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INDUCED SHEAVES AND GROTHENDIECK TOPOLOGIES

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INTRODUCTION. The theory of sheaves, as it is exposed in the classical book of R. Godement {2}, has been generalized in sucessive stops. Ending this process, M. Artin introduced the notion of Grothendieck topology and developed the fundamental part of the theory in a functorial way (cf. {1}). Although, the concept of Grothendieck topology seems to be insufficient to relate certain aspects of the theory of sheaves; for example, the notion of subspace (not necessarily open !) is omited and so, induced sheaves and relative cohomology must be ignored.

The purpose of this paper is to obtain the essential results about induced sheaves (the concept of topological category enable us to work in this direction; cf. §1). Topological methods play an important role in the problems in question, as Godement shows (cf . {2} , Ch. II, §2.9). Therefore, we are forced to introduce a various kind of axioms, valids -of course- in the classical situation of a topological space. We mention that the results of this paper are useful also in not conventional cases, namely, the "étale" Grothendieck topology for preschemes (cf. {1} , Ch. III).

Results and notations of Artin's seminar ({1} , Ch. I, II) are continuosly used, frequently without specific reference. This results are stated in {1}for sheaves of abelian groups, but all of them could be generalized taking an arbitrary category of values and inserting axioms where necessary.Here, we have followed the abstract formulation (the basic facts about limit of functors, existence of injectives, adjoint situations, derived functors of a composition, etc. are stated in the usual literature; for example, cf. {3}) . Of course, the reader could suppose that all sheaves in this paper are abelian sheaves.

1. TOPOLOGICAL CATEGORIES AND INDUCED SHEAVES.

This section is of introductory character. Its aim is to lay down the terminology used throughout this paper and to collect the basic facts. We begin with the following:

DEFINITION 1.1. *A topological category is a triple* (M,T,ϕ) *such that* M *is a category,* T *is a family of Grothendieck topologies* $(T_M)_{M\in \mathrm{Ob}M}$ ϕ *is a family of morphisms in* M $(\phi_M : M \to X)_{M\in \mathrm{Ob}M}$ *and the following axioms are satisfied:*

tc1) For all object M *in* $\mathcal{M}$ *Cat* $\mathcal{T}_M$ *is a full subcategory of* $\mathcal{M}$. X *is an object of Cat* $\mathcal{T}_X$ *and* ϕ_X *is the identity morphism* e_X *of* X.

tc2) The diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "\phi_U"'] & V \arrow[d, "\phi_V"] \\ & X \end{tikzcd}$$

is commutative, for all morphism $f\colon U \to V$ *in Cat* $\mathcal{T}_X$.

tc3) $\mathcal{M}$ *has fibered products of the form* ${}_{\phi_U}U \times_{\phi_M} M$ *(briefly noted* $U \times_X M$*), where* U *is an object of Cat* $\mathcal{T}_X$ *and* M *is an object of* $\mathcal{M}$, *such that*

$$U \in \mathrm{Ob}(Cat\ \mathcal{T}_X) \Longrightarrow U \times_X M \in \mathrm{Ob}(Cat\ \mathcal{T}_M)$$

$(f_i\colon U_i \to U)_{i\in I} \in Cov\ \mathcal{T}_X \Longrightarrow (f_i \times_X e_M\colon U_i \times_X M \to U \times_X M)_{i\in I} \in Cov\ \mathcal{T}_M$ *for all object* M *of* $\mathcal{M}$.

REMARKS 1.2. i) Axioms tc1 and tc2 tell us that M is an object of Cat $\mathcal{T}_M$, because $X \times_X M = M$ and X is an object of Cat $\mathcal{T}_X$.

ii) Recall that if X is an object in a category $\mathcal{M}$, then is called prefinal (resp. final) iff $\mathrm{Hom}_{\mathcal{M}}(M,X) \neq \emptyset$ (resp. $\mathrm{Hom}_{\mathcal{M}}(M,X)$ is a set of one element), for all $M \in \mathrm{Ob}\mathcal{M}$. If $< X >$ is the discret subcategory of $\mathcal{M}$ associated to X, one easily checks that the following statements are equivalent (cf. {1} Ch. I, §0):

a) $\mathcal{M}$ satisfies axiom L1 and $< X >$ is a final subcategory of $\mathcal{M}$.

b) X is a final object of $\mathcal{M}$.

c) $\mathcal{M}$ satisfies axiom L2 and X is a prefinal object of $\mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{M}}(X,X) = \{e_X\}$.

Clearly, if $(\mathcal{M},\mathcal{T},\phi)$ is a tc (topological category) then X is a prefinal object of $\mathcal{M}$.

Let $\mathcal{M}$ be a category with final object X and let ϕ be the family of morphisms canonically associated to X. If $\mathcal{T}$ is a family of topologies satisfying tc1 and tc3, respect to $\mathcal{M}$ and ϕ, then $(\mathcal{M},\mathcal{T},\phi)$ is a tc of the following type:

DEFINITION 1.3. *A topological category* $(\mathcal{M},\mathcal{T},\phi)$ *is called* tc^0 *iff it satisfies:*

tc2') For all morphism $f\colon M \to N$ *in* $\mathcal{M}$ *the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\phi_M"'] & N \arrow[d, "\phi_N"] \\ & X \end{tikzcd}$$

is commutative.

Given a tc $C = (M, T, \phi)$, we shall be using a naive nomenclature:

M is called the category of subspaces of C (consistently, an object M in M is called a subspace of C). The object X is referred to as the space of C and so, an object M of M is also called a subspace of X refering to ϕ_M as the inclusion morphism of M in X.

If M is a subspace of M, T_M is called the relative topology of M and Cat T_M is called the category of relative open objects of M. Abusing language, T_X is called the topology of C and Cat T_X is called the category of open subspaces of C.

A morphism f: M $\longrightarrow$ N in M is called a ϕ-morphism iff the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\phi_M"'] & & N \arrow[dl, "\phi_N"] \\ & X & \end{tikzcd}$$

is commutative. We define a category M_ϕ putting:

$$\mathrm{Ob}M_\phi = \mathrm{Ob}M$$

$$\mathrm{Hom}\ M_\phi : \phi\text{-morphisms of } M$$

M_ϕ is a subcategory of M and clearly is a full subcategory (equivalently, is equal to M) iff C is tc^0.

DEFINITION 1.4. *A morphism of topological categories* F: $C \longrightarrow C'$ *is a functor* F: $M \longrightarrow M'$ *such that:*

mtc1) For all object M *of* M $F/\mathrm{Cat}\ T_M$ *is a morphism of topologies, of* T_M *in* $T'_{F(M)}$.

mtc2) $F(\phi) = \phi'$ *(i.e.* $F(X) = X'$ *and* $F(\phi_M) = \phi'_{F(M)}$, *for all* $M \varepsilon \mathrm{Ob}M$*).*

mtc3) F *preserves fibered products of the form* $U \times_X M, U \varepsilon \mathrm{Ob}(\mathrm{Cat}\ T_X)$ *and* $M \varepsilon \mathrm{Ob}M$.

REMARKS 1.5. i) Now we can talk about the category of small topological categories.

ii) If F: $C \longrightarrow C'$ is a mtc then we have

$$U \varepsilon \mathrm{Ob}(\mathrm{Cat}\ T_X) \Longrightarrow F(U) \times_{X'} F(M) \varepsilon \mathrm{Ob}(\mathrm{Cat}\ T'_{F(M)})$$

$$(f_i : U_i \longrightarrow U)_{i \varepsilon I} \varepsilon \mathrm{Cov}\ T_X \Longrightarrow (F(f_i) \times_{X'} e_{F(M)} : F(U_i) \times_{X'} F(M) \longrightarrow$$

$$\rightarrow F(U) \times_{X'} F(M))_{i \varepsilon I} \varepsilon \mathrm{Cov}\ T'_{F(M)}$$

for all subspace M of C.

iii) A mtc $F: \mathcal{C} \longrightarrow \mathcal{C}'$ induces for each subspace M of $\mathcal{C}$ a mor-phism of topologies $F/\mathrm{Cat}\, \mathcal{T}_M: \mathcal{T}_M \longrightarrow \mathcal{T}'_{F(M)}$ and so, induces the u-sual functors (direct and inverse image) between the corresponding categories of presheaves or of sheaves.

Let M be a subspace of a tc $\mathcal{C}$. If $\mathcal{A}$ is an arbitrary category, the category of presheaves $\mathcal{P}(\mathcal{T}_M,\mathcal{A})$ is briefly denoted by $\mathcal{P}_M$, and a presheaf in $\mathcal{P}_M$ is called a presheaf over M. Similarly, if $\mathcal{A}$ is a category with products, $\mathcal{S}_M$ denotes the category of sheaves $\mathcal{S}(\mathcal{T}_M,\mathcal{A})$, and a sheaf in $\mathcal{S}_M$ is called a sheaf over M.

$\rho_M: \mathcal{T}_X \longrightarrow \mathcal{T}_M$ is the morphism of topologies defined by the assignment of objects $U \longrightarrow U \times_X M$.

A category $\mathcal{A}$ will be called:

o) A0 iff it is a complete category (respect to functorial direct limits) with products and zero object.

i) A1 iff it is A0 and abelian.

ii) A2 iff it is A1 and satisfies the Grothendieck axiom A.B.5.

iii)A3 iff it is A2 and has a generator.

Let $\mathcal{C}$ be a tc and let $\mathcal{A}$ be an A1 category (as category of values).

DEFINITION 1.6. *If* M *is a subspace of* $\mathcal{C}$ *and* F *is a sheaf over* X *, then we call* ρ_{M_s} (F) *the sheaf induced by* F *over* M*, and we denote it by* F/M.

DEFINITION 1.7. *If* M *is a subspace of* $\mathcal{C}$ *and* $\alpha: F \longrightarrow G$ *is a morphism of sheaves over* X*, then we call* $\rho_{M_s}(\alpha)$ *the morphism induced by* α *over* M*, and we denote it by* α/M.

REMARKS 1.8. i) Since $\rho_X = e_{\mathcal{T}_X}$ it is clear that $F/X = F$ and $\alpha/X = \alpha$

ii) Since ρ_{M_s} is a functor it is clear also that $e_F/M = e_{F/M}$ and $(\beta\alpha)/M = (\beta/M)(\alpha/M)$.

iii) Remark that expresions of the type $(F/M)/N$ have no sense here, because the "absolute" topology $\mathcal{T}_X$ plays a special role in our developments.

Now we need to prove some previous results. In the next lemma, and only in the next, $\mathcal{A}$ may be an arbitrary category.

LEMMA 1.9. *If* $f: \mathcal{K} \longrightarrow \mathcal{K}'$ *is a morphism of small categories, the following statements are true*:

i) If f *is a full and representative functor, then* $f^P: P' \longrightarrow P$ *is full.*

ii) If f *is representative and* $\alpha \in Hom\ P'$ *is such that* $f^P(\alpha)$ *is an isomorphism, then* α *is also an isomorphism.*

Proof i) We want to show that the function $Hom_{P'}(P_1, P_2) \longrightarrow$ $\longrightarrow Hom_P(f^P(P_1), f^P(P_2))$, $\alpha \longrightarrow f^P(\alpha)$, is surjective. Given $\beta \in Hom_P(f^P(P_1), f^P(P_2))$ we define a morphism $\alpha \in Hom_{P'}(P_1, P_2)$ in the following way: since f is representative, given an object V in K' there exists an object U in K such that f(U) = V; therefore, we take $\alpha(V) = \beta(U)$. The good definition of α is obtained by the following argument: if U' is an object of K such that f(U')=V, since f is a full functor there exists a morphism m: $U' \longrightarrow U$ such that $f(m) = e_V$. Now, since β is a morphism of presheaves, we have the commutative diagram

$$\begin{tikzcd} f^P(P_1)(U) \arrow[r, "\beta(U)"] \arrow[d, "f^P(P_1)(m)"'] & f^P(P_2)(U) \arrow[d, "f^P(P_2)(m)"] \\ f^P(P_1)(U') \arrow[r, "\beta(U')"'] & f^P(P_2)(U') \end{tikzcd}$$

i.e. we have the commutative diagram

$$\begin{tikzcd} P_1(V) \arrow[r, "\beta(U)"] \arrow[d, "e_{P_1(V)}"'] & P_2(V) \arrow[d, "e_{P_2(V)}"] \\ P_1(V) \arrow[r, "\beta(U')"'] & P_2(V) \end{tikzcd}$$

and so, $\beta(U) = \beta(U')$. It is trivial that $f^P(\alpha) = \beta$.

ii) Given $\alpha \in$ Hom P' and $V \in ObK'$, observe that

$V = f(U) \Longrightarrow \alpha(V) = f^P(\alpha)(U)$.

COROLLARY 1.10. *If* f: $T \longrightarrow T'$ *is a morphism of topologies, the following statements are true:*

i) If f *is a full and representative functor, then* $f^s: S' \longrightarrow S$ *is full.*

ii) If f *is representative and* $\alpha \in Hom\ S'$ *is such that* $f^s(\alpha)$ *is an isomorphism, then* α *is also an isomorphism.*

Proof : Apply the lemma, taking in mind that $if^s \simeq f^p i'$, where i (resp. i') is the inclusion functor of S (resp. S') in P (resp. P')

COROLLARY 1.11. *If* $f: T \longrightarrow T'$ *is a full and representative morphism of topologies, then* $f_s: S \longrightarrow S'$ *is a representative functor.*

Proof: Since f_s is left adjoint to f^s, there exists a canonical morphism of functors $\Lambda: f_s \circ f^s \longrightarrow e_{S'}$. Now, since f^s is full by 1.10,i, we have that $f^s(\Lambda_{F'}): f^s(f_s \circ f^s(F')) \longrightarrow f^s(F')$ is an isomorphism, for all sheaf F' in S'. Therefore, applying 1.10 , ii , $\Lambda_{F'}: f_s f^s(F') \longrightarrow F'$ is also an isomorphism; and so,given a sheaf F' in S' the sheaf $f^s(F')$ is a preimage by f_s of F'.

The situation above suggest us the following

DEFINITION 1.12. *A tc C is called* tc^1 *iff the morphism* ρ_M *is a full and representative functor, for all subspace* M *of* C.

REMARK 1.13. If C is tc^1 we can apply both corollaries to the morphism $\rho_M: T_X \longrightarrow T_M$. In particular, 1.11 tell us that the restriction functor $./M: S_X \longrightarrow S_M$ is representative, for all subspace M of a tc^1 C.

In order to obtain the classical theorems about " characteristic " sheaves (cf. {2} , Ch. II § 2.9) our first result is

LEMMA 1.14. *Let* M *be a subspace of a* tc^1 C *and let* F *be a sheaf over* X. *If we define the sheaf* F^M *by* $F^M = f^s f_s(F)$ *then* $F^M/M \simeq F/M$. *(* f *is* ρ_M *).*

Proof: Adjointness gives us a canonical morphism of functors $\Delta: e_S \to f^s f_s$; since f is a full functor (cf. 1.10,i) $f_s(\Delta_F): f_s(F) \longrightarrow f_s(F^M)$ is an isomorphism. Hence, $F/M \simeq F^M/M$.

The technique concerning to open subspaces will be obtained using the following type of categories

DEFINITION 1.15. *A tc C is called* tc^2 *iff for all open subspace* A *of* C *the following conditions are satisfied:*

i) T_A *is a subtopology of* T_X.

ii) If V *is open in* A, *then* $V \times_X A = V$ *(i.e.* $e_V \times_X \phi_A : V \times_X A \longrightarrow V \times_X X = V$ *is an isomorphism).*

LEMMA 1.16. *If* A *is an open subspace of a* tc^2 C *and* V *is open in* A, *then* $<(V,h_V)>$ *is initial in* I_V^f, *where* $f: T_X \longrightarrow T_A$ *is the morphism* ρ_A *and* $h_V: V \longrightarrow f(V)$ *is the inverse morphism of* $e_V \times_X \phi_A$

Proof: Let (U,n) be any object in I_V^f. If $p_U: f(U) \longrightarrow U$ denotes the first projection, we define a morphism $m: V \longrightarrow U$ by $m = p_U n$. We claim that $m: (V,h_V) \longrightarrow (U,n)$ is a morphism in I_V^f; to prove this we are reduced to check that the diagram

$$\begin{tikzcd} f(V) \arrow[rr, "f(m)"] & & f(U) \\ & V \arrow[ul, "h_V"] \arrow[ur, "n"'] & \end{tikzcd}$$

is commutative. In fact, if $g_V: f(V) \longrightarrow V$ is the morphism $e_V \times_X \phi_A$ we have the commutative diagram

$$\begin{tikzcd} f(V) \arrow[r, "ng_V"] \arrow[d, "g_V"'] & f(U) \arrow[d, "p_U"] \\ V \arrow[r, "m"'] & U \end{tikzcd}$$

Therefore, recalling that g_V is the first projection of $f(V)$, an uniqueness result on fibered products yields $f(m) = ng_V$. Hence, $f(m)h_V = n$.

The last thing to check is that $\mathrm{End}((V,h_V))$ is a set of one element. In fact, if $r: (V,h_V) \longrightarrow (V,h_V)$ is a morphism in I_V^f, then the diagram

$$\begin{tikzcd} f(V) \arrow[rr, "f(r)"] & & f(V) \\ & V \arrow[ul, "h_V"] \arrow[ur, "h_V"'] & \end{tikzcd}$$

is commutative, i.e. $f(r) = e_{f(V)}$. Therefore, since the diagram

$$\begin{tikzcd} f(V) \arrow[r, "f(r)"] \arrow[d, "g_V"'] & f(V) \arrow[d, "g_V"] \\ V \arrow[r, "r"'] & V \end{tikzcd}$$

is also commutative, results $r = e_V$.

COROLLARY 1.17. *If* A *is a complete category, Cat* T_X *has fibered products and* V *is open in* A, *then the following statements are true:*

i) If P *is a presheaf over* X, *then* $f_p(P)(V) \approx P(V)$.

ii) If in addition A *has products and* F *is a sheaf over* X, *then* $f_p i(F)$ *is a sheaf over* A*(*i: $S_X \longrightarrow P_X$ *is the inclusion functor).*

iii)If A *is abelian too and* F *is a sheaf over* X, *then* $F/A(V) \approx F(V)$.

Proof: i) Since Cat T_X has fibered products and f preserves fibered products (because is a morphism of type ρ_M), the category I_V^f satisfies axiom L1* (cf. {1} II, Th. 4.14). Therefore, applying the lemma, we see that (V,h_V) is an initial object in I_V^f, and so $f_p(P)(V) = \varinjlim P_V \approx P_V((V,h_V)) = P(V)$.

ii) Applying i, check the definition of sheaf.

iii) It is clear by ii that $F/A \approx f_p i(F)$. Hence, i yields the desired result.

2. COMPLEMENTED TOPOLOGICAL CATEGORIES AND CLOSED SUBSPACES.

At this point, we need the notion of closed object in a topological category. Since we have the concept of open object, thinking in the closed sets of a topological space it is enough to find a notion replacing the set-theoretic operation of complement. Thus, we give the following

DEFINITION 2.1. *A complemented topological category is a tc* C, *together with a functor* c: $M^* \longrightarrow M$ *such that, if* $\theta = cX$ *and* $U_X = \mathrm{Cat}T_X$, *the following axioms are satisfied:*

ctc1) c *is an involution functor (i.e.* $c^* \circ c = e_{M^*}$*).*

ctc2) $\theta \in \mathrm{Ob}U_X$ *and there exists* $(U_i \longrightarrow \theta)_{i\in I} \in \mathrm{Cov}\, T_X$ *such that* $I=\emptyset$

ctc3) $cM \times_X M = \theta$ *and* $\theta \times_X M = \theta$, *for all* $M \in \mathrm{Ob}M$.

REMARKS 2.2. i) $F_X = cU_X$ is called the category of closed sub - spaces of C, and c is called the complement operator of C.

ii) Axiom ctc3 says that the diagrams

$$\begin{tikzcd} \theta \arrow[r, "c(\phi_{cM})"] \arrow[d, "c(\phi_M)"'] & M \arrow[d, "\phi_M"] \\ cM \arrow[r, "\phi_{cM}"'] & X \end{tikzcd} \qquad \begin{tikzcd} \theta \arrow[r, "c(\phi_{cM})"] \arrow[d, "c(\phi_X)=e_\theta"'] & M \arrow[d, "\phi_M"] \\ \theta \arrow[r, "\phi_\theta"'] & X \end{tikzcd}$$

are fibered products. We recall that (in the following proofs) we only need the first condition of ctc3 for closed subspaces, and the second for open subspaces.

iii) ctc1 tell us:

a) the complement of a closed object is open.
(The dual proposition is trivially true). From the definition of θ and ctc2 we obtain:

b) θ is open and closed.

Therefore:

c) X is open and closed.

Using ctc2 and the second condition of ctc3, we see:

d) If M is a subspace of C, then $\theta \in \mathrm{Ob}U_M$ and there exists a covering $(V_i \longrightarrow \theta)_{i \in I}$ in T_M such that $I = \emptyset$.

Recalling that, in a category with zero object, the product of an empty family of objects is the zero object, we obtain:

e) If A is a category with products and zero object and P is a presheaf over a subspace M of C, then

$$P \text{ monopresheaf} \Longrightarrow P(\theta) = 0$$

In particular,

$$P \text{ sheaf} \Longrightarrow P(\theta) = 0$$

(In the sense of {1}, a monopresheaf is a presheaf satisfying (+)).

Now, we have the necessary technique in order to prove one of the crucial results of this paper.

THEOREM 2.3. *If* A *is an* A1 *category and* M *is a subspace of a* tc^1C, *then for any sheaf* F *over* X *we have:*

i) The sheaf F^M *defined by* $F^M = f^s f_s(F)$, *where* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M, *satisfies* $F^M/M \simeq F/M$.

If A *is an* A2 *category,* C *is, in addition,* tc^2 *and has a complement operator* c*, Cat* T_X *has fibered products, and* X *is final in Cat* T_X*, then for any sheaf* F *over* X *we have:*

ii) If M *is a closed subspace of* C*, then* $F^M/cM \simeq 0$*; if the sheaf* F_{cM} *is defined by the exactness of the sequence* $0 \longrightarrow F_{cM} \longrightarrow F \xrightarrow{\Delta_F} F^M$ *(i.e.* $F_{cM} = Ker\ \Delta_F$*), then* $F_{cM}/cM \simeq F/cM$ *and* $F_{cM}/M \simeq 0$.

Proof: i) it is 1.14, exactly.

ii) We begin with the first statement. Since cM is an open subspace of C, applying 1.17, iii it is enough to show that $F^M(V) = 0$, for any V open in cM. Recalling that f^p preserves sheaves, because f is a morphism of topologies, we see that $F^M \simeq f^p i_M f_s(F)$. Therefore, we have $F^M(V) \simeq f_s(F)(f(V))$; but $f(V) = \theta$, because V is an object of T_{cM} and ρ_M is a representative functor (see axiom ctc3), and $f_s(F)(\theta) = 0$, because $f_s(F)$ is a sheaf (cf. 2.2, iii, e). Hence, $F^M/cM \simeq 0$.

Now, we prove the second statement. Since Cat T_X has fibered products, X is a final object in Cat T_X, A is an A2 category, and $\rho_A\colon T_X \longrightarrow T_A$, where A is any subspace of C, preserves the "spaces" of the topologies and fibered products, then $\rho_{A_s}\colon S_X \longrightarrow S_A$ is an exact functor (cf. {1} II, th. 4.14). Hence, f_s and g_s are exact functors ($f = \rho_M$ and $g = \rho_{cM}$).

In S_X we have the exact sequence

$$0 \longrightarrow F_{cM} \longrightarrow F \xrightarrow{\Delta_F} F^M$$

Thus, the sequence

$$0 \longrightarrow g_s(F_{cM}) \longrightarrow g_s(F) \longrightarrow g_s(F^M)$$

is exact, or equivalently, is exact the sequence

$$0 \longrightarrow F_{cM}/cM \longrightarrow F/cM \longrightarrow F^M/cM \simeq 0$$

Hence, $F_{cM}/cM \simeq F/cM$.

In a similar way, we obtain the exact sequence

$$0 \longrightarrow F_{cM}/M \longrightarrow F/M \longrightarrow F^M/M$$

Since Δ_F/M is an isomorphism, the exactness of this sequence yields $F_{cM}/M \simeq 0$.

The main purpose of the latter part of this section is to prove that, under certain restrictive conditions, $\Delta_F\colon F \longrightarrow F^M$ is an epimorphism. Until this moment, (C,c) will denote a fixed ctc.

If A is a category with products and zero object, we give the following

DEFINITION 2.4. *If* M *is a subspace of* C *and* F *is a sheaf over* X, *we say that* F *is null outside* M *iff for all open subspace* U *of* C *we have:*

$$U \times_X M = \theta \Longrightarrow F(U) = 0$$

If M is any subspace of C, $S(M)$ will denote the full subcategory of S_X defined by the sheaves null outside M. If $f\colon T_X \longrightarrow T_M$ is a morphism of topologies and A is an $\Lambda 1$ category, $f_\circ\colon S(M) \longrightarrow S_M$ will denote the functor $f_s/S(M)$.

THEOREM 2.5. *(A of type* A1*). If* M *is a subspace of* C *and* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M *, then the functor* $f_\circ\colon S(M) \longrightarrow S_M$ *has a right adjoint* $f^\circ\colon S_M \longrightarrow S(M)$.

Proof: Since f^s is right adjoint to f_s, it is enough to show that the image of f^s is a subcategory of $S(M)$. (Then, f° is f^s with $S(M)$ as codomain).

Given a sheaf G over M, notice that $f^s(G) \simeq f^p i_M(G)$, where i_M is the inclusion functor of S_M in P_M , and so, we only need to show that

$$U \times_X M = \theta \Longrightarrow f^p i_M(G)(U) = 0$$

for any open subspace U of C. In fact, we have

$$f^p i_M(G)(U) = i_M(G)(f(U)) = i_M(G)(\theta) = G(\theta) = 0$$

(the last equality is true because G is a sheaf).

Now, we wish to obtain a theorem of equivalence between the categories $S(M)$ and S_M. A similar result of Artin concerning to closed subschemes (cf. {1} III, Th. 2.2), guide us in the generalization process.

LEMMA 2.6. *Let* $f\colon K \longrightarrow K'$ *be a functor and let* V *be an object of* K' *such that for any* $A \in \mathrm{Ob}K$ *and any* $\eta \in Hom_{K'}(V,f(A))$ *there exist*

$U \in ObK$ *and* $m \in Hom_K(U,A)$ *satisfying* $V \xrightarrow[h]{\sim} f(U)$ *and* $f(m)\circ h = n$.
Then, the full subcategory $I^f(V)$ *of* I^f_V *defined by the class* $\{(U,h);$ $U \in ObK.\ h \in Iso_{K'}(V,f(U))\}$ *is initial in* I^f_V.

Proof: Let (A,n) be any object in I^f_V ; applying the hypothesis on V to the morphism $n: V \longrightarrow f(A)$, we can find a morphism in K m : $U \longrightarrow A$ and an isomorphism in K' $V \xrightarrow{h} f(U)$ such that $f(m)h = n$. Therefore, the diagram

$$\begin{tikzcd} f(U) \arrow[rr, "f(m)"] & & f(A) \\ & V \arrow[ul, "h"] \arrow[ur, "n"'] & \end{tikzcd}$$

is commutative and so, $m: (U,h) \longrightarrow (A,n)$ is a morphism in I^f_V .

COROLLARY 2.7. *If* $f: K \longrightarrow K'$ *is a full and representative functor, then* $I^f(V)$ *in* I^f_V *, for all object* V *of* K'.

COROLLARY 2.8. *(A is a complete category). Let* $f: K \longrightarrow K'$ *be a morphism of small categories such that K has fibered products and f is a full and representative functor which preserves fibered products. Then, any presheaf* P *in* $P(K,A)$ *satisfies* $f_p(P)(V) \simeq \varinjlim p_V/I^f(V)^*$, $V \in ObK'$.

Proof: It is enough to notice that I^f_V satisfies the axiom $L1^*$, because K has fibered products and f preserves fibered products.

DEFINITION 2.9. (C,c) *is called*·

i) ctc^1 *iff C is* tc^1 *and for any closed subspace M the following conditions are satisfied:*

a) *Any covering in* T_M *is induced by* ρ_M *from a covering in* T_X .
b) *If U and U' are open subspaces of C such that* $U \times_X M \simeq U' \times_X M$ *and F is a sheaf null outside M, then* $F(U) \simeq F(U')$.

ii) ctc^2 *iff C is* tc^2 *and any closed subspace M satisfies:*

$$U \times_X M = \theta \Longrightarrow U \times_X cM = U$$

for all open subspace U of C.

iii) ctc^3 *iff (C,c) is* ctc^1 *and* ctc^2.

LEMMA 2.10. *(A of type A0). If (C,c) is* ctc^1*, Cat* T_X *has fibered* *products,* F *is a sheaf null outside a closed subspace* M *of* C*, and* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M*, then the following statements are true, for any open subspace* U *of* C*:*

i) $f_p i(F)(U \times_X M) \simeq F(U)$.

ii) $f_p i(F)$ *is a sheaf over* M.

iii) $F/M(U \times_X M) \simeq F(U)$.

Proof: i) Since Cat T_X has fibered products, applying 2.9 we see that $f_p i(F)(U \times_X M) \simeq \varinjlim i(F)_{U \times_X M}/I^f(U \times_X M)^*$. Now, since (C,c) satisfies 2.9, i, b it is obvious that the values of the functor $i(F)_{U \times_X M}$ are all isomorphic, because any one is isomorphic to F(U). Hence, $\varinjlim i(F)_{U \times_X M}/I^f(U \times_X M)^* \simeq F(U)$.

ii) Since (C,c) satisfies 2.9, i, a, applying i it follows easily that $f_p i(F)$ is a sheaf (one only needs to check the definition of sheaf).

iii) Because of ii we have $f_s(F) \simeq f_p i(F)$. Therefore, i yields the desired result.

Now, it is almost obvious how to prove:

THEOREM 2.11. *(A of type* A1*). If* (C,c) *is* ctc^1*, Cat* T_X *has fibered products,* M *is a closed subspace of* C*, and* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M*, then the functor* $f_\circ\colon S(M) \longrightarrow S_M$ *is an equivalence of categories, which inverse is* $f^\circ\colon S_M \longrightarrow S(M)$.

Proof: By adjointness (see 2.5), there are natural transformations $\Phi\colon e_{S(M)} \longrightarrow f^\circ f_\circ$ and $\Psi\colon f_\circ f^\circ \longrightarrow e_{S_M}$. It is a straightforward matter, which we leave to the reader, to check that Φ and Ψ are functorial isomorphisms.

COROLLARY 2.12. *If* F *and* F' *are sheaves null outside* M*, then*

$$F/M \simeq F'/M \Longrightarrow F \simeq F'$$

THEOREM 2.13. *(A of type* A1*). If* (C,c) *is* ctc^2*, Cat* T_X *has fibered products and* M *is a closed subspace of* C*, then*

$$F/cM \simeq 0 \Longrightarrow F \text{ is null outside } M$$

for any sheaf F *over* X.

Proof: Let U be an open subspace of C such that $U \times_X M = \emptyset$; taking in mind that (C,c) is ctc^2 and applying 1.17,iii we see that $F(U) \simeq$ $\simeq F(U \times_X cM) \simeq F/cM(U \times_X cM) = 0$.

COROLLARY 2.14. *If (C,c) is ctc^3 and* F *and* F' *are sheaves over* X, *then*

$$F/M \simeq F'/M \quad , \quad F/cM \simeq 0 \simeq F'/cM \Longrightarrow F \simeq F'$$

Now, we can obtain the desired result:

THEOREM 2.15. *(A of type A2). If (C,c) is ctc^3, Cat T_X has fiber-ed products,* X *is final in Cat T_X and* M *is a closed subspace of* C, *then the following statements are true, for any sheaf* F *over* X :

i) $\Delta_F : F \longrightarrow F^M$ *is an epimorphism.*

ii) F^M *is uniquely determined by* F.

Proof: i) Recall that f_s and g_s are exact functors (see the proof of 2.3). Let C be the sheaf over X defined by the exactness of the sequence

$$F \xrightarrow{\Delta_F} F^M \longrightarrow C \longrightarrow 0$$

Then, we have the exact sequence

$$g_s(F) \longrightarrow g_s(F^M) \longrightarrow g_s(C) \longrightarrow 0$$

or equivalently

$$F/M \longrightarrow \underset{\overset{\shortparallel}{0}}{F^M/cM} \longrightarrow C/cM \longrightarrow 0$$

and so, $C/cM = 0$.

In a similar way we obtain the exact sequence

$$F/M \longrightarrow F^M/M \longrightarrow C/M \longrightarrow 0$$

Therefore, since Δ_F/M is an isomorphism, we conclude that $C/M = 0$. Now, 2.14 yields that $C = 0$.

ii) If F' is a sheaf over X satisfying $F'/M \simeq F/M$ and $F'/cM \simeq 0$, then $F'/M \simeq F^M/M$ and $F'/cM \simeq 0 \simeq F^M/cM$. Hence, 2.14 yields that $F' \simeq F^M$.

We end this section with a well known result on "characteristic" sheaves.

THEOREM 2.16. *(A of type A2). If (C,c) is tc^1 and tc^2, $Cat T_X$ has fibered products, X is final in $Cat\ T_X$ and M is a closed subspace of C, then for any sheaf G over M there exists a sheaf F over X such that $F/M \simeq G$ and $F/cM \simeq 0$. If (C,c) is ctc^3, then F is uniquely determined by G.*

Proof: Since C is tc^1, $./M: S_X \longrightarrow S_M$ is a representative functor (see 1.13) and so, given a sheaf G over M we can find a sheaf H over X such that $H/M \simeq G$. Then, taking $F = H^M$, 2.3 enable us to conclude that $F/M \simeq G$ and $F/cM \simeq 0$.

3. RELATIVE COHOMOLOGIES.

This section is devoted to realize an analysis of the cohomological effects of induced sheaves. Of course, the well known results exposed in the book of Godement (cf. {2} Ch. II, §4.9, §4.10, Th. 5.11.1) are obtained here, employing functorial methods. The compact exposition of cohomological theory presented in the Artin's seminar ({1} Ch. II) is continuosly used. Sheaves and presheaves are considered in this order.

I) COHOMOLOGY OF SHEAVES.

Let A be an A3 category and let C be a tc such that $Cat\ T_X$ has fibered products and X is final in $Cat\ T_X$. (Notice that the hypothesis on $Cat\ T_X$ yield the exactness of the restriction functors).

We begin introducing the "true" cohomology.

DEFINITION 3.1. *If M is a subspace of C, for each integer $n \geq 0$ we define the functor $H^n_M: (Cat\ T_X)^* \times S_X \longrightarrow A$ by:*

$$H^n_M = H^n_{T_M}(.\ \times_X M\ ,\ ./M)$$

THEOREM 3.2. *The following statements are true:*

i) $U \times_X M \simeq U' \times_X M\ ,\quad F/M \simeq F'/M \Longrightarrow H^n_M(U,F) \simeq H^n_M(U',F')$.

ii) $H^n_X \simeq H^n_{T_X}$.

iii) $H^*_M(U,\ )$ *is an exact cohomological functor.*

If the functor $./M: S_X \longrightarrow S_M$ carries injective sheaves into flask sheaves, then *iv)* $H^n_M(U,\ ) \simeq R^n\, H^\circ_M(U,\ )$.

If C *has a complement operator* c *such that* (C,c) *is* ctc^1 *and* M *is a closed subspace of* C, *then for any sheaf* G *over* M *we have:*

v) $H^n(U\times_X M, G) \simeq H^n(U, \rho^s_M(G))$.

and for any sheaf F *over* X *we have:*

vi) $H^n_M(U,F) \simeq H^n(U,F^M)$.

If C *has a complement operator* c *such that* (C,c) *is* ctc^3 *and* M *is a closed subspace of* C, *then for any sheaf* F *over* X *we have:*

vii) If F *is null outside* M, *then* $H^n_M(U,F) \simeq H^n(U,F)$.

viii) There is a cohomological exact sequence of general term

$$H^n(U,F_{cM}) \longrightarrow H^n(U,F) \longrightarrow H^n(U,F^M)$$

Proof: i) and ii) are trivial.

iii) Notice that $H^*_{T_M}(U\times_X M,\ )$ is an exact cohomological functor and $./M\colon S_X \longrightarrow S_M$ is an exact functor.

iv) Since $R^n\, H^\circ_{T_M}(U\times_X M,\ ) \simeq H^n_{T_M}(U\times_X M,\ )$ and f^s, where $f\colon T_X \longrightarrow T_M$ is the morphism ρ_M, is an exact functor, which carries injectives into $H^\circ_{T_M}(U\times_X M,\ )$ - acyclics, the proposition follows easily:

$$R^n H^\circ_M(U,\ ) = R^n(H^\circ_{T_M}(U\times_X M,\ )\circ f_s) \simeq (R^n H^\circ_{T_M}(U\times_X M,\ ))\circ f_s \simeq H^n_{T_M}(U\times_X M,\ )\circ f_s =$$
$$= H^n_M(U,\ ) .$$

v) We claim that f^s is an exact functor; since the diagram

$$\begin{tikzcd} S_M \arrow[rr, "f^s"] \arrow[dr, "f^\circ"'] & & S_X \\ & S(M) \arrow[ur, "j_M"'] & \end{tikzcd}$$

where $j_M\colon S(M) \longrightarrow S_X$ is the inclusion functor, is commutative, it is enough to show that f° and j_M are exact functors. The exactness of f° is clear by reasons of equivalence (see 2.11), and the exactness of j_M follows from the fact that $S(M)$ is closed in S_X under taking kernels and cokernels, as it is easily deduced from the definitions.

The spectral theorem of Artin-Leray, applied to the morphism $f\colon T_X \longrightarrow T_M$, tell us that

$$H^p(U, R^q f^s(G)) \underset{p}{\Longrightarrow} H^n(U\times_X M, G)$$

for any sheaf G over M. Therefore, recalling that

$$q > 0 \Longrightarrow R^q f^s = 0$$

(because f^s is exact), we obtain

$$H^n(U,f^s(G)) \approx H^n(U \times_X M,G).$$

vi) Applying the above result, we have

$$H^n(U,F^M) = H^n(U,f^s f_s(F)) \approx H^n(U \times_X M,f_s(F)) = H^n_M(U,F).$$

vii) If (C,c) is ctc[3] , then for any sheaf F over X we have

$$F \text{ null outside } M \Longrightarrow F \approx F^M$$

In fact, from 2.13 follows that F^M is null outside M and so, since $F/M \approx F^M/M$, 2.12 yields that $F \approx F^M$.

Applying this result and i, we obtain

$$H^n(U,F) \approx H^n(U,F^M)$$

Hence, vi yields the desired result.

viii) By 2.3, ii and 2.15,i the sequence of sheaves over X

$$0 \longrightarrow F_{cM} \longrightarrow F \xrightarrow{\Delta_F} F^M \longrightarrow 0$$

is exact and so, iii yields the desired result.

REMARKS 3.3. i) Notice that the relative (read local) character of the cohomology just defined appears clearly in 3.2,i, 3.2,ii and 3.2,vii.

ii) Of course, the hypothesis on 3.2,iv can not be removed. Sufficient conditions in the classical case are well known (cf. {2} , II §3.3).

iii) Observe that the statement (notations as in 3.2, vi)

$$H^n_{cM}(U,F) \approx H^n(U,F_{cM})$$

is not true, in general. Then, if we introduce the notation:

$${}_{cM}H^n(U,F) = H^n(U,F_{cM})$$

under the hypothesis on 3.2,viii, we obtain an exact cohomological sequence of general term

$${}_{cM}H^n(U,F) \longrightarrow H^n(U,F) \longrightarrow H^n_M(U,F)$$

Now, we focus our attention in the cohomology with presheaves values.

DEFINITION 3.4. *If* M *is a subspace of* C, *for each integer* $n \geq 0$ *we define the functor* $H^n_M : S_X \longrightarrow P_M$ *by:*

$$H^n_M = H^n_{T_M}(./M)$$

THEOREM 3.5. *The following statements are true:*

i) $F/M \simeq F'/M \Longrightarrow H^n_M(F) \simeq H^n_M(F')$.

ii) $H^n_X \simeq H^n_{T_X}$

iii) H^*_M *is an exact cohomological functor.*

If the functor $./M : S_X \longrightarrow S_M$ *carries injective sheaves into flask sheaves, then*

iv) $H^n_M \simeq F^n H^\circ_M$

Without assumptions, we have for any sheaf F *over* X:

v) $H^n_M(F) \circ \rho_M \simeq H^n_M(\ ,F)$.

If C *has a complement operator* c *such that* (C,c) *is* ctc^1 *and* M *is a closed subspace of* C, *then for any sheaf* F *over* X *we have:*

vi) $H^n_M(F) \circ \rho_M \simeq H^n(F^M)$.

If (C,c) *is* ctc^3, *we also have:*

vii) If F *is null outside* M, *then* $H^n_M(F) \circ \rho_M \simeq H^n(F)$.

viii) There is an exact cohomological sequence of general term

$$H^n(F_{cM}) \longrightarrow H^n(F) \longrightarrow H^n(F^M)$$

Proof: i), ii), iii) and iv) can be obtained as in 3.2.

v) Knowing that $H^n_{T_M}(G) \simeq H^n_{T_M}(\ ,G)$, for any sheaf G over M, the proposition follows easily:

$$H^n_M(F)(U \times_X M) = H^n_{T_M}(F/M)(U \times_X M) \simeq H^n_{TM}(U \times_X M, F/M) = H^n_M(U,F).$$

vi) Applying v and 3.2,vi, we obtain:

$$H^n_M(F)(U \times_X M) \simeq H^n_M(U,F) \simeq H^n(U,F^M) \simeq H^n(F^M)(U).$$

vii) The statement in question can be obtained as vi, applying now v and 3.2,vii. Also, it can be proved in the following way: since (C,c) is ctc^3, for any sheaf F over X we have

$$F \text{ null outside } M \Longrightarrow F \simeq F^M$$

Hence, $H^n(F) \simeq H^n(F^M)$ and so, the proposition follows from vi.

viii) It can be obtained as 3.2,vii.

II) COHOMOLOGY OF PRESHEAVES.

Let A be an A1 category and let C be an arbitrary tc. First, we consider the cohomology of a covering. In order to conserve a spectral result and to obtain a new one, we adopte the following

DEFINITION 3.6. *If* M *is a subspace of* C *and* K_U *is the category of coverings of an open subspace* U *of* C*, for each integer* $n \geq 0$ *we define the functor* $H^n_M : K^*_U \times P_M \longrightarrow A$ *by*

$$H^n_M = H^n_{T_M}(.\ \times_X M,\ .)$$

THEOREM 3.7. *The following statements are true:*

i) $(U_i \to U)_{i\in I} \simeq (U'_j \to U)_{j\in J} \Longrightarrow H^n_M((U_i \to U)_{i\in I},\ ) \simeq H^n_M((U'_j \to U)_{j\in J},\ )$

ii) $H^n_X \simeq H^n_{T_X}$

iii) $H^*_M((U_i \to U)_{i\in I},\ )$ *is an exact cohomological functor.*

If A *is an A3 category, then we have:*

iv) $H^n_M((U_i \longrightarrow U)_{i\in I},\ ) \simeq R^n H^0_M((U_i \longrightarrow U)_{i\in I},\ )$.

v) $H^p_M((U_i \longrightarrow U)_{i\in I}, H^q_M(F)) \underset{p}{\Longrightarrow} H^n_M(U,F)$.

If A *is an A1 category,* C *has a complement operator* c *such that* (C,c) *ctc*^{1}*, Cat* T_X *has fibered products and* M *is a closed subspace of* C *, then for any sheaf* F *over* X *we have:*

vi) If F *is null outside* M*, then* $H^n_M((U_i \longrightarrow U)_{i\in I}, F/M) \simeq H^n_M((U_i \to U)_{i\in I}, F)$

If A *is an A3 category and* X *is final in Cat* T_X*, we also have:*

vii) If F *is null outside* M*, then*

$$H^p_M((U_i \longrightarrow U)_{i\in I}, \rho_{M_p}(H^q(F))) \underset{p}{\Longrightarrow} H^n(U,F)$$

Proof. i) and ii) are trivial.

iii) Notice that $H^*_{T_M}((U_i \times_X M \longrightarrow U \times_X M)_{i\in I},\ )$ is an exact cohomological functor.

iv) Since A is an A3 category, $H^n_{T_M}((U_i\times_X M \longrightarrow U \times_X M)_{i\in I},\ ) \simeq$
$\simeq R^n H^\circ_{T_M}((U_i \times_X M \longrightarrow U \times_X M)_{i\in I},\ )$.

v) Since A is an A3 category, the cohomologies of sheaves are de fined and we have

$$H^p_{T_M}((U_i\times_X M \longrightarrow U\times_X M)_{i\in I}\ ,\ \mathcal{H}^q_{T_M}(F/M)) \underset{p}{\Longrightarrow} H^n_{T_M}(U\times_X M, F/M)$$

vi) It follows easily from 2.10,iii, by a direct analysis of the complex which gives the cohomology.

vii) Observe that $S(M)$ has injectives, because it is equivalent to the category of sheaves over M (see 2.11) (since A is an A3 category, S_M has injectives). Also, observe that the functor j_M carries injectives into flasks, because $f^s = j_M \circ f^\circ$, where $f\colon T_X \longrightarrow T_M$ is the morphism ρ_M, and f^s has this property.

Now, consider the (two) functors given by the commutative diagram

$$\begin{tikzcd}
S(M) \arrow[r, "\bullet"] \arrow[d, "j_M"'] & P_M \arrow[r, "H^\circ_M((U_i\longrightarrow U)_{i\in I},\ )"] & A \\
S_X \arrow[r, "i"'] & P_X \arrow[u, "f_p"'] &
\end{tikzcd}$$

Let us evaluate its derived functors. By iv, we have

$$R^p H^\circ_M((U_i\longrightarrow U)_{i\in I}\ ,\ \ ) \simeq H^p_M((U_i\longrightarrow U)_{i\in I}\ ,\ \ )$$

Recalling that f_p is an exact functor, by the hypothesis on Cat T_X, and that j_M is an exact functor, which carries injectives into i-acy clics, we obtain

$$R^q(f_p i j_M) \simeq f_p \circ R^q(i j_M) \simeq f_p\circ (R^q i)\circ j_M = f'_p \circ \mathcal{H}^q \circ j_M$$

Both results elucidate the first member of the spectral convergence in question. Concerning to the second member, 2.10,i implies

$$H^\circ_M((U_i\longrightarrow U)_{i\in I}\ ,\ \ )\circ (f_p i j_M) \simeq H^\circ((U_i\longrightarrow U)_{i\in I}\ ,\ \ )\circ(i j_M)$$

Therefore, recalling that $H^\circ((U_i\longrightarrow U)_{i\in I}\ ,\ \ )\circ i \simeq \Gamma_U$, we obtain

$$H^\circ_M((U_i\longrightarrow U)_{i\in I}\ ,\ \ )\circ (f_p i j_M) \simeq \Gamma_U\circ j_M$$

and so, since j_M is an exact functor which carries injectives into Γ_U-acyclics, we have

$$R^n(H^\circ_M((U_i\longrightarrow U)_{i\in I}\ ,\ \ )\circ (f_p i j_M)) \simeq R^n(\Gamma_U j_M) \simeq (R^n\Gamma_M)\circ j_M =$$
$$= H^n(U,\ \ )\circ j_M$$

We introduce the limit cohomology of presheaves by a more general procedure than the one used by Artin in {1} . Of course, both definitions agree in the case that the category of values is A3.

Let A be an A1 category and let T be an arbitrary topology. If U is an object of Cat T and K is a subcategory of K_U, for each integer $n \geq 0$ we define the functor $H^n_T(K, \ ): P(T,A) \longrightarrow A$ by:

$$H^n_T(\ ,P) = \varinjlim H^n_T(\ ,P) \circ k^* \ , \quad P \in \mathrm{Ob}P \quad ,$$

where $k: K \longrightarrow K_U$ is the inclusion functor (notice that $H^n_T(\ ,P)$: $K^*_U \longrightarrow A$).

It is straightforward,to check the following propositions:

i) If A is an A2 category and K is filtrant, then $H^*_T(K, \ )$ is an exact cohomological functor.

ii) If A is an A3 category and K is filtrant, then $H^n_T(K, \ ) \simeq R^n H^\circ_T(K, \ )$. (Concerning to i, the usual statement about the exactness of the limit is required; and for ii, the proposition i tell us that it is enough to show that $H^n_T(K, \ )$ vanishes on injectives, if $n > 0$).

Notice that all the other results of {1}, concerning to limit cohomology, are preserved by our definition.

The Čech cohomology of presheaves is introduced following {1}.

Now, we focus our attention in the relative limit cohomology.

DEFINITION 3.8. *If* M *is a subspace of* C *and* K *is a subcategory of* K_U *for each integer* $n \geq 0$ *we define the functor* $H^n_M(K, \ ): P_M \longrightarrow A$ *by*

$$H^n_M(K, \ ) = H^n_{T_M}(K \times_X M, \ )$$

(Notice that $K \times_X M$ is a subcategory of $K_U \times_X M$, which is a subcategory of $K_{U \times_X M}$).

THEOREM 3.9. *The following statements are true:*

i) $K \times_X M \simeq K' \times_X M \implies H^n_M(K, \ ) \simeq H^n_M(K', \ )$.

ii) $H^n_X(K, \ ) \simeq H^n_{T_X}(K, \ )$.

If $k: K \longrightarrow K_U$ *is the inclusion functor, then for any presheaf* P *over* M *we have*

iii) $H^n_M(K,P) \simeq \varinjlim H^n_M(\ ,P) \circ k^*$

If K^* *is filtrant, then:*

iv) $H^\circ_M(K, \ ) \circ i_M \simeq \Gamma_{U \times_X M}$

If A *is an A2 category and* K^* *is filtrant, then:*

v) $H^*_M(K,\ )$ *is an exact cohomological functor.*

If A *is an A3 category and* K^* *is filtrant, then:*

vi) $H^n_M(K,\ ) \approx R^nH^\circ_M(K,\ )$.

vii) $H^p_M(K,H^q_M(F)) \underset{p}{\Longrightarrow} H^n_M(U,F)$

If A *is an A1 category,* K^* *is filtrant,* C *has a complement operator* c *such that* (C,c) *is ctc*[1]*, Cat* T_X *has fibered products and* M *is a closed subspace of* C*, then for any sheaf* F *over* X *we have:*

viii) If F *is null outside* M*, then* $H^n_M(K,F/M) \approx H^n(K,F)$.

If A *is an A3 category and* X *is final in Cat* T_X*, then we also have:*

ix) If F *is null outside* M*, then* $H^p_M(K,\rho_M H^q(F)) \underset{p}{\Longrightarrow} H^n(U,F)$.

Proof: i) and ii) are trivial.

iii) If $k_M: K\times_X M \longrightarrow K_{U\times_X M}$ is the inclusion functor, by definition we have for any presheaf P over M

$$H^n_{T_M}(K\times_X M,P) \approx \varinjlim H^n_{T_M}(\ ,P) \circ k^*_M$$

and it is clear that

$$\varinjlim H^n_{T_M}(\ ,P) \circ k^*_M \approx \varinjlim H^n_{T_M}(.\times_X M,P) \circ k^*$$

iv) Notice that

$$K^* \text{ filtrant} \Longrightarrow K\times_X M^* \text{ filtrant}$$

and so, we have

$$H^\circ_{T_M}(K\times_X M,\ ) \circ i_M \approx \Gamma_{U\times_X M}$$

v) Since $K\times_X M^*$ is filtrant, then $H^*_{T_M}(K\times_X M,\ )$ is an exact cohomological functor.

vi) Recalling that K^* is filtrant, we have

$$H^n_{T_M}(K\times_X M,\ ) \approx R^nH^\circ_{T_M}(K\times_X M,\ )$$

vii) By the same reasons, we have the spectral convergence

$$H^p_{T_M}(K\times_X M\ ,\ H^q_{T_M}(F/M)) \underset{p}{\Longrightarrow} H^n_{T_M}(U\times_X M,F/M).$$

viii) Applie 3.7, vi and pass to the limit over K^*_U, using the proposition iii.

ix) It can be obtained as 3.7,vii.

We end this section introducing the relative Čech cohomology of pre sheaves. The definition is not the expected one, because the natural definition do not preserves the relative character (see 3.11,i). However, in the special case 3.11,xi both procedures agree.

DEFINITION 3.10. *If* M *is a subspace of* C *and* U *is open, for each integer* $n \geq 0$ *we define the functor* $\check{H}^n_M(U,\ ): P_M \longrightarrow A$ *by:*

$$\check{H}^n_M(U,\ ) = \check{H}^n_{T_M}(U\times_X M,\ )$$

THEOREM 3.11. *The following statements are true:*

i) $U\times_X M \simeq U'\times_X M \Longrightarrow \check{H}^n_M(U,\ ) \simeq \check{H}^n_M(U',\ )$

ii) $\check{H}^n_X(U,\ ) \simeq \check{H}^n_{T_X}(U,\ )$

iii) $\check{H}^n_M(U,P) \simeq \varinjlim H^n_M(\ ,P)$

iv) $\check{H}^\circ_M(U,\ ) \circ i_M \simeq \Gamma_{U\times_X M}$

If A *is an A2 category, then:*

v) $\check{H}^*_M(U,\ )$ *is an exact cohomological functor.*

If A *is an A3 category, then:*

vi) $\check{H}^n_M(U,\ ) \simeq R^n\check{H}^\circ_M(U,\ )$

vii) $\check{H}^p_M(U,\mathcal{H}^q_M(F)) \underset{p}{\Longrightarrow} H^n_M(U,F)$

viii) $\check{H}^1_M(U,F/M) \simeq H^1_M(U,F)$, $\check{H}^2_M(U,F/M) \underset{\sim}{\subset} H^2_M(U,F)$

If A *is an A1 category,* C *has a complement operator* c *such that* (C,c) *is* ctc^1, *Cat* T_X *has fibered products and* M *is a closed subspace of* C, *then for any sheaf* F *over* X *we have:*

ix) *If* F *is null outside* M, *then* $\check{H}^n_M(U,F/M) \simeq \check{H}^n(U,F)$

If A *is an A2 category and* X *is final in Cat* T_X, *then we also have:*

x) *If* F *is null outside* M, *then* $\check{H}^p_M(U,\rho_{M_p}(\mathcal{H}^q(F))) \underset{p}{\Longrightarrow} H^n(U,F)$.

If A *is an A1 category and* M *is a subspace of* C *such that any covering of* $U\times_X M$ *is induced by* ρ_M *from a covering of* U, *then we have:*

xi) $\check{H}^n_M(U,\ ) \simeq H^n_M(K_U,\ )$.

Proof: i), ii), iii), iv), v), vi) and vii) can be obtained as the homologous propositions of 3.9.

viii) It follows immediately from

$$\check{H}^1_{T_M}(U\times_X M,F/M) \simeq H^1_{T_M}(U\times_X M,F/M)\ ,\quad \check{H}^2_{T_M}(U\times_X M,F/M) \underset{\sim}{\subset} H^2_{T_M}(U\times_X M,F/M).$$

It should be pointed out that viii could be obtained from vii just as in the absolute cohomology case.

ix) and x) can be obtained as 3.9, viii and 3.9,ix, resp.

xi) The hypothesis on M tell us that $K_U \times_X M \simeq K_{U \times_X M}$. Hence, we have

$$H^n_{T_M}(K_{U\times_X M}, \ ) \simeq H^n_{T_M}(K_U \times_X M, \ )$$

or equivalently

$$\check{H}^n_M(U, \ ) \simeq H^n_M(K_U, \ )$$

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