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# PURITY AND ALGEBRAIC CLOSURE

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Throughout this paper $R$ denotes an associative ring with identity. We shall study the following properties associated to $R$.

a) the purity of the inclusion $R \subset M$ of $R$ in an injective $R$-module $M$ containing it.

b) the algebraic closure of M. Hall, of submodules of free $R$-modules.

c) a weak injectivity property of $R$ as an $R$-module.

Section 2 contains the main results. In Section 3 we characterize von Neumann rings in terms of purity.

## 1. PRELIMINAIRES.

i) Purity. Let $M$ and $N$ be right $R$-modules. An exact sequence $0 \to N \to M$ of $R$-modules will be said pure if for every left $R$-module $A$, the induced sequence $0 \to N \otimes A \to M \otimes A$ is exact ($\otimes = \otimes_R$). If $N$ is a submodule of $M$, we say that $N$ is pure in $M$ if the exact sequence $0 \to N \xrightarrow{i} M$, where i denotes the inclusion map, is pure. Let $N$ be a right $R$-module. Then the following conditions are easily seen to be equivalent (and we shall therefore say simply that $N$ is pure),

1) $N$ is pure in any injective module containing it

2) $N$ is pure in its injective hull

3) $N$ is pure in any module containing it.

ii) Conditions (h°), (c°), (a°), (b°). Let $A$ be a left (resp. right) $R$-module and $n \in N$. $A^n$ denotes the left (resp. right) $R$-module, direct sum of n copies of A. If $a \in A^n$ we write $a = [a_1,..,a_n]$ in terms of its coordinates. With $R'^n$ (resp. $R''^n$) we denote the previous situation for $A = R$. Let $A$ be a left $R$-module. We define a left pairing $R''^n \times A^n \to A$ by $r.a = \sum_{i=1}^n r_i.a_i$.

For any non-empty set $S \subset R'^n$, $S^r$ denotes the right annihilator of $S$ in $R''^n$, that is

$$S^r = \{r \;/\; r \in R''^n \text{ and } s.r = 0 \text{ if } s \in S\}$$

In analogous way we define the left annihilator $T^l \subset R'^n$ of a non-empty set $T \subset R''^n$.

According to M. Hall {1} , a submodule $S$ of $R'^n$ will be said to be closed if $S = (S^r)^l = S^{rl} = \bar{S}$ . We can now state

CONDITION $(h°)_1$ : Every finitely generated submodule of $R'^n$ is closed.

CONDITION $(c^\circ)_1$: Every finitely generated left ideal of $R$ is closed.
CONDITION $(c^\circ)_1$ is the special case of $(h^\circ)_1$ when $n = 1$.

Next we define the weak injectivity referred above. This is
CONDITION $(a^\circ)_1$: Every $R$-homomorphism of a finitely generated left ideal of $R$, into $R$, is realized by a right multiplication by an element of $R$.
CONDITION $(b^\circ)_1$: Let $U$ and $T$ be left ideals of $R$, then

$$(U \cap T)^r = U^r + T^r \qquad \text{holds.}$$

We also define analogous conditions for right objects, we write them $(h^\circ)_r$, $(c^\circ)_r$, etc. ...

On restricting the previous conditions to principal ideals or cyclic submodules we introduce conditions $(h^{\circ\circ})_1$, $(h^{\circ\circ})_r$, etc. ...

The following results will be used in the sequel.

PROPOSITION 1.1. (Ikeda-Nakayama {2}, Th. 1). *The following implications hold in $R$:*

*i)* $(a^{\circ\circ})_1 \Longleftrightarrow (c^{\circ\circ})_r$

*ii)* $(a^\circ)_1 \Longleftrightarrow (b^\circ)_1,\ (c^{\circ\circ})_r$

PROPOSITION 1.2. ({3}) 1, §2, Exer. 24). *Let $M$ be a right $R$-module and $M'$ a submodule of $M$. Then $M'$ is pure in $M$ if and only if for any set of elements* $m'_i \in M'$, $x_j \in M$, $r_{ij} \in R$ $(i=1,\dots,m; j=1,\dots,n)$ *such that*

$$m'_i = \textstyle\sum_{j=1}^n x_j . r_{ij}$$

*there exist elements* $x'_j \in M'$, $j=1,\dots,n$ *satisfying*

$$m'_i = \textstyle\sum_{j=1}^n x'_j . r_{ij}$$

As an immediate consequence of Prop. 1.2 we have the following

PROPOSITION 1.3. *Let $R_r$ be an injective hull of $R$, as right $R$-modules. Assume that $R$ is pure in $R_r$. Then any homomorphism $\mu: U \longrightarrow R$ of a finitely generated submodule $U$ of $R''^n$ into $R$ admits an extension to $R''^n$.*

*Proof:* Clearly $\mu$ admits an extension to $\mu': R''^n \longrightarrow R_r$. Therefore if $u_1,\dots,u_m$ denote a set of generators of $U$ and $e_1,\dots,e_n$ the canonical basis of $R''^n$, we have

$$\mu(u_i) = \textstyle\sum_{j=1}^n \mu'(e_j).r_{ij} \qquad i=1,\dots,m$$

By the purity of $R$ in $R_r$ there exist elements $x'_j$, $j=1,\dots,n$ in $R$

satisfying

$$\mu(u_i) = \sum_{j=1}^{n} x'_j \cdot r_{ij} \qquad i=1,\dots,m$$

Consequently the mapping defined by

$$e_j \to x'_j$$

gives an extension of $\mu$.

PROPOSITION 1.4. *Let A be a left R-module. Then A is injective if and only if every homomorphism $U \to A$ of a submodule U of $R'^n$ into A is realized by an element of $A^n$, that is, there exists $y \in A^n$ such that $\mu(u) = u.y$ for all $u \in U$.*

2. MAIN RESULTS.

Let $R_r$ denote an injective right R-module containing R

THEOREM 1. *The following implications hold in R:*

*R is right pure in* $R_r$ $\iff$

$(h^\circ)_1$ $\implies$

$(h^{\circ\circ})_1$ $\iff$

$(a^\circ)_r$

*Proof:* R is right pure in $R_r \implies (h^\circ)_1$

Let H be a finitely generated submodule of $R'^n$ and let

$$z'_i = [z_{1i},\dots,z_{ni}] \in R'^n \quad , \quad i=1,\dots,m$$

be a set of generators of it. Let $a = [a_1,\dots,a_n] \in R'^n$ be an element of $H^r$, that is, such that

(1) $\qquad u \in R'^n \, , \, z_i.u = 0 \, , \, i=1,\dots,m \implies a.u = 0$

Let $H''$ be the submodule of $R''^n$ generated by the vectors

$$z'_i = [z_{i1},\dots,z_{im}] \quad , \, i=1,\dots,n$$

Then (1) says precisely that

$$\mu: z'_i \to a_i$$

defines a homomorphism

$$\mu: H'' \to R$$

There exists then by Prop. 1.4, $b = [b_1,\dots,b_m] \in R_r$ satisfying

$$a_i = \mu(z'_i) = b.z'_i \qquad i=1,\dots,n$$

By the purity of R in $R_r$ we find $u \in R'^n$ with

$$a_i = u.z'_i \qquad i=1,\dots,m$$

that is

$$a_i = \sum_{j=1}^{m} u_j \cdot z_{ij} \qquad \text{or} \qquad a = \sum_{j=1}^{m} u_j \cdot z_j$$

which amounts to saying that $a \in H$, as we wanted to prove.

$(h^\circ)_1 \Longrightarrow R$ is right pure in $R_r$.

Let $a_i \in R$, $z_i \in R'^n$, $u \in R_r^n$ , $i=1,\dots,m$ satisfy

(2) $\qquad a_i = u.z_i \qquad i=1,\dots,m$

If $b \in R''^m$ satisfies $z_i^!.b = 0$, then by (2) we have $a.b = 0$ and by condition $(h^\circ)_1$ we have that there exist $r_i \in R$, $i=1,\dots,m$

with $\qquad a = \sum_{i=1}^{m} r_i \cdot z_i^!$

that is $\qquad a_i = r.z_i \qquad i=1,\dots,m$

with $r = [r_1,\dots,r_n]$. This proves our claim.

$(h^\circ)_1 \Longrightarrow (h^{\circ\circ})_1$ is trivial.

Finally we prove the equivalence $(h^{\circ\circ})_1 \Longleftrightarrow (a^\circ)_r$

$(h^{\circ\circ})_1 \Longrightarrow (a^\circ)_r$

Let $I = \langle a_1,\dots,a_n\rangle$ be a right ideal of $R$ generated by $a_1,\dots,a_n$

Let $\phi$: $I \longrightarrow R$ be a homomorphism of $I$ into $R$, as right $R$-modules.

Let $b_i = \phi(a_i)$, $i=1,\dots,n$. Since $\phi$ is a homomorphism, for any $t_1,\dots,t_n$ in $R$ $\quad \sum_{i=1}^{n} a_i \cdot t_i = 0 \Rightarrow \sum_{i=1}^{n} b_i \cdot t_i = 0$

This means that $[b_1,\dots,b_n] \in [a_1,\dots,a_n]^{rl} = \langle[a_1,\dots,a_n]\rangle$.

So there is $k \in R$ satisfying

$$[b_1,\dots,b_n] = k.[a_1,\dots,a_n]$$

that is $\quad \phi(a_i) = b_i = k.a_i$

and this proves $(a^\circ)_r$.

$(a^\circ)_r \Longrightarrow (h^{\circ\circ})_1$

This implication will be proved following the scheme of the proof of Th. 5.1 in {1}. We recall that by PROP. 1.1 (or its dual), $(a^\circ)_r \Longrightarrow (b^\circ)_r,(c^{\circ\circ})_1$. Let $S$ be a submodule of $R'^n$ generated by $a_1,\dots,a_n$.

The proof will proced by induction on n. For $n = 1$ , $S$ is a principal left ideal of $R$ and by $(c^{\circ\circ})_1$ we have that $S = \bar{S}$. Let $2 \leq n$ and assume that every cyclic submodule of $R'^{(n-1)}$ is closed. Let

$$T_1 = \{[x_1^!,0,\dots,0] \in R''^n \;/\; a_1.x_1^! = 0\}$$

$$T_2 = \{[0,x_2^!,\dots,x_n^!] \in R''^n \;/\; a_2.x_2^! + \dots + a_n.x_n^! = 0\}$$

Clearly $T_1$, $T_2 \subset S^r$

Then for every $U = [u_1,\dots,u_n] \in \bar{S}$ we have $U \in T_1^1$, so $u_1.x_i^! = 0$ and by the closeness of $\langle a_1\rangle$ we get $u_1 = t.a_1$ , $t \in R$.

Now $U - t[a_1,\ldots,a_n] = [0,v_2,\ldots,v_n] = V \in \bar{S} \subset T_2^1$. By the closure of the principal left submodule generated by $[a_2,\ldots,a_n]$ we have

$$[0,v_2,\ldots,v_n] = r[0,a_2,\ldots,a_n]$$

Let $I_1 = \langle a_1\rangle$, $I_2 = \langle a_2,\ldots,a_n\rangle$. Then $w \in I_1 \cap I_2$ if and only if there exist $x_1,\ldots,x_n \in R$ such that

$$w = a_1.x_1 = -(a_2.x_2 +\ldots+ a_n.x_n)$$

But

$$V = [0,r.a_2,\ldots,r.a_n] \in \bar{S} \text{ and } w \in I_1 \cap I_2 \text{ as above}$$

give

$$0 = r.a_2.x_2 +\ldots+ r.a_n.x_n = -r.a_1.x_1$$

that is

$$r \in (I_1 \cap I_2)^1$$

and since we have condition $(b°)_1$, r can be written as

$$r = m_1 + m_2 \quad , m_i \in I_i^1 , i=1,2$$

Hence

$$V = [0,r.a_2,\ldots,r.a_n] = [0,m_1.a_2,\ldots,m_1.a_n] = m_1.[a_1,\ldots,a_n]$$

and

$$U = V + t.[a_1,\ldots,a_n] = (m_1 + t).[a_1,\ldots,a_n] \in S$$

Theorem is now proved.

**AN EXAMPLE.**

Let $R$ be a right Ore domain (that is, a ring without zero divisors $\neq 0$ and with the right common multiple property). Then if h(R) is the injective hull of R, h(R) carries a ring structure which makes it isomorphic to the left field of quotients of R. Clearly R is right pure in h(R) if and only if h(R) = R is a division ring. More generally, for any $n \in N$, $M_n(R)$ is right pure in $M_n(h(R))$ if and only if R = h(R), ($M_n(\ )$ denotes the full ring of matrices). In fact, if $M_n(R)$ is right pure in $M_n(h(R))$, then by THEOREM 1, $M_n(R)$ satisfies condition $(a°)_r$ . But this readily implies that condition $(a°)_r$ holds in R. We are done, since a ring without zero divisors $\neq 0$ and satisfying $(a°)_r$ is necessarily a division ring.

**THEOREM 2.** *Let R be a left semihereditary ring. Then*

$$(a°)_1, (c°)_1 \Longrightarrow (h°)_1$$

*Proof:* Let $S$ be submodule of $R'^n$ generated by the vectors

$$A_i = [a_{i1},\dots,a_{in}] \qquad i=1,\dots,s$$

Let $S^o$ be the submodule of $S$ consisting of all vectors with 0 in the first component. Then

LEMMA 1. *$S^o$ is finitely generated*

*Proof:* Let

$$A = [a_{11},a_{21},\dots,a_{s1}] \in R''^s$$

and *assume*, for the time being, that the left annihilator of $A$ in $R'^s$ be generated by

$$B^i = [b_1^i,\dots,b_s^i] \qquad i=1,\dots,m$$

Then if $x \in S^o$ we have $r_1,\dots,r_s \in R$ satisfying

$$x = \sum_{i=1}^{s} r_i.A_i = [0,\sum_{i=1}^{s} r_i.a_{i2},\dots,\sum_{i=1}^{s} r_i.a_{in}]$$

therefore

$$[r_1,\dots,r_s] = \sum_{j=1}^{m} t_j.B^j \qquad t_j \in R$$

that is

$$r_k = \sum_{j=1}^{m} t_j.b_k^j \qquad k=1,\dots,s$$

But then

$$x = \sum_{k=1}^{s} r_k.A_k = \sum_{k=1}^{s} (\sum_{j=1}^{m} t_j.b_k^j).A_k$$
$$= \sum_{j=1}^{m} t_j.(\sum_{k=1}^{s} b_k^j.A_k)$$

We now claim that

$$A_j' = \sum_{k=1}^{s} b_k^j.A_k \qquad j=1,\dots,m$$

generate $S^o$. In fact, notice that x was an arbitrary element of $S^o$ and that the first component of $A_j'$ is $\sum_{k=1}^{s} b_k^j.a_{k1} = 0$
Our claim follows.

Now, in order to complete the proof of Lemma 1 we need to prove that we can assume that the left annihilator of $A$ in $R'^n$ is finitely generated. For this we shall use the hypothesis that $R$ is a left semihereditary ring. Let $F$ be a free left $R$-module generated by $f_1,\dots,f_s$ and $0 \to K \to F \xrightarrow{\phi} L \to 0$ be an exact sequence where $L$ is the left ideal of $R$ generated by $a_{11},\dots,a_{s1}$ and $\phi$ be the homomorphism defined by $\phi : f_j \to a_{j1}$. Notice that $K$ is isomorphic to the left annihilator of $A$ in $R'^s$. Since $L$ is projective, that sequence splits and $K$ is then a direct summand of a finitely generated $R$-module, therefore is finitely generated. This ends the proof of Lemma 1.

We proceed the proof of THEOREM 2 by induction in the length of the vectors in $S$. If $n = 1$, then $S$ is a finitely generated left ideal of $R$, and so by condition $(c^\circ)_1$ is closed. Let $2 \leqslant n$ and assume that every finitely generated submodule of $R'^{(n-1)}$ is closed. In particular, the submodule $B \subset R'^{(n-1)}$ associated to $S^\circ$, dropping the first coordinate of the elements in $S^\circ$, is closed. Next we need to prove another partial result

LEMMA 2. *If* $[x_2', \ldots, x_n'] \in B^r$, *then there exists* $x_1 \in R$ *such that*

$$[x_1, x_2', \ldots, x_n'] \in S^r$$

*Proof*: Let $r_1, \ldots, r_s \in R$ satisfy $\sum_{i=1}^{s} r_i \cdot a_{i1} = 0$. Then

$$\sum_{i=1}^{s} r_i \cdot A_i = [0, \textstyle\sum_{i=1}^{s} r_i a_{i2}, \ldots, \sum_{i=1}^{s} r_i a_{in}] \in S^\circ$$

and by the hypothesis we have

$$0 = \sum_{k=2}^{n} (\sum_{i=1}^{s} r_i a_{ik}) \cdot x_k'$$

$$= \sum_{i=1}^{s} r_i \cdot (\sum_{k=2}^{n} a_{ik} x_k')$$

which says that

$$\phi : a_{i1} \rightarrow \textstyle\sum_{k=2}^{n} a_{ik} x_k'$$

defines an $R$-homomorphism of the left ideal generated by $a_{i1}, i=1,\ldots,s$ into $R$. By property $(a^\circ)_1$ there is $-x_1 \in R$ realizing $\phi$, that is

$$a_{i1}x_1 + a_{i2}x_2' + \ldots + a_{in}x_n' = 0 \qquad i=1,\ldots,s$$

and this ends the proof of LEMMA 2.

To complete the prcof of THEOREM 2 we follow the scheme of proof of THEOREM 5.2 of {1}. Let $U = [u_1, \ldots, u_n] \in \bar{S}$. $S^r$ contains all those vectors

$$[x_1, 0, \ldots, 0] \text{ such that } a_{i1}x_1 = 0 \text{ , } i=1,\ldots,s$$

Therefore

$$x_1 \in \langle a_{11}, a_{21}, \ldots, a_{s1} \rangle^r$$

$$u_1 \in \langle a_{11}, a_{21}, \ldots, a_{s1} \rangle^{rl} = \langle a_{11}, a_{21}, \ldots, a_{s1} \rangle$$

(by condition $(c^\circ)_1$),

$S_\circ$, $\quad u_1 = r_1 a_{11} + \ldots + r_s a_{s1}$ , $r_i \in R$

Let $\quad U' = \sum_{i=1}^{s} r_i \cdot A_i$

$U'$ belongs to $S$ and moreover $V = U - U' = [0, v_2, \ldots, v_n] \in \bar{S}$

satisfies

(') $$v_2x_2 + \ldots + v_nx_n = 0$$

for any $[x_1,x_2,\ldots,x_n] \in S^r$.

Let $\mathcal{D}$ denote the submodule of $R''^{(n-1)}$ of all elements $x_2,\ldots,x_n$ for which there is $x_1 \in R$ satisfying $[x_1,x_2,\ldots,x_n] \in S^r$.

Clearly we have $\mathcal{D} \subset B^r$. But by LEMMA 2, $B^r \subset \mathcal{D}$. So $\mathcal{D} = B^r$.

Furthermore $[v_2,\ldots,v_n] \in \mathcal{D}^l = B^{rl} = B$ according to the inductive hypothesis. Of course we need to know that $B$ is finitely generated, but this follows from LEMMA 1 and the definition of $B$.
We have then that $[0,v_2,\ldots,v_n] \in S^\circ \subset S$ and finally

$$U = V + U' \in S$$

This means that $\bar{S} \subset S$ and THEOREM 2 is proved.

COROLLARY. *Let R be a left and right semihereditary ring. Assume that* $(c^\circ)_l$ *and* $(c^\circ)_r$ *holds. Then* $(a^\circ)_l \Longleftrightarrow (a^\circ)_r$.

*Proof:* Assume that $(a^\circ)_l$ holds. Then

$$(a^\circ)_l \Longleftrightarrow (h^\circ)_l \quad \text{, by Theorem 2}$$
$$\Longrightarrow (h^{\circ\circ})_l$$
$$\Longrightarrow (a^\circ)_r \quad \text{, by Theorem 1}$$

The other implication follows in the same way.

## 3. VON NEUMANN RINGS.

In this section we give characterizations of von Neumann rings in terms of purity. We recall that a von Neumann ring is a ring $R$ satisfying: for every $a \in R$ there is $x \in R$ such that $a.x.a = a$

We shall say that a ring is absolutely flat (resp. pure) if any right $R$-module is flat (resp. pure).

THEOREM 3. *Let R be a ring. The following conditions are all equivalent:*

*a) R is absolutely pure*

*b) R is a von Neumann ring*

*c) R is absolutely flat*

*d) every cyclic right R-module is pure*

*Proof:* a) $\Longrightarrow$b) Let $z \in R$. Then the right ideal $z.R$ is pure in $R$. Since $R$ has identity we can write $z = 1.z$. By the purity there is $x \in R$ such that $z = (z.x).z$ as we wanted to prove.

b) $\Longrightarrow$ c) is a well known result
c) $\Longrightarrow$ d) and c) $\Longrightarrow$ a) are clear
d) $\Longrightarrow$ b)
Let $I$ be a right ideal of $R$, $a \in R$ and $\phi: \langle a\rangle \to R/I$ be a homomorphism of the right ideal $\langle a\rangle$ generated by a into the cyclic module $R/I$. Let $S$ be an injective right module containing $R/I$.
There exists $s \in S$ satisfying

$$\phi(a) = s.a$$

and since $R/I$ is pure in $S$, we can find $c \in R/I$ such that

$$\phi(a) = c.a$$

This means that $\phi$ can be extended to a homomorphism of $R$ into $R/I$. Being $I$ and $a \in R$ arbitrary we can apply Th. 3 of {2} to conclude that $R$ is a von Neumann ring.
Proof of Theorem 3 is now complete.

REMARK 1. Using the absolute purity of von Neumann rings, as shown in THEOREM 3, we can give an immediate answer to a question posed in {4 }, §25.(1). Namely: Let $A$ be a right $R$-module, where $R$ is a von Neumann ring. Suppose that $A$ is generated by n elements. Then every finitely generated submodule of $A$ is generated by n elements. In fact, let $A'$ be a finitely generated submodule of $A$, $a_1,\dots,a_n$ a set of generators of $A$ and $a_1',\dots,a_m'$ a set of generators of $A'$. We have $r_{ji} \in R$ satisfying

$$a_i' = \sum_{j=1}^{n} a_j.r_{ji} \quad , \quad i=1,\dots,m$$

Being $A'$ pure in $A$ there exist $x_j' \in A'$, $j=1,\dots,n$ satisfying

$$a_i' = \sum_{j=1}^{n} x_j'.r_{ji}$$

Clearly, $x_j'$ is a set of generators of $A'$.
Next we characterize those right semihereditary rings which are von Neumann rings.

LEMMA. (Compare {3}, Chap. I, §2, Exer. 18 a)). *Let $R$ be a right semihereditary ring and let $B$ be an injective right $R$-module containing $R$ such that $R$ is pure in $B$. Then any finitely generated submodule of a projective right $R$-module is a direct summand of it.*

*Proof:* Let $P$ be a projective right $R$-module and let $M$ be a finitely generated submodule of it. Without loss of generality we can assume that $P$ is finitely generated and free. In fact, if $F$ is a free module of which $P$ is a submodule then we can write $F = F_1 \oplus F_2$, with $F_1$ free, finitely generated and containing $M$. If $M$ is a direct summand of $F_1$, it is also a direct summand of $F$ and therefore of $P$. Being $R$ right semihereditary, $M$ is a projective module. Let $a_1,\ldots,a_n$ be elements of $M$ and $\phi'_1,\ldots,\phi'_n$ mappings of $M$ into $R$, satisfying

$$a = \sum_{i=1}^{n} a_i \cdot \phi'_i(a)$$

for every $a \in M$.

Since $R$ is pure in $B$, by PROP. 1.3, the mappings $\phi'_i$ can be extended to mappings $\phi_i$: $P \longrightarrow R$. Let $\phi$: $P \longrightarrow M$ be the mapping defined by

$$\phi: x \longrightarrow \sum_{i=1}^{n} a_i \cdot \phi_i(x)$$

Clearly $\phi$ defines a projection of $P$ onto $M$. $M$ is then a direct summand of $P$.

REMARK 2. The previous Lemma permits to give an immediate answer to a question posed in {4}, §25.(1). Namely, let $R$ be a von Neumann ring. Then if every torsion free $R$-module is projective, $R$ is a left self-injective ring. In fact, let h($R$) be an injective hull of $R$. Then h($R$) is torsion free, therefore it is projective. Let $I = \langle e \rangle$ be a principal non-zero left ideal of $R$, e an idempotent. By the previous Lemma $I$ is a direct summand of h($R$) and so $I$ is injective. Since $I$ was arbitrary, we have also that $J = \langle 1-e \rangle$ is injective. Therefore $R = I \oplus J$ is injective as we wanted to prove.

THEOREM 4. *Let R be a ring. Then R is a von Neumann ring if and only if R is right semihereditary and pure (in some injective right R-module containing it).*

*Proof:* Apply the previous Lemma to $P = R$ to get that every finitely generated right ideal of $R$ is a direct summand of $R$. This is enough to assure that $R$ be a von Neumann ring.

Base in the same Lemma we have

PROPOSITION 3.1. *Let R be a ring. Then R is a von Neumann ring iff R is right semihereditary and satisfies condition* $(a°)_r$

*Proof:* To prove part "if" we proceed as in the proof of the Lemma applied to the situation $P = R$ and using condition $(a°)_r$ to extend the mappings $\phi'_i$.

COROLLARY. *Let R be a left noetherian, left hereditary ring sat-*

*isfying condition* $(a°)_r$. *R is then a semisimple (d.c.c.) ring.*

*Proof*: According to a result by L.W. Small ({5} , COROLLARY 3) the two first hypothesis imply that R is right semihereditary. Condition $(a°)_r$ and the previous proposition prove our claim, since a left noetherian von Neumann ring is necessarily semisimple (d.c.c.).

## REFERENCES


1. Marshall Hall. A Type of Algebraic Closure. Annals of Mathematics 40 (1939), 360-369.
2. M. Ikeda and T. Nakayama. On Some Characteristic Properties of Quasi-Frobenius and Regular Rings. Proceedings of the A.M.S 5 (1954), 15-19.
3. N. Bourbaki. Algebre Commutative. Chap. I and 2. (1961) Hermann, Paris.
4. R.S. Pierce. Modules over Commutative Regular Rings. American Mathematical Society Memoir 70 (1967).
5. L.W. Small. Semihereditary Rings. Bulletin of the A.M.S. 73 (1967) 656-658.



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