that $\int f\, dP = 1$. In case 1) a and c exist satisfying this equation and $c \neq a$. In case 2) choose c = 0. In both cases: $1_D(a\, 1_C + c\, 1_{C'})$ is $\mathcal{B}$-measurable, and then C.D is $\mathcal{B}$-measurable. This means that for every $C \in \mathcal{C}$, CD is $\mathcal{B}$-measurable. Then on D, $\mathcal{C} \subset \mathcal{B}$, contradicting the maximality of T_1. QED.

This theorem does not solve the problem proposed above since the question was on the set structure and the preceding result is on the Boolean structure. But it may be this last one is the right structure where the problem should be posed. Next we describe the equivalent measures to P for which we can afford to ask commutation whenever this is present for P.

THEOREM 2. *Assume* EF = FE, $\int f\, dP = 1, 0 \leq f$, $dQ = f\, dP$, $Q \sim P$. *Then,* $E_Q F_Q = F_Q E_Q$ *iff* f = gh *where* g *is* $\mathcal{B}$*-measurable and* h $\in \mathcal{C}$*-measurable, both non negative.*

COROLLARY. *If* EF = FE *then every* $f \in L^1(\Omega)$ *is of the form* f = gh, $g \in \mathcal{B}$*-measurable,* $h \in \mathcal{C}$*-measurable iff* Ω *is decomposable into two disjoint sets* T_0, T_1 *with the properties described in Th. 1.*

The corollary follows from theorems 1 and 2. To prove the preceding theorem we shall make use of the following auxiliary proposition.

PROPOSITION 2. $E_Q = E$ *iff* f *is* $\mathcal{B}$*-measurable, where* f = dQ/dP.

Proof: Let us see the necessity.
$E(g) = E(fg)/E(f) = E(g)\ \forall g$ implies $E(fg) = E(gE(f))\ \forall g$ and therefore $f = E(f)$. The sufficiency is easier: $E_Q(g) = E(fg)/E(f) = f\, E(g)/f = E(g)$.

Proof of theorem 2. First we observe that formula (2) of section 2 holds for every non negative function f, a.e. finite, and every non negative function h, say, not greater than one. Second observation: the functions g and h that appear in the hypothesis can always be supposed finite everywhere. Third observation: Proposition 2 can be extended with the same proof to the following: If Q is σ-finite

and equivalent to P and f = dQ/dP is $\mathcal{B}$-measurable, then $E_Q = E$, and conversely, if $0 < f < \infty$ a.e. [P] and dQ = f dP, the equality implies the $\mathcal{B}$-measurability of f.
Fourth: g and h can always be supposed to be greater than 0 everywhere, since this can be admitted for f.
Let us consider the σ-finite measure: dK = h dP ; from the observations we see that: $E_Q\ s = E_K\ s = E\ hs/Eh \in \mathcal{C}$-measurable if s is $\mathcal{C}$-measurable, nonnegative and not greater than one. This is the proof of the sufficiency. Let us see the necessity. G(f) = G(gh) = EF(gh) = = E(h)F(g) and Ef.Ff = f.Eh.Fg imply:

$$(2) \qquad f = \frac{Ef\ Ff}{Gf} = g'h'.$$

This provides a canonic decomposition of f since $g' = Ef / \sqrt{Gf}$ is $\mathcal{B}$-measurable.
If E_Q and F_Q commute, using formula 2, section (2):

$$(3) \qquad E_Q F_Q \psi = E(f\ F(f\psi)/Ff)/Ef.$$

If $\psi = \phi\ Ef/f$, $\phi \in \mathcal{C}$-measurable, (3) equals to

$$(4) \qquad E(f\phi Gf/Ff)/Ef\ .$$

Changing in (3), E with F, for that ψ we get (3) equal to:

$$(5) \qquad F(fE\phi)/Ff\ .$$

Since $E\phi$ is $\mathcal{D}$-measurable, from (4) = (5) we obtain:

$$(6) \qquad E\phi = \frac{Gf}{Ef}\ E(f\phi/Ff)\ , \quad \text{for } \phi \in \mathcal{C}\text{-measurable.}$$

If $\phi = 1_C$, integrating (6):

$$(7) \qquad P(BC) = \int_{BC} f \frac{Gf}{EfFf}\ dP \quad , \forall\ B \in \mathcal{B}\ , \ \forall\ C \in \mathcal{C}\ .$$

Therefore (7) holds replacing BC by any set $A \in \mathcal{A}$. Then the integrand is equal to 1 which implies $f = g'h'$, $g' = Ef/\sqrt{Gf}$, $h' = Ff/\sqrt{Gf}$.

11. GENERATOR INDEPENDENCE. $\mathcal{B}$ will be said g-independent of $\mathcal{C}$ iff $\forall\ B \in \mathcal{B}$ (or equivalently, $B \in \mathcal{B} - \mathcal{B} \wedge \mathcal{C}$) there exists $\mathcal{E} \subset \mathcal{B}$ a σ-algebra independent of $\mathcal{C}$ and such that $\mathcal{D}$ and $\mathcal{E}$ generate a σ-algebra $\mathcal{D} \oplus \mathcal{E}$ that contains B.
Then $\cup\ \mathcal{E}(B)$, $B \in \mathcal{B} - \mathcal{D}$, is a family independent of $\mathcal{B}$ which together with $\mathcal{D}$ generate $\mathcal{B}$. But that union is a family of generators of a σ-algebra $\mathcal{B}'$ such that $\mathcal{B} = g(\mathcal{D},\mathcal{B}')$. g-independence is implied by

independence. This section will be devoted to the proof of the following theorem and previous results. We shall pause on same of them since they are highly interesting in themselves.

THEOREM 1. *i) If $\mathcal{B}$ is g-independent of $\mathcal{C}$ then* EF = FE.
ii) If Ω does not possess $\mathcal{D}$-atoms with respect to $\mathcal{B}$ and EF = FE *then $\mathcal{B}$ is g-independent of $\mathcal{C}$.*
iii) If Ω does not possess $\mathcal{D}$-atoms with respect to $\mathcal{B}$ neither to $\mathcal{C}$ and EF = FE *then $\mathcal{B}$ is g-independent of $\mathcal{C}$ and $\mathcal{C}$ of $\mathcal{B}$.*

Proof: i) Given $B \in \mathcal{B}$ let us see that $F\,1_B$ is $\mathcal{D}$-measurable. Take $\mathcal{E} \subset \mathcal{B}$ such that $\mathcal{D} \oplus \mathcal{E} \ni B$. Consider the product $(X\text{x}Y, \bar{\mathcal{C}} \times \bar{\mathcal{E}}, \bar{P}=dxdy)$, (where dxdy stands for the product of two measures represented by dx and dy) isomorphic to $(\Omega, \mathcal{C} \oplus \mathcal{E}, P)$. Call $\bar{B}$ a set in XxY corresponding to B. $E(\cdot\, /\bar{\mathcal{C}})$ is obtained by integration in the second variable:

$$(") \qquad E(1_{\bar{B}}\ /\bar{\mathcal{C}}) = \int 1_{\bar{B}}(x,y)\ dy.$$

$1_{\bar{B}}$ is $\bar{\mathcal{D}} \times \bar{\mathcal{E}}$ -measurable and therefore (") is a $\bar{\mathcal{D}}$-measurable function. This proves i). Before proving ii) we shall go into auxiliary results Theorem 1 was part of {MP_2}. Next theorem is due to Maharam (cf.{M}, lemma 1) but the statement is slightly different because it uses the idea of conditional atom.

THEOREM 2. *i) Let (Ω, M, P) be a probability space, $L \subset M$, a σ-subalgebra such that M does not contain L-atoms. Assume $M \in M - L$. There exists a σ-subalgebra $\mathcal{B}$ of M, σ-isomorphic to* $\mathcal{B}(0,1)$, *such that* $M \in g(L,\mathcal{B})$ *and $\mathcal{B}$ is independent of L. Therefore,* $g(L,\mathcal{B}) = L \oplus \mathcal{B} \ni M$.
ii) Every homogeneous probability space is σ-isomorphic (in Boolean sense with preservation of measure) to a $(\mathcal{B}_\Lambda, \bar{P})$, $\bar{P}$ = Lebesgue measure.
iii) A non atomic (i.e. without atoms) probability space is represented in one and only one way as a direct sum $\sum_\Lambda (\Omega_\Lambda, \mathcal{B}_\Lambda, c_\Lambda P_\Lambda)$ where $0 < c_\Lambda \leq 1$, $\sum c_\Lambda = 1$, and the Λ's are infinite ordinals of different cardinality.

Proof: i) Given a partition $0 < 2^{-n} < \ldots < k2^{-n} < \ldots 1$, define the functions $x_k^{(n)} = \{(P^L(M) \wedge k.2^{-n}) \vee (k-1).2^{-n}\} - (k-1).2^{-n}, 1 \leq k \leq 2^n$.

Then $\sum_k x_k^{(n)} = P^L(M)$ and $x_{2k-1}^{(n+1)} + x_{2k}^{(n+1)} = x_k^{(n)}$.

Define $M_k^{(n)} \subset M - \sum_{j=1}^{k-1} M_j^{(n)}$ verifying $P^L(M_k^{(n)}) = x_k^{(n)}$,

$M_{2k-1}^{(n+1)} + M_{2k}^{(n+1)} = M_k^{(n)}$.

This is possible because of theorem 2, section 7.
Define now the functions $\eta_k^{(n)}$ in the same way but in relation to the set Ω - M. So $\eta_k^{(n)} + \chi_k^{(n)} = 2^{-n}$, and call $N_k^{(n)}$ the sets associated. Then the sets $M_k^{(n)} + N_k^{(n)}$ are independent of the algebra L and the same is true for the σ-algebra $\mathcal{B}$ generated by those sets, which is isomorphic to $\mathcal{B}(0,1)$ as it is easy to see from the construction.
Let us see that $M \in L \oplus \mathcal{B} = g(L,\mathcal{B})$.

Calling $C_k^{(n)} = \{\omega;\ P^L(M) \geq k\ 2^{-n}\}$, the set $T^{(n)} = \sum_k C_k^{(n)}(M_k^{(n)} + N_k^{(n)})$

belongs to $g(L,\mathcal{B})$ and verifies $P(M \Delta T^{(n)}) \leq 2^{-n}$. We leave this easy verification to the reader. From $P(M \Delta T^{(n)}) \xrightarrow[n]{} 0$, we get i).
ii) Assume A is the Boolean σ-algebra quotient of A with the sets of measure zero. Assume also that the generators of A are ordered in the same way as that described in th. 1, iii), section 2, and let Λ be the ordinal (the least one with a given cardinality) involved in the ordering.
Call B_j the Boolean σ-algebra associated to $\mathcal{B}(\prod_{0\leq i<j} (0,1)_{(i)})$ and suppose it is σ-isomorphic to a subalgebra L of A, such that L contains at least the generators g_i , $i < j < \Lambda$. Since card $j <$ dim A = card Λ from proposition 3, §7, and i), L can be extended to a σ-subalgebra σ-isomorphic to B_{j+1} containing the generator with least index not contained in L.
This implies that A is isomorphic to a B_Γ.card $\Gamma <$ card Λ is impossible because dim B_Γ = card Γ. On the other hand, from the construction $\Gamma \leq \Lambda$, and therefore $\Gamma = \Lambda$.
iii) Follows from ii) and theorem 2, §2. QED.

COROLLARY 1. *If* m *is the dimension of the σ-algebra* M *associated to* (Ω,M,P) *and it has no atom, then* card M = $m^{\aleph_0}$.

The proof is left to the reader. We observe only that ii) and theorem 3, §4, are of similar nature.

COROLLARY 2. *If* $L = \{\emptyset,\Omega\}$ *and* M *is non-atomic then* M *contains a σ-subalgebra isomorphic to* $\mathcal{B}(0,1)$.

In fact, in this context atom and L-atom are equivalent concepts. The corollary follows immediately from i).

COROLLARY 3. *Let* (Ω,A,P) *be a probability space without atoms,* $\mathcal{B}$

and C, σ-subalgebras of A, all of them homogeneous, completed in A such that $A = g(B,C)$. *If* $EF = FE$ *and* $\dim B > \dim D < \dim C$ *then B, C, D and A are Boolean σ-isomorphic to product σ-algebras* B_2, B_3, B_4 *and* B_1, *respectively, in such a way that*

$$B_1 = B(\prod_{0\leq i<\beta} (0,1)_{(i)}) \times B(\prod_{\beta\leq i<\delta} (0,1)_{(i)}) \times B(\prod_{\delta\leq i<\gamma} (0,1)_{(i)}) ,$$

$$B_2 = B(\prod_{0\leq i<\beta}) \times B(\prod_{\beta\leq i<\delta}) , \quad B_4 = B(\prod_{\beta\leq i<\delta}) ,$$

$$B_3 = B(\prod_{\beta\leq i<\delta}) \times B(\prod_{\delta\leq i<\gamma}).$$

Proof: With a slight modification the proof of ii) of the preceding theorem works to prove the isomorphism of B, C, D with B_2, B_3, B_4 respectively. For the isomorphism of A with B_1 it is only necessary to demonstrate that the first factor (the last one in B_3 is treated in the same way) in B_2 is independent of B_3. Of this takes care the following proposition, which holds in the general setting we proposed ourselves along this paper.

PROPOSITION 1. *If* $EF = FE$ *and* $B \supset g(D, B_\circ)$ *with* $B_\circ$ *independent of* D, *then* $B_\circ$ *is independent of* C.

Proof: Let $B_\circ \in B_\circ$. We know that $G\, 1_{B_\circ} = P(B_\circ)\, 1_\Omega$. If $C \in C$,

$$P(CB_\circ) = \int_C F\, 1_{B_\circ}\, dP = \int_C G\, 1_{B_\circ}\, dP = \int_C P(B_\circ)\, dP = P(B_\circ)P(C). \qquad \text{QED.}$$

Proof of ii) of theorem 1. Using i) of theorem 2 after replacing M by B and L by D we see that for any $M \in B - D$ there exists $B_\circ$ isomorphic to $B(0,1)$ such that $B_\circ \subset B$ and is independent of D. Because of proposition 1 it is independent of C, and this proves ii).
As a matter of intellectual curiosity we could add the following corollary to theorem 2, (cf. theorem 4, §7).

PROPOSITION 2. *If* (Ω, M, P) *does not contain any* (L,S)*-atom then i) of theorem 2 holds with* $B \subset g(S,M)$.

It can be proved in the same way as shown before, but we shall not prove it since this result will not be used in the sequel.

DEFINITION: *When four σ-algebras* B_i , $i=1,2,3,4$ *are related as in corollary 3 with* B_4 *non-trivial we shall say that* B_2 *and* B_3 *satisfy a relation of spurious independence in strict sense. If it is not*

known that B_4 is not trivial we shall say that between B_2 and B_3 there is spurious independence.
(If $B_4 = \{\emptyset,\Omega\}$ spurious independence coincides with independence).

12. DECOMPOSITION OF A PROBABILITY SPACE. In the usual setting with commutation of the associated conditional expectation operators it is possible to decompose the space in pieces where there is inclusion, or independence or g-independence, beeing only necessary, sometimes, to increase $\mathcal{D}$ with a partition of Ω by sets of $\mathcal{B} \vee \mathcal{C}$ = the algebra generated by $\mathcal{B}$ and $\mathcal{C}$.
First step: Isolate the atoms of $\mathcal{D}$. On them $\mathcal{B}$ and $\mathcal{C}$ are independent.
Second Step: In the complement we still have commutation of the corresponding conditional expectation operators. Isolate the set of $\mathcal{D}$ where $\mathcal{B} \subset \mathcal{C}$, maximal with respect to this property.
Third step: In the remaining part we still have commutation. There, isolate the maximal set of $\mathcal{D}$ where $\mathcal{C} \subset \mathcal{B}$.
The three steps are possible because of the results of section 9. The remarks of section 8, in particular 3) and 5), are specially illuminating at this moment.
From the results of lemma 3, §8, we see that the $\mathcal{D}$-atomic part with respect to $\mathcal{B}$ in the set Ω_3 of $\mathcal{D}$ that remained after the third step, can be put as à union of a denumerable family $\{B_n\}$ of $\mathcal{B} \wedge \mathcal{C}$-atoms of $\mathcal{B}$. Adjoin $\{B_n\}$ to $\mathcal{C}$, the σ-algebra $\mathcal{C}' = g(\mathcal{C},\{B_n\})$ commute with $\mathcal{B}$ as can be seen from lemma 1 of section 9. Therefore they also commute on Ω_3. In $\Omega_4 = \Omega_3 - \Sigma B_n$ there not exist $\mathcal{D}$-atoms with respect to $\mathcal{B}$. After adjoining the B_n's to $\mathcal{C}$ we have increased "the set of $\mathcal{D}$ where $\mathcal{B} \subset \mathcal{C}$". In fact, $\mathcal{C}'$ is obtained from $\mathcal{C}$ adjoining sets of $\mathcal{B}$: $\{B_n\}$, in consequence they belong to $\mathcal{D}' = \mathcal{B} \wedge \mathcal{C}'$ and in each of them, $\mathcal{B} \subset \mathcal{C}'$. That is, the $\mathcal{D}$-atoms of $\mathcal{B}$ are a hidden form of inclusion.
Fourth step: Adjoin to $\mathcal{C}$ a denumerable partition by sets of $\mathcal{B}$ which are conditional atoms of the $\mathcal{D}$-atomic part of $\mathcal{B}$ contained in Ω_3. The new σ-algebra $\mathcal{C}'$ generated by those sets together with $\mathcal{C}$ commute with $\mathcal{B}$. In those sets $\mathcal{B} \subset \mathcal{C}'$. Observe now that in Ω_4, $\mathcal{B}$ is g-independent of $\mathcal{C}$ (cf. Th. 1, section 11).

THEOREM 1. *After adjoining to $\mathcal{C}$ a denumerable family of disjoint sets of $\mathcal{B}$ it is obtained a σ-algebra $\mathcal{C}'$ commuting with $\mathcal{B}$ such that, in a partition of Ω by sets of $\mathcal{D}' = \mathcal{B} \wedge \mathcal{C}'$, or $\mathcal{B} \subset \mathcal{C}'$ or $\mathcal{C} \subset \mathcal{B}$ or $\mathcal{B}$ is independent of $\mathcal{C}$ or $\mathcal{B}$ is g-independent of $\mathcal{C}'$. The second and third situation occur at sets of $\mathcal{D}$. The first, second and fourth situation occur at exactly one set of the partition.*

This theorem was in essence correctly understood in {MP_2}. A technical mistake brought to the authors to the belief that the $\mathcal{D}$-atomic part with respect to $\mathcal{B}$: ΣB_n , was $\mathcal{D}$-measurable when EF = FE. This is not so as can be easily seen from example V, section 3, were the conditional atomic part is reduced to only one atom. The situation was still clarified in {HN} , proposition of §2, where the $\mathcal{D}$-atoms of $\mathcal{B}$,with $\mathcal{B}$ generated by $\mathcal{D}$ and a partition of $\mathcal{B}$-sets , are characterized.

<u>Fifth step</u>: Repeat the procedure of the fourth step in the set Ω_4 but changing $\mathcal{B}$ with $\mathcal{C}'$ and $\mathcal{C}$ with $\mathcal{B}$. That is, eliminate in Ω_4 the $\mathcal{D}'$-atoms of $\mathcal{C}'$ adjoining to $\mathcal{B}$ a denumerable partition-of the $\mathcal{D}'$-atomic part relative to $\mathcal{C}'$-by sets of $\mathcal{C}'$ which are conditional atoms. Therefore, the new σ-algebra $\mathcal{B}'$ commutes with $\mathcal{C}'$, $\mathcal{B}' = g(\mathcal{B},\{C_n'\})$, and on $\Omega_5 = \Omega_4 - \Sigma C_n'$, $\mathcal{C}'$ is g-independent of $\mathcal{B}'$ because of theorem 1 of section 11. Besides all the sets adjoined to $\mathcal{B}$ or $\mathcal{C}$ belong to $\mathcal{B} \vee \mathcal{C}$ as well as Ω_3, Ω_4 and Ω_5.

<u>m^{th} step</u>: $m > 3$ is a denumerable ordinal. Call $\mathcal{B}_n$, $\mathcal{C}_n$ the σ-algebras in the n^{th} step, $3 < n < m$; assume they commute and that they are obtained from $\mathcal{B}$, $\mathcal{C}$, adjoining a partition P_n of sets of $\mathcal{B} \vee \mathcal{C}$, P_{n+1} refining P_n. Call Ω_n a set in P_n specially chosen of the form B.C, $B \in \mathcal{B}$, $C \in \mathcal{C}$, $\Omega_n \supset \Omega_{n+1}$.

If m is a limit ordinal define $\mathcal{B}_m$ and $\mathcal{C}_m$ as the limit σ-algebras generated, respectively, by $\mathcal{B}_n$, $n < m$ and $\mathcal{C}_n$, $n < m$.

Call $\Omega_m = \bigcap_n (\Omega_n : n < m)$.

If m is not a limit ordinal, and even, adjoin to $\mathcal{C}_{m-1}$ a denumerable partition by sets of $\mathcal{B}_{m-1}$ of the $\mathcal{D}_{m-1}$-atomic part of $\mathcal{B}_{m-1}$ contained in Ω_{m-1} taking care that the sets of the partition be conditional atoms. If Ω_{m-1} is void the procedure stops.

If m is an ordinal and not a limit one, and odd, the construction is the same changing $\mathcal{B}_{m-1}$ with $\mathcal{C}_{m-1}$. In both cases Ω_m is defined as Ω_{m-1}minus the conditional atomic part. Obviously Ω_m is of the form B.C, and from §9 we see that $\mathcal{B}_m$ commutes with $\mathcal{C}_m$.

Since this is an exhaustion procedure it stops after a denumerable family of steps. Then, two possibilites force the stopping: a) a certain $\Omega_m = \emptyset$, b) in a certain Ω_m the corresponding $\mathcal{D}_m$-atomic part is void. In this last case, in that set, there not exist $\mathcal{D}_m$-atoms with respect to $\mathcal{B}_m$ neither to $\mathcal{C}_m$. That is, the restrictions of $\mathcal{B}$, $\mathcal{C}$ to Ω_m have no conditional atom with $(\mathcal{B} \wedge \Omega_m) \wedge (\mathcal{C} \wedge \Omega_m) = \mathcal{D}_m$, the conditional algebra. Then we have:

THEOREM 2. *After adjoining to $\mathcal{C}$ and $\mathcal{B}$ a denumerable partition by sets of $\mathcal{B} \vee \mathcal{C}$ they are obtained σ-algebras $\mathcal{C}'$ and $\mathcal{B}'$ still commuting such that in each set of the partition: i) one of the last two alge* *bras is subordinated to the other, or ii) $\mathcal{B}$ is independent of $\mathcal{C}$, or iii) $\mathcal{B}'$ is g-independent of $\mathcal{C}'$ and $\mathcal{C}'$ is g-independent of $\mathcal{B}'$. The independence occurs at sets of the partition belonging to $\mathcal{D}$. iii) occurs at exactly one set.*

13. G-INDEPENDENCE AND HOMOGENEITY. In this paragraph we shall work on the space $(\Omega,\mathcal{A},P)$ with $g(\mathcal{B},\mathcal{C}) = \mathcal{A}$ and $\mathcal{B}$ commuting with $\mathcal{C}$ is such that on Ω, $\mathcal{B}$ is g-independent of $\mathcal{C}$ and $\mathcal{C}$ of $\mathcal{B}$. This is the situation at which we arrived in one set of the partition in th. 2 of the preceding section. We shall try to go deeper on the structure of the algebras when g-independence in both senses is present. A first auxiliary proposition whose easy proof we leave to the reader follows next.

PROPOSITION. *Let* $A \in \mathcal{A}$, $B \in \mathcal{B}$. *If* $B.A$ *is an atom of* $\mathcal{B} \wedge A$ *then* $\nabla_1(B.A)$ *is an atom of* $\mathcal{B}$, *and conversely.*

This proposition is quite general as is the following result.

LEMMA 1. *Assume* $(\Omega,\mathcal{A},P)$ *has no atom in* $\mathcal{D} = \mathcal{B} \wedge \mathcal{C}$, *and as always* $\mathcal{A} = g(\mathcal{B},\mathcal{C})$, $EF = FE$.
There exists a finite partition $A_0,\ldots A_N$ *of* Ω *by sets of* K *(the least family containing* $\mathcal{B}$ *and* $\mathcal{C}$ *and closed by differences) such that on one of them, say* A_0, $\mathcal{B}_0$, $\mathcal{C}_0$, $\mathcal{D}_0$ *are simultaneously homogeneous. Besides the* σ*-algebras:* $\mathcal{B}' = g(\mathcal{B},A_0,\ldots,A_N)$, $\mathcal{C}' = g(\mathcal{C},A_0,\ldots A_N)$ *commute, i.e.,* $F'E' = E'F'$.

Proof: Call D_1 the set of $\mathcal{D}$ of Maharam's representation with least dimension. This set can be identified as the maximal set with least dimension. It exists, i.e., is not void, since $\mathcal{D}$ is non atomic. Let B_1 be the maximal set with least dimension of $\mathcal{B} \wedge D_1$; since $\mathcal{D}$ has no atom neither $\mathcal{B}$ nor $\mathcal{C}$ nor $\mathcal{A}$ have atoms and the mentioned theorem can be applied again to D_1 because of the proposition proved above. Call C_1 the maximal set with least dimension of $\mathcal{C} \wedge B_1$. Next do the same with $\mathcal{D}$ and C_1 obtaining D_2, and repeat the process. It is obtained so a sequence $D_1 \supset B_1 \supset C_1 \supset D_2 \supset B_2 \supset C_2 \supset D_3 \supset \ldots$. Let $\beta_i = \dim B_i$, $\gamma_i = \dim C_i$.
From theorem 1, §2 we know that $\beta_n \geq \beta_{n+1}$ and $\gamma_n \geq \gamma_{n+1}$. Because of the well-ordering of the ordinals there exists p such that if $n \geq p$, $\beta_n = \beta_{n+1}$, $\gamma_n = \gamma_{n+1}$. Let us see that $D_{p+1} = B_{p+1} = C_{p+1}$ which will prove the triple homogeneity of that set. From $\beta_{p+1} \leq \dim_{\mathcal{B}} D_{p+1} \leq \beta_p$ we have $\dim_{\mathcal{B}} D_{p+1} = \beta_{p+1}$ and since B_{p+1} is maximal

with dimension β_{p+1} we must have $B_{p+1} = D_{p+1}$ if we know that this last set is $\mathcal{B}$-homogeneous. But this follows from the fact that in no subset of D_{p+1} the $\mathcal{B}$-dimension can be less than β_p and cannot be greater because of what we have already seen. From $C_{p+1} \subset B_{p+1} \subset \subset C_p$ it follows in the same way that B_{p+1} is $\mathcal{C}$-homogeneous and $\dim_{\mathcal{C}} B_{p+1} = \gamma_p = \gamma_{p+1}$ and since C_{p+1} is maximal with dimension γ_{p+1}, we have $C_{p+1} = B_{p+1} = D_{p+1}$.
Consider now the sets $B_1 \supset C_1 \supset B_2 \supset \ldots \supset C_p \supset D_{p+1}$. From section 9 we know that $\mathcal{B}$ and $g(\mathcal{C},B_1)$ commute. Since C_1 belongs to this last algebra $g(\mathcal{B},C_1)$ and $g(\mathcal{C},B_1)$ commute. From $C_1 \in g(\mathcal{C},B_1)$ it follows that $g(\mathcal{C},B_1,C_1)$ commutes with $g(\mathcal{B},B_1,C_1)$. Following so we get the commutation of $g(\mathcal{B},B_1,\ldots,B_p,C_p)$ with $g(\mathcal{C},B_1,\ldots,B_p,C_p)$. Since D_{p+1} belongs to both σ-algebras it can be added to them.
Now observe that all the sets $B_1,\ldots,C_p,D_{p+1}$, belong to $\mathcal{K}$ and same thing occurs with the sets of the partition that they determine. QED.

LEMMA 2. *i)* $A \in \mathcal{A}$: $\mathcal{B}' = g(\mathcal{B},A)$, $\mathcal{C}' = g(\mathcal{C},A)$. *With all generality it holds:*

$$(\mathcal{B} \wedge A) \wedge (\mathcal{C} \wedge A) = (\mathcal{B}' \wedge \mathcal{C}') \wedge A.$$

ii) If $A \in \mathcal{B} \vee \mathcal{C}$ *and* $\mathcal{B}$ *commutes with* $\mathcal{C}$ *it holds the following equality, which is false in general:*

$$(\mathcal{B} \wedge A) \wedge (\mathcal{C} \wedge A) = (\mathcal{B} \wedge \mathcal{C}) \wedge A.$$

iii) If A *belongs to the union of* $\mathcal{B}$ *and* $\mathcal{C}$ *and they commute:*

$$(\mathcal{B} \wedge \mathcal{C}) \wedge A = (\mathcal{B}' \wedge \mathcal{C}') \wedge A.$$

Proof: i) follows from the definitions and iii) from this proposition and ii). Let us prove ii). The right member is always included in the left member. Put $H = B_1.A = C_1.A$, $B_1 \in \mathcal{B}$, $C_1 \in \mathcal{C}$, and assume $A \in \mathcal{C}$. Then $H \in \mathcal{C}$ and applying ∇_2 we obtain: $H = A.\nabla_2 B_1 = A.\nabla B_1 \in \in (\mathcal{B} \wedge \mathcal{C}) \wedge A$. QED.

LEMMA 3. *Let* $A_0,\ldots,A_n$ *be a partition of* Ω, $\mathcal{B}_i = g(\mathcal{B}_{i-1},A_{i-1})$, $\mathcal{B}_0 = \mathcal{B}$, $i=1,2,\ldots,n+1$, $\mathcal{C}_i = g(\mathcal{C}_{i-1},A_{i-1})$, $\mathcal{C}_0 = \mathcal{C}$. *Assume that* $\mathcal{B}$ *commutes with* $\mathcal{C}$ *and that* $A_j \in \mathcal{B}_j \cup \mathcal{C}_j$. *Then* $\mathcal{B}_{n+1}$ *and* $\mathcal{C}_{n+1}$ *commute and* (1) $(\mathcal{B}_{n+1} \wedge \mathcal{C}_{n+1}) \wedge A_n = (\mathcal{B} \wedge \mathcal{C}) \wedge A_n$.

Proof: $\mathcal{B}_j$ and $\mathcal{C}_j$ commute (cf. §9). Consider the equalities:

(2) $(\mathcal{B}_k \wedge \mathcal{C}_k) \wedge A_{k-1} = (\mathcal{B} \wedge \mathcal{C}) \wedge A_{k-1}$, $(\mathcal{B}_k \wedge \mathcal{C}_k) \wedge T_k = (\mathcal{B} \wedge \mathcal{C}) \wedge T_k$,

where $k=0,1,\ldots,n$, $T_k = \sum_{h=k}^{n} A_h$. They hold for $n=1$ because they coincide in this case with iii) of the preceding lemma 2. Intersecting the second equality with A_k we get: $(\mathcal{B}_k \wedge \mathcal{C}_k) \wedge A_k = (\mathcal{B} \wedge \mathcal{C}) \wedge A_k$.
Since $A_k \in \mathcal{B}_k \cup \mathcal{C}_k$, from the preceding lemma we obtain:

$$(B_{k+1} \wedge C_{k+1}) \wedge A_k = (B_k \wedge C_k) \wedge A_k .$$

This and the preceding equality prove the first part of (2) for k+1. Intersecting with $T_{k+1} \in B_k \cup C_k$, we obtain in an analogous way the second part of (2) for k+1. Therefore (1) is proved. QED.

LEMMA 4. *Under the hypothesis of the preceding lemma, it holds:*

i) $(B_j \wedge C_j) \wedge A_k = (B_{k+1} \wedge C_{k+1}) \wedge A_k$,

$(B_j \wedge C_j) \wedge T_{k+1} = (B_{k+1} \wedge C_{k+1}) \wedge T_{k+1}$, $j=0,1,\dots,k$, $k=0,\dots,n$.

ii) For $0 \leq k \leq n$, $0 \leq j \leq k$:

$$(B \wedge C) \wedge A_j = (B_1 \wedge C_1) \wedge A_j = \dots = (B_{k+1} \wedge C_{k+1}) \wedge A_j .$$

Proof: i) is proved in the same way as the preceding lemma. ii) follows by induction. Assume for $j < h$, $h < k$, that

(3) $(B_h \wedge C_h) \wedge A_j = (B \wedge C) \wedge A_j .$

It is immediate that for $j < k-1$, $h \leq k$ we have: $(B_k \wedge C_k) \wedge A_j =$ $= (B_{k-1} \wedge C_{k-1}) \wedge A_j$, and therefore for these indices we proved ii). For $j = k - 1$, it coincides with i). QED.

THEOREM 1. *Assume that B and C commute, together generate A, all of them are completed in A. If Ω has no atom on D there exists a partition* $\Sigma \subset A$ *such that* $\bar{B} = g(B,\Sigma)$, $\bar{C} = g(C,\Sigma)$ *commute and the sets of* Σ *are homogeneous for* $\bar{B}$, $\bar{C}$ *and* $\bar{D} = \bar{B} \wedge \bar{C}$. *Moreover, on each* $A \in \Sigma : (\bar{B} \wedge \bar{C}) \wedge A = (B \wedge C) \wedge A$, $\bar{B} \wedge A = B \wedge A$, $\bar{C} \wedge A = C \wedge A$, *wich implies that each such A is B, C and D-homogeneous.*

DEFINITION. *Suppose that in Ω is present the usual setting with commutation. If A, B, C, D are homogeneous and dim D = dim B or = dim C then we shall say that B and C are quasi-independent.*

Theorem 1 asserts that if there is no atom of *D* in each set of Σ we have spurious independence or quasi-independence (cf. section 11). Applying this to the decomposition in th. 2 of preceding section, we obtain:

THEOREM 2. *After adjoining to B and C a denumerable partition of Ω by sets of A, they are obtained σ-algebras B" and C" still commuting such that in each set of the partition: i) one of the two algebras is subordinated (included into) to the other; ii) B is independent of C; iii) B and C are strictly spurious independent; iv) they are quasi-independent and not independent. The independ-*

ence occurs at sets of the partition belonging to $\mathcal{D}$. i) occurs at sets of $B \vee C$ belonging to the partition.

Th. 2 is obtained, as we already said, from theorem 2 of section 12 and corollary 3 of section 11, and the preceding theorem applied to the g-independent part of the decomposition of §12 . iv) is a consequence of remark 3, §8. It can be questioned if iv) is a reasonably description of the situation that it tries to isolate. We think it is not and, first of all, it does not keep up with i)-iii). Observe that subordination could be present and still be described as quasi-independence. The matter requires further investigation.

Proof of theorem 1. Applying lemma 1 we find a partition Σ_o, finite, such that for a certain $V_o \in \Sigma_o, B, C, \mathcal{D}$, are homogeneous and $B_o = g(B, \Sigma_o)$, $C_o = g(C, \Sigma_o)$, commute. On V_o, $B = B_o$, $C = C_o$, $\mathcal{D} = \mathcal{D}_o = B_o \wedge C_o$. In fact, in the proof of lemma 1 we had to add to B and C the partition determined by $B_1 \supset C_1 \supset \ldots \supset D_{p+1} = V_o$ which is constituted by sets of K, but moreover, this partition satisfies the hypothesis of lemma 3 and 4 if we define: $A_o = \complement B_1$, $A_1 = B_1 - C_1$, $A_2 = C_1 - B_2$,..., $A_n = V_o$. If all the elements of Σ_o are homogeneous simultaneously the procedure stops and if not we choose $J \in \Sigma_o$, one of the sets not simultaneously homogeneous, and repeat the preceding process on it. This provides a new finite partition refining Σ_o, Σ_1, obtained from Σ_o by partitioning of J and such that $g(B,\Sigma_1)$ commutes with $g(C,\Sigma_1)$. On one set of Σ_1, V_1 , contained in J there is simultaneous homogeneity. Using ii) of lemma 4 we see that $(g(B,\Sigma_1) \wedge g(C,\Sigma_1)) \wedge V_1 = (g(B,\Sigma_o) \wedge g(C,\Sigma_o)) \wedge V_1 = [(g(B,\Sigma_o) \wedge g(C, \Sigma_o)) \wedge J] \wedge V_1 = (B \wedge C) \wedge V_1$, Besides since $V_o \wedge J = \emptyset$ and $V_o \in g(B,\Sigma_o) \wedge g(C,\Sigma_o) \subset g(B,\Sigma_1) \wedge g(C,\Sigma_1)$, this last algebra intersects V_o in an algebra coinciding with $(B \wedge C) \wedge V_o$. Following so we obtain a sequence of partitions $\Sigma_o < \Sigma_1 < \Sigma_2 < \ldots$. Call $\Sigma^o = \bigcup_n \Sigma_n$, $B^o = g(B,\Sigma^o)$, $C^o = g(C, \Sigma^o)$. From §9, we know that the last two algebras commute and trivially it follows that $(B^o \wedge C^o) \wedge X_k = (B \wedge C) \wedge X_k$ for each $X_k \in \Sigma^o$. On an infinite family of X_k there is simultaneous homogeneity. Let them be $H_1, H_2, \ldots$. If $\sum_1^\infty P(H_j) = 1$ then the theorem is proved. If not, we can repeat the process subdividing the remaining sets of the partition, getting so a new partition $\Sigma^1 > \Sigma^o$ with the same properties as Σ^o, and more sets H'_1 , H'_2 , H'_3 ,... . On a certain (denumerable) step the process ends by exhaustion, which proves the theorem.

14. FINAL REMARK. In the theorem of Burkholder and Chow that we

mentioned in the introduction, EF = G or only in the limit, $\lim (EF)^n = G$. In fact, $(EF)^n = G$ implies EF = G. This follows from: in a Hilbert space H, let S_i be a subspace with projector P_i, i=1,2; if $P_1P_2f \in S = S_1 \cap S_2$, and $f \in S_1$ then $P_1P_2f = P_2f$. In fact, if $n_1 \perp S_1$, $s_1 \in S_1$ and $P_2f = s_1 + n_1$, it follows $P_1P_2f = s_1 \in S$. Therefore, if $g' = f - s_1$, $\|g' - n_1\|^2 = \|g'\|^2 + \|n_1\|^2$ since $f - n_1 \in S_1$. On the other hand, $P_2g' = n_1$ which implies $\|g'\|^2 = \|n_1\|^2 + \|g' - n_1\|^2$. Then $n_1 = 0$, and $P_2f = s_1 = P_1P_2f$. Analogously, $F(EF)^n = G$ implies EF = G. Prof. R. Maronna observed that EF = FE iff $\mathcal{B}$ and $\mathcal{C}$ are conditionally independent given $\mathcal{D}$. In fact, if EF = FE, $0 \leq x \in L^2(\mathcal{B})$, $0 \leq y \in L^2(\mathcal{C})$, $D \in \mathcal{D}$, we have: $\int_D G(xy)\,dP = \int EF(1_D xy)dP = \int E(1_D y(F(1_D x))dP = \int G(1_D x)G(1_D y)dP = \int_D G(x)G(y)\,dP$, and therefore, $G(xy) = G(x)G(y)$. Conversely, if this equality holds, from $\int xy\,dP = \int G(xy)\,dP = \int G(x)G(y)\,dP = \int y\,G(x)\,dP$, we obtain $x - Gx \perp L^2(\mathcal{C})$. Then Fx = Gx, which implies FE = G.

If all the algebras are completed in $\mathcal{A}$, and $\mathcal{B}$ and $\mathcal{C}$ are conditionally independent given $\mathcal{E}$, then $\mathcal{E} \supset \mathcal{D}$, as it is easy to see from $E(1_D^2|\mathcal{E}) = E(1_D|\mathcal{E})^2$. Therefore, EF = FE iff $\mathcal{B}$ and $\mathcal{C}$ are conditionally independent with respect to the minimal σ-algebra for which this is possible.

Since $x - Gx \perp L^2(\mathcal{C})$, then $x - Gx \perp y - Gy$, then $L^2(\mathcal{C}) \ominus L^2(\mathcal{D}) \perp L^2(\mathcal{B}) \ominus L^2(\mathcal{D})$.

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