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PARALLEL FAMILIES OF ∞^1 HYPERSURFACES IN RIEMANNIAN SPACE V_n.(1)

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1. ELEMENTARY CONCEPTS CONCERNING A POSITIONAL FIELD OF FORCE ϕ.(2)
Consider a field of force ϕ, whose force vector acting at any point $x = (x^i)$, of a Riemannian space V_n, is $\phi = \phi^i = \phi_i$. The force vector ϕ, is considered to be a vector function of position only, and to be of class two over a certain region of V_n. Thus the force vector ϕ is a continuous function and possesses continuous partial derivatives with respect to the x^i of first and second orders throughout the given region of V_n. Such a field of force ϕ, is termed a *positional field of force* ϕ in the Riemannian space V_n.

The positional field of force ϕ may be given by either the covariant components ϕ_i, or the contravariant component $\phi^i = g^{ij}\phi_j$. Here the $g^{ij} = g^{ij}(x)$, are the contravariant components of the fundamental metric tensor of the given Riemannian space V_n. It is assumed that $|\phi| = |\varepsilon\, g^{ij}\phi_i\phi_j|^{1/2} > 0$, $\varepsilon = \pm 1$. The trivial case when ϕ is identically zero, which leads to the ∞^{2n-2} geodesics of V_n, is omitted from consideration.

If a particle of constant mass $m > 0$, is constrained to move along a path C of the given region of V_n, then its speed v is governed by the law

$$(1.1) \qquad T = \frac{mv^2}{2} = \int_{C\,x_0}^{x} \phi_i dx^i + \frac{mv_0^2}{2} \qquad ,$$

where the initial speed at the point x_0, is $v_0 > 0$. The integral appearing in (1.1), is termed the *work* W performed in moving from x_0 to x along the curve C. The quantity $T = 1/2.mv^2$, is called the *kinetic energy* of the particle.

If the work W is independent of the path C, then the positional field of force ϕ is said to be *conservative*. In this case there exists a potential function $V = V(x)$, such that

$$(1.2) \qquad \phi_i = -\frac{\partial V}{\partial x^i} = -\mathrm{grad}(V).$$

For a conservative field of force ϕ, the relation (1.1), becomes the energy equation, namely

$$(1.3) \qquad T + V = \frac{mv^2}{2} + V(x) = E \quad ,$$

where E is the total constant energy.

For any positional field of force ϕ, conservative or not, the simple family of ∞^{n-1} *Faraday lines of force* C, is composed of the ∞^{n-1} integral solutions C of the system of n first order ordinary differential equations

$$(1.4) \qquad \frac{dx^i}{ds} = \frac{g^{ij}\phi_j}{|\phi|} = \frac{\phi^i}{|\phi|} \quad ,$$

where s denotes the arc length along any Faraday line of force C.

THEOREM 1.1. *If* ϕ *is a conservative field of force in* V_n, *let* L *be a curve such that at each point* P *of* L, *there exists a Faraday line of force* C, *tangent to* L. *Assume that these* ∞^1 *Faraday lines of force are all transversal to some curve* L*. *Let* AD *denote the line of force at the initial point* A *on* L *and terminal point* D *on* L*. *Similarly* BE *denotes the line of force at* B *on* L, *terminating at* E *on* L*. *Then*

$$(1.5) \qquad \int_{L_{AB}} |\phi| ds = \int_{C_{AD}} |\phi| ds - \int_{C_{BE}} |\phi| ds.$$

Another way of writing this is in terms of the characteristic function Γ of V_n. Thus

$$(1.6) \qquad \int_{L_{AB}} \Gamma(x;\frac{\partial V}{\partial x}) ds = \int_{C_{AD}} \Gamma(x;\frac{\partial V}{\partial x}) ds - \int_{C_{BE}} \Gamma(x;\frac{\partial V}{\partial x}) ds.$$

It is noted that if p_i is an absolute covariant vector, then $\Gamma(x;p) = |\varepsilon g^{ij} p_i p_j|^{1/2}$, $\varepsilon = \pm 1$. If the characteristic function $\Gamma(x;\frac{\partial V}{\partial x}) = +1$, then (1.6), becomes an analogue of the Jacobi string condition for V_n. Also $\Gamma(x;\frac{\partial V}{\partial x}) = 1$, is then the Hamilton Jacobi partial differential equation of first order.

2. THE LAMÉ DIFFERENTIAL PARAMETERS OF FIRST AND SECOND ORDERS.[3]

If u(x) and v(x) are two absolute scalar functions of at least class C^1, then their gradients $\frac{\partial u}{\partial x^i}$ and $\frac{\partial v}{\partial x^i}$, are two absolute covariant vectors orthogonal to the two (n-1) dimensional hypersurfaces u(x)= constant and v(x)=constant. The inner product between these two gradients is called the *Lamé differential parameter of order one*. It is written as

$$(2.1)\qquad \Delta_1(u,v) = (\text{grad } u, \text{grad } v) = g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} .$$

In particular if $u(x) = v(x)$, this becomes

$$(2.2)\qquad \Delta_1(u) = \Delta_1(u,u) = g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} .$$

Now since in a conservative field of force there is a potential function $V = V(x)$, such that $\phi_i = -\frac{\partial V}{\partial x^i}$, it is seen that

$$(2.3)\qquad |\phi| = |\text{grad } V| = |\varepsilon\Delta_1(V)|^{1/2} > 0 .$$

If ϕ_i is the covariant form of the force vector ϕ, then the *divergence* of ϕ is defined by the absolute scalar

$$(2.4)\qquad \theta = \text{div}(\phi) = g^{ik}\phi_{i,k} = g^{ik}\left[\frac{\partial \phi_i}{\partial x^k} - \Gamma^j_{ik}\,\phi_j\right] .$$

A field of force ϕ is said to be *solenoidal* if and only if the divergence of the force vector ϕ is zero throughout the region of V_n under consideration.

If $g = |g_{ij}|$, the determinant of the g_{ij}, then another form of (2.4), is

$$(2.5)\qquad \theta = \text{div}(\phi) = \frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial x^i}\sqrt{|g|}\,\phi^i\right] .$$

It is noted that the form of the divergence given in (2.5), is that most convenient for applications.

The Lamé differential parameter of the second order is the *Laplacean*. It is defined by

$$(2.6)\qquad \Delta_2(u) = \nabla^2 u = g^{ik}\left(\frac{\partial u}{\partial x^i}\right)_k = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\,g^{ij}\frac{\partial u}{\partial x^j}\right) ,$$

where $u = u(x)$ is an absolute scalar function of position.

A positional field of force is said to be *Laplacean* if and only if it is both conservative and solenoidal. Thus a conservative field of force ϕ, with a potential function $V(x)$ is Laplacean if and only if $V(x)$ is a harmonic function. That is $\nabla^2 v = 0$.

3. CONSERVATIVE FIELDS OF FORCE ϕ FOR WHICH THE EQUIPOTENCIAL HYPERSURFACES FORM A PARALLEL FAMILY. Let ϕ be a conservative field

of force with potential function $V = V(x)$. The family $V(x)$ = constant represents a set of ∞^1 hypersurfaces, called *equipotential hypersurfaces*, each of which is of deficiency one. It is assumed that $\Delta_1(V) \neq 0$. Then a unit vector orthogonal to these hypersurfaces is

$$(3.1) \qquad \xi_i = \frac{1}{\sqrt{|\Delta_1(V)|}} \frac{\partial V}{\partial x^i} .$$

The contravariant form of this vector is

$$(3.2) \qquad \xi_i = \frac{1}{\sqrt{|\Delta_1(V)|}} g^{ij}\frac{\partial V}{\partial x^j} .$$

Therefore the orthogonal trajectories C of this set of ∞^1 equipotential hypersurfaces $V(x)$ = constant are the ∞^{n-1} Faraday lines of force given as the integral solutions C of the system of n first order ordinary differential equations $\frac{dx^i}{ds} = \xi^i$

The ∞^{n-1} curves C and the ∞^1 hypersurfaces V = constant are said to form a *normal family*. Also a set of ∞^1 hypersurfaces $V(x)$ = constant is said to be *parallel* if and only if it cuts orthogonally a set of ∞^{n-1} geodesics C of V_n.

THEOREM 3.1. *The set of ∞^1 equipotential hypersurfaces $V(x)$ = constant of a conservative field of force ϕ, is a parallel family if and only if the Lamé parameter $\Delta_1(V)$ is a function of the potential function $V = V(x)$ alone. Thus*

$$(3.3) \qquad \Delta_1(V) = g^{ij}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j} = \varepsilon F(V) \quad ,$$

where $\varepsilon = \pm 1$, according as $\Delta_1(V) \geq 0$, or $\Delta_1(V) < 0$.

For, in this case the ∞^{n-1} orthogonal trajectories are the ∞^{n-1} geodesics C of V_n. Hence parallel displacements of the unit tangent vectors along these curves C are also tangent to the respective curves C. Therefore

$$(3.4) \qquad \left(\frac{dx^i}{ds}\right)_{,k} \frac{dx^k}{ds} = \left(\Delta_1^{-1/2} g^{ij}\frac{\partial V}{\partial x^j}\right)_{,k} g^{k\alpha}\frac{\partial V}{\partial x^\alpha} = 0 \quad .$$

where $\Delta_1 = \Delta_1(V)$, and the symbol $(\ )_{,k}$ means the covariant derivative of the quantity inside the parenthesis with respect to the x^k. Thus

$$(3.5)\qquad \left[-\frac{1}{2}\Delta_1^{-3/2}\frac{\partial\Delta_1}{\partial x^k}\frac{\partial V}{\partial x^j}+\Delta_1\left(\frac{\partial V}{\partial x^j}\right)_{,k}\right]g^{k\alpha}\frac{\partial V}{\partial x^\alpha}=0$$

Since $|g^{ij}| \neq 0$, it is seen that

$$(3.6)\qquad -\frac{1}{2}\frac{\partial\Delta_1}{\partial x^k}g^{k\alpha}\frac{\partial V}{\partial x^j}\frac{\partial V}{\partial x^\alpha}+\Delta_1 g^{k\alpha}\frac{\partial V}{\partial x^\alpha}\left(\frac{\partial V}{\partial x^j}\right)_{,k}=0$$

However it is easily seen that

$$(3.7)\qquad g^{k\alpha}\frac{\partial V}{\partial x^\alpha}\left(\frac{\partial V}{\partial x^j}\right)_{,k}=g^{k\alpha}\frac{\partial V}{\partial x^\alpha}\left(\frac{\partial V}{\partial x^k}\right)_{,j}=\frac{1}{2}\frac{\partial\Delta_1}{\partial x^j}$$

Substitute this result into (3.6). Then interchange the dummy indices k and α and substract. It is evident that

$$(3.8)\qquad \frac{\partial V}{\partial x^j}g^{k\alpha}\left|-\frac{\partial\Delta_1}{\partial x^k}\frac{\partial V}{\partial x^\alpha}+\frac{\partial\Delta_1}{\partial x^\alpha}\frac{\partial V}{\partial x^k}\right|=0$$

Since at least one $\frac{\partial V}{\partial x^j} \neq 0$, and since $|g^{k\alpha}| \neq 0$, these conditions become

$$(3.9)\qquad \frac{\partial\Delta_1}{\partial x^\alpha}\frac{\partial V}{\partial x^k}-\frac{\partial\Delta_1}{\partial x^k}\frac{\partial V}{\partial x^\alpha}=\frac{\partial(\Delta_1,V)}{\partial(x^k,x^\alpha)}=0$$

Hence $\Delta_1(V)$ and V are functionally dependent. As these steps are reversible the proof of Theorem 3.1, is complete.

Now recall that since V(x) is an absolute scalar function it is known that $\Gamma(x;\frac{\partial V}{\partial x}) = |\varepsilon\Delta_1(V)|^{1/2} \geq 0.$

THEOREM 3.2. *If V(x) = constant, with $\Delta_1(V) \neq 0$, form a set of ∞^1 equipotential hypersurfaces of a conservative field of force ϕ, each of deficiency one, which form a parallel family, there exists a non-constant function $\chi = \chi(V)$ which obeys the Hamilton-Jacobi partial differential equation of first order, namely*

$$(3.10)\qquad \Gamma(x;\frac{\partial\chi}{\partial x})=|\varepsilon\Delta_1(\chi)|^{1/2}=\left|\varepsilon g^{ij}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^j}\right|^{1/2}=+1$$

Conversely if $\chi(x)$ = constant obeys the Hamilton-Jacobi equation then it represents a parallel family.

For, in Theorem 3.1 set $\chi(V) = \int\frac{dV}{\sqrt{|F(V)|}}$, where F(V) is a non-constant function of V = V(x), and note that V(x) = constant is parallel if and only if

$$(3.11) \qquad \left[\varepsilon\Delta_1(\chi)\right]^{1/2} = \left[\varepsilon g^{ij}\frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^j}\right]^{1/2} = \mathcal{I}(x;\frac{\partial\chi}{\partial x}) = +1$$

Thus. Theorem 3.2 is proved.

4. SOME CONDITIONS FOR PARALLEL FAMILIES OF ∞^1 HYPERSURFACES IN RIEMANNIAN SPACE V_n. If λ_i is an absolute covariant vector in V_n, its covariant first order partial derivative is given by

$$(4.1) \qquad \lambda_{i,j} = \frac{\partial\lambda_i}{\partial x^j} - \Gamma^a_{ij}\lambda_a$$

In particular if $V = V(x)$, is an absolute scalar function

$$(4.2) \qquad V_{,i} = \frac{\partial V}{\partial x^i}, \qquad V_{,ij} = \frac{\partial^2 V}{\partial x^i\partial x^j} - \Gamma^a_{ij}\frac{\partial V}{\partial x^a}$$

Thus the ordinary partial derivative of the first order Lamé differential parameter $\Delta_1 = \Delta_1(V)$, with respect to x^α is

$$(4.3) \qquad \frac{\partial\Delta_1}{\partial x^\alpha} = 2\,g^{ij}\,V_{,i}\,V_{,j\alpha} = 2\,g^{ij}\left[\frac{\partial^2 V}{\partial x^j\partial x^\alpha}\frac{\partial V}{\partial x^i} - \Gamma^a_{j\alpha}\frac{\partial V}{\partial x^a}\frac{\partial V}{\partial x^i}\right].$$

Let $T_{\alpha\beta}$, denote the absolute covariant tensor of second order defined by

$$(4.4) \qquad T_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial V}{\partial x^\beta}\frac{\partial\Delta_1}{\partial x^\alpha} - \frac{\partial V}{\partial x^\alpha}\frac{\partial\Delta_1}{\partial x^\beta}\right] = g^{ij}V_{,i}\left[V_{,\beta}\,V_{,j\alpha} - V_{,\alpha}\,V_{,j\beta}\right].$$

Now supposing that $\Delta_1(V) > 0$, and since $g = |g_{ij}| \neq 0$, the following result is evident.

THEOREM 4.1. *Let the potential function* $V = V(x)$, *of a conservative field of force* ϕ, *be of at least class two in a given open region of Riemannian space* V_n, *with* $\Delta_1(V)$ *positive. Then the family of* ∞^1 *hypersurfaces, each of deficiency one, defined as the equipotential hypersurfaces* $V(x)$ = *constant of the field of force* ϕ, *is a parallel family if and only if*

$$(4.5) \qquad T_{\alpha\beta} = g^{ij}\,V_{,i}\left[V_{,\beta}\,V_{,j\alpha} - V_{,\alpha}\,V_{,j\beta}\right] = 0.$$

Moreover, each component of this tensor, when $\alpha \neq \beta$, α,β *fixed, namely*

$$(4.6) \qquad V_{,\beta}\,V_{,j\alpha} - V_{,\alpha}\,V_{,j\beta},$$

is orthogonal to $V(x)$ = *constant.*

Now the condition (4.5) can be written in the form

$$(4.7) \qquad V_{,\beta}\, g^{ij}\, V_{,i}\, V_{,j\alpha} = V_{,\alpha}\, g^{ij}\, V_{,i}\, V_{,j\beta}.$$

Let ρ be defined so that

$$(4.8) \qquad g^{ij}\, V_{,i}\, V_{,j\alpha} = \rho V_{,\alpha} \quad , \quad g^{ij}\, V_{,i}\, V_{,j\beta} = \rho V_{,\beta}.$$

The second set of equations in (4.8) follows from the first since at least one $V_{,\alpha}$ is not zero.

THEOREM 4.2. *A family of* ∞^1 *equipotential hypersurfaces* $V(x)$ = *constant of a conservative field of force* ϕ, *with potential function* $V = V(x)$, *and* $\Delta_1(V) > 0$, *is a parallel family if and only if*

$$(4.9) \qquad \frac{1}{2}\frac{\partial \Delta_1}{\partial x^\alpha} = g^{ij}\, V_{,i}\, V_{,j\alpha} = \rho\, V_{,\alpha} \quad ,$$

where $\alpha=1,2,\ldots,n$, *and* ρ *is an absolute scalar function. Thus if* $\Delta_1 > 0$, *is not a constant the two families* $\Delta_1 =$ *constant and* $V(x)$= *constant are identical.*

It is noted that if $\Delta_1 = c > 0$, is identically constant, then by a suitable homothetic map T, the conservative field of force ϕ, with potential function $V = V(x)$, is such that the force vector $\phi_i = -\dfrac{\partial V}{\partial x^i}$, has unit magnitude.

5. CARTOGRAMS AND PARALLEL FAMILIES. Let V_n and $\bar{V}_n$, be two Riemannian spaces whose metrics are given by the two definite quadratic differential forms: $ds^2 = g_{ij}dx^i dx^j$, and $d\bar{s}^2 = \bar{g}_{ij}dx^i dx^j$, where each of the $2n^2$ functions $g_{ij}(x)$, $\bar{g}_{ij}(x)$, is of at least class three in an n dimensional region of points of V_n, or $\bar{V}_n$. Two points, one of V_n and the other of $\bar{V}_n$, are said to correspond if and only if they are determined by the *same* curvilinear coordinates $x=(x^1,x^2,\ldots,x^n)$. This establishes a point to point transformation between V_n and $\bar{V}_n$. Such a transformation between V_n and $\bar{V}_n$, is called a *cartogram* T.

If ρ is a positive scalar point function of at least class three in the given region of V_n such that $d\bar{s} = \rho ds$, then the cartogram T is said to be *conformal* and the two Riemannian spaces V_n and $\bar{V}_n$ are said to be *conformally equivalent*. Hence $\bar{g}_{ij} = \rho^2 g_{ij}$, and $\bar{g}^{ij} = \frac{1}{\rho^2} g^{ij}$. In this case $\bar{V}_n$ is called a *conformal image* of the space V_n.

In particular if the positive scalar point function ρ is a constant then $\bar{V}_n$ is called a *homothetic image* of V_n and T is called a *homothetic transformation* T. If $\rho = +1$, $\bar{V}_n$ is called an *isometric image* of V_n and T is an *isometric correspondence*. In this case $\bar{V}_n$ is said to be *applicable* to V_n.

Now let $V(x)$ = constant represent a parallel family of equipotential hypersurfaces of a conservative field of force ϕ in V_n. The condition for this is

$$(5.1) \qquad g^{ij} V_{,i} V_{,j\alpha} = \rho V_{,\alpha} ,$$

for $\alpha=1,2,\ldots,n$, and ρ a scalar function.

Suppose that this parallel family in V_n corresponds to a parallel family of ∞^1 hypersurfaces in $\bar{V}_n$. Thus the condition (5.1) must hold in both V_n and $\bar{V}_n$. Hence for some scalar functions $\rho(x)$ and $\bar{\rho}(x)$

$$(5.2) \qquad g^{ij}\frac{\partial V}{\partial x^i}\left[\frac{\partial^2 V}{\partial x^j \partial x^\alpha} - \Gamma^\lambda_{j\alpha}\frac{\partial V}{\partial x^\lambda}\right] = \rho\frac{\partial V}{\partial x^\alpha}, \quad \bar{g}^{ij}\frac{\partial V}{\partial x^i}\left[\frac{\partial^2 V}{\partial x^j \partial x^\alpha} - \bar{\Gamma}^\lambda_{j\alpha}\frac{\partial V}{\partial x^\lambda}\right] = \bar{\rho}\frac{\partial V}{\partial x^\alpha} .$$

Hence

$$(5.3) \qquad (\bar{\rho} g^{ij} - \rho\bar{g}^{ij})\frac{\partial V}{\partial x^i}\frac{\partial^2 V}{\partial x^j \partial x^\alpha} - \frac{\partial V}{\partial x^i}(\bar{\rho} g^{ij}\Gamma^\lambda_{j\alpha} - \rho\bar{g}^{ij}\bar{\Gamma}^\lambda_{j\alpha})\frac{\partial V}{\partial x^\lambda} = 0$$

Now ρ and $\bar{\rho}$ are independent of the partial derivatives of $V(x)$. Clearly (5.3) is an identity for all such partial derivatives. Hence necessarily

$$(5.4) \qquad \bar{\rho} g^{ij} - \rho\bar{g}^{ij} = 0 , \quad \bar{\rho} g^{ij}\Gamma^\lambda_{j\alpha} - \rho\bar{g}^{ij}\bar{\Gamma}^\lambda_{j\alpha} = 0$$

Set $u^2 = \frac{\bar{\rho}}{\rho} > 0$. Then from the first of equations (5.4) it is seen that $\bar{g}^{ij} = \frac{1}{u^2} g^{ij}$. Thus we have established that V_n and $\bar{V}_n$ are necessarily related by a conformal cartogram T.

Now if T is a conformal cartogram the Christoffel symbols of the second kind correspond by the equations

$$(5.5) \qquad \bar{\Gamma}^\lambda_{j\alpha} = \Gamma^\lambda_{j\alpha} + \frac{1}{u}\left[\delta^\lambda_j\frac{\partial u}{\partial x^\alpha} + \delta^\lambda_\alpha\frac{\partial u}{\partial x^j} - g_{j\alpha}g^{\lambda\beta}\frac{\partial u}{\partial x^\beta}\right]$$

However the second of equations (5.4) yields

$$\bar{g}^{ij}\,\bar{\Gamma}^{\lambda}_{j\alpha} = \frac{1}{u^2}\,g^{ij}\Gamma^{\lambda}_{j\alpha} \tag{5.6}$$

Upon comparing (5.5) and (5.6), it is found that

$$\frac{1}{u^3}\,g^{ij}\left[\delta^{\lambda}_{j}\frac{\partial u}{\partial x^{\alpha}} + \delta^{\lambda}_{\alpha}\frac{\partial u}{\partial x^{j}} - g_{j\alpha}\,g^{\lambda\beta}\frac{\partial u}{\partial x^{\beta}}\right] = 0. \tag{5.7}$$

Since $u > 0$ and $g = |g_{ij}| \neq 0$, these reduce to

$$\delta^{\lambda}_{j}\frac{\partial u}{\partial x^{\alpha}} + \delta^{\lambda}_{\alpha}\frac{\partial u}{\partial x^{j}} - g_{j\alpha}\,g^{\lambda\beta}\frac{\partial u}{\partial x^{\beta}} = 0. \tag{5.8}$$

Multiply through by $g^{j\alpha}$. Upon simplification (5.8) reduces to

$$g^{j\alpha}\frac{\partial u}{\partial x^{j}} = 0. \tag{5.9}$$

Again since $1/g = |g^{j\alpha}| \neq 0$, it is seen that $\frac{\partial u}{\partial x^{j}} = 0$, for $j=1,2,.,n$.

THEOREM 5.1. *Let* V_n *and* $\bar{V}_n$ *be two Riemannian spaces for which the fundamental quadratic differential forms are* $ds^2 = g_{ij}dx^i dx^j$ *and* $d\bar{s}^2 = \bar{g}_{ij}dx^i dx^j$ *respectively. Suppose that* $V(x) =$ *constant is parallel family of* ∞^1 *hypersurfaces in* V_n, *where* $V(x)$ *is the potential function of a conservative field of force* ϕ *in* V_n. *Then every such family will correspond to a parallel family of* ∞^1 *hypersurfaces in* $\bar{V}_n$, *by a cartogram* T, *if and only if* T *is a homothetic map.*

For, in this case the scale function $u(x)$ obeys $\frac{\partial u}{\partial x^{\alpha}} = 0$, for $\alpha = 1,2,\ldots,n$. Since the scale function is always positive, it is seen that $u = u(x) \equiv c > 0$, where c is a constant. Thus the proof of Theorem 5.1 is complete.

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