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NOTE ON GALOIS EXTENSION OVER THE CENTER

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In {2} S.U.Chase, D.K.Harrison and A.Rosenberg obtained a Galois Theory for strongly Galois extensions of commutative rings (CHR-Galois). This was generalized to non commutative rings by F.R. Demeyer {3} , T.Kanzaki {6} , H.F.Kreimer {8} and others. Recently, O.E.Villamayor and D.Zelinsky obtained in {11} a weak Galois theory of commutative rings in order to study the strong one from a different point of view.

In the first section of this short paper we shall use similar arguments to those of {11} to show that if an algebra Λ over a commutative ring R is a strongly Galois extension of R, then Λ and its center C are weakly Galois extensions over C and R, respectively . If Λ is a weakly Galois extension over C, Λ is the sum of all C-modules J_σ (see below or {10} for the definition of J_σ). By means of some properties of the J_σ's, we shall study in section 2, a Galois theory over the center, the argument being similar to that of {7} and {5}, Theorem 1.

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1. GALOIS EXTENSION OVER R.

Let R be a commutative ring with identity, Λ an algebra over R, C the center of Λ and G a finite group of automorphisms of Λ. We say that Λ is a Galois extension with respect to G of its G-fixed ring Λ^G if there exist elements x_i , y_i i=1,2,...,n, in Λ such that $\sum_{\sigma\varepsilon G}\sigma(x_i)y_i = \delta_{\sigma,1}$. We note that if $\Lambda^G = R$, then Λ is a finitely generated and separable R-algebra by {9} , Lemma 2.

Let Γ be an R-subalgebra of Λ and G the group of all automorphisms of Λ leaving invariant the elements of Γ.We quote here the definition of weakly Galois extension of {11}: Λ is said to be a (right) weakly Galois extension of Γ if the following two conditions are satisfied:

a) Λ is a finitely generated projective right Γ-module.

b) $G\Lambda_\ell = \mathrm{Hom}_\Gamma(\Lambda,\Lambda)$, where $\sum g_i x_{i\ell}(\lambda) = \sum g_i(x_i\lambda)$ for $g_i\varepsilon G, x_i, \lambda\varepsilon\Lambda$.

LEMMA 1. *Let* Λ *be a strongly Galois extension of* R *with group* G. *If* R *has no proper idempotents, then* Λ *and its center* C *are weakly*

Galois extensions of C *and* R, *respectively.*

Proof: Since Λ is a finitely generated R-module, there exist mutually orthogonal primitive central idempotents e_i such that $\sum_{i=1}^{n} e_i = 1$. Then $\mathrm{Hom}_C(\Lambda,\Lambda) = \sum_i \oplus \mathrm{Hom}_{Ce_i}(\Lambda e_i, \Lambda e_i)$ and $\mathrm{Hom}_R(C,C) = \sum_i \oplus \mathrm{Hom}_{Re_i}(Ce_i, Ce_i)$. Let $T_i = \{g|\varepsilon G, g(e_i)=e_i\}$. We can easily see that Λe_i is a strongly Galois extension of Re_i with group T_i (see {3}). Furthermore, Λe_i and Ce_i are strongly Galois extensions over Ce_i and Re_i with respect to H_i and T_i/H_i by {3} and {9}, where $H_i = \{g|\varepsilon T_i, g(d)=c \ \forall\ c\varepsilon Ce_i\}$. Hence $\mathrm{Hom}_{Ce_i}(\Lambda e_i, \Lambda e_i) = H_i(\Lambda e_i)_\ell$ and $\mathrm{Hom}_{Re_i}(Ce_i, Ce_i) = (T_i/H_i)(Ce_i)_\ell$. Now, we put $H_i^* = \{h|\varepsilon G, h|\Lambda e_i = h' \text{ for some } h'\varepsilon H_i, h|\Lambda e_j = I_{\Lambda e_j} \text{ for } i\neq j\}$. Then $H_i^* \subseteq H = \{h|\varepsilon G, h|C = I_C\}$. Therefore, $\mathrm{Hom}_C(\Lambda,\Lambda) = H\Lambda_\ell$. Similarly we obtain $\mathrm{Hom}_R(C,C) = G'C_\ell$, where G' is the group of all automorphisms of C over R. Condition a) follows from {1}, since Λ is separable over R.

THEOREM 1. *Let* Λ *be a strongly Galois extension of* R *with finite group* G. *Then* Λ *and its center* C *are weakly Galois extensions of* C *and* R, *respectively.*

Proof: We shall use the same notation and argument of {11}. Since Λ is $\Lambda \otimes \Lambda^*$-projective, $C_x = (\mathrm{Hom}_{\Lambda^\ell}(\Lambda,\Lambda))_x = \mathrm{Hom}_{\Lambda_x^\ell}(\Lambda_\alpha,\Lambda_\alpha)$ (cf. {11}, (2.7)). Furthermore, R_x has no proper idempotents by {11}, (2.13). Hence, $H(x)(\Lambda_x)_\ell = \mathrm{Hom}_{C_x}(\Lambda_x,\Lambda_x)$ and $G'(x)(C_x)_\ell = \mathrm{Hom}_{Rx}(C_x,C_x)$, where $H(x)$, $G'(x)$ are as above in Λ_x and C_x. Since C is R-finitely generated, all elements of $H(x)$ and $G'(x)$ are induced by elements of H and G', respectively (by {11}, (2.14)). Hence $(H\Lambda\ell)_x = H(x)(\Lambda_x)_\ell = \mathrm{Hom}_{Cx}(\Lambda_x,\Lambda_x) = (\mathrm{Hom}_C(\Lambda,\Lambda))_x$ since Λ is C-finitely generated and projective. Therefore, $H\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$. Similarly $G'C_\ell = \mathrm{Hom}_R(C,C)$.

We shall give latter an example in which a strongly Galois extension of R is not a strongly Galois extension over its center with respect to the corresponding subgroup.

2. GALOIS EXTENSION OVER CENTER.

In this section we always assume that Λ is a separable algebra over its center C. If $H\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$, then for any element x in Λ^H, x_ℓ belongs to the center of $\mathrm{Hom}_C(\Lambda,\Lambda) = C$; hence $\Lambda^H = C$. We shall study some properties are treated in {7} and {10}. For $\sigma \varepsilon H$ let $J_\sigma = \{x|\varepsilon\Lambda, yx = x\sigma(y) \text{ for all } y\varepsilon\Lambda\} = \mathrm{Hom}_{\Lambda^\ell}(\Lambda,\Lambda\sigma)$, where $\Lambda\sigma$ is the same module as Λ as left Λ-module and the operation of

Λ as right Λ-module is defined by $x*y = x\sigma(y)$. Furthermore, $\Lambda_\sigma = \Lambda \otimes_C J_\sigma$ and $\Lambda = \Lambda J_\sigma$ (see {10}). For any element $x_\ell\sigma \in J_{\sigma\ell}\sigma \subseteq \Lambda_\ell H(x_\ell\sigma)(y) = x_{\sigma(y)} = yx_r$ for every $y \in \Lambda$. Hence, $(J_\sigma)_\ell^\sigma = (J_\sigma)_r$ and $\Lambda_\ell\sigma = \Lambda_\ell J_{\sigma\ell}\sigma = \Lambda_\ell(J_\sigma)_r = \Lambda_\ell \otimes_C (J_\sigma)_r$ since $\Lambda_\ell\Lambda_r = \Lambda_\ell \otimes_C \Lambda_r$ by {1}.

PROPOSITION 2. *Let* Λ *be separable over its center* C *and* S *a subset of* H. *Then* $S\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$ *if and only if* $\Lambda = \sum_{\sigma\in S} J_\sigma$.

Proof: $S\Lambda_\ell = \sum_{\sigma\in S}\Lambda_\ell\sigma = \sum\Lambda_\ell \otimes (J_\sigma)_r$. Since C is a C-direct summand of Λ and $\mathrm{Hom}_C(\Lambda,\Lambda) = \Lambda_\ell \otimes_C \Lambda_r$, the proposition follows.

COROLLARY. Λ *is a weakly Galois extension if and only if* $\Lambda = \sum_{\sigma\in S} J_\sigma$, *where* S *is a finite subset of* H. *Furthermore,* Λ *is generated by units as* C*-module if and only if* $S\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$ *and the elements of* S *are inner-automorphisms.*

Proof: It is clear.

Let S be a subset of H. We call S strongly distinct if there exists a family of elements $\{x_i^{(\sigma)}, y_i^{(\sigma)}\}_{i=1}^{n=n(\sigma)}$, $\sigma \in S$ such that

$$\sum\tau(x_i^{(\sigma)})y_i^{(\sigma)} = \delta_{\tau,\sigma} \quad \text{for all } \sigma,\tau \in S.$$

It is clear that this condition is equivalent with the existence of Galois generators if S is a group.

THER@REM 3. *Let* S $, J_\sigma$ *be as above and* $\Gamma = \sum_{\sigma\in S} J_\sigma$. *If* S *is strongly distinct, then* $\Gamma = \sum_{\sigma\in S}\oplus J_\sigma$. *Conversely, if* $\Gamma = \sum_{\sigma\in S}\oplus J_\sigma$ *and* Γ *is a direct summand of* Λ *as* C*-module, then* S *is strongly distinct.*

Proof: Assume that $\Gamma = \sum_\sigma\oplus J_\sigma$ and Γ is a direct summand of Λ as C-module. $\Lambda_\ell \otimes \Lambda_r = \mathrm{Hom}_C(\Lambda,\Lambda)$, since Λ is C-separable. Let p_σ be a projection of Λ onto J_σ. Then $p_\sigma \in \mathrm{Hom}_C(\Lambda,\Lambda)$. Hence, there exist elements $\{x_i^{(\sigma)},y_i^{(\sigma)}\}_{i=1}^{n=n(\sigma)}$ in Λ such that $\sum x_{i\ell}^{(\sigma)} \otimes y_{ir}^{(\sigma)} = p_\alpha$. Therefore, $0 = p_\sigma(J_\tau) = J_\tau\sum\tau(x_i^{(\sigma)})y_i^{(\sigma)}$ for $\sigma\neq\tau$. Since $\Lambda J_\tau = \Lambda$, $\sum\tau(x_i^{(\sigma)})y_i^{(\sigma)} = 0$. Similarly, $J_\sigma(I - \sum\sigma(x_i^{(\sigma)})y_i^{(\sigma)}) = 0$. Hence, $I = \sum\sigma(x_i^{(\sigma)})y_i^{(\sigma)}$. Conversely, assume S is strongly distinct. We assume $0 = \sum_{\sigma\in S} z_\sigma$, $z_\sigma \in J_\sigma$. Then $0 = \sum_i\sum_\tau x_i^{(\sigma)} z_\tau y_i^{(\sigma)} = \sum_\sigma z_\sigma(\sum_i \tau(x_i^{(\sigma)})y_i^{(\sigma)}) = z_\sigma$ .Hence, $\Gamma = \sum\oplus J_\sigma$.

LEMMA 2. *Let* $A \supset B$ *be* R*-algebras. If* B *is* R*-separable, then* $V_A(B)$

is a direct summand of A *as two-sided* $V_A(B)$*-module, where* $V_A(B) = \{a | \varepsilon A\ ,\ ba = ab$ *for all* $b \in B\}$.

Proof: We consider A as a left $B \otimes_R B^*$-module. Since B is R-separable, there exist elements $\{x_i, y_i\}_i$ in Λ such that $\sum x_i y_i = 1$ and $\sum bx_i \otimes y_i = \sum x_i \otimes y_i b$ for all $b \in B$. Now we define a map $\phi : A \longrightarrow A$ by setting $\phi(a) = \sum (x_i \otimes y_i^*) a = \sum x_i a y_i$ for $a \in A$. From the above relations of $\{x_i, y_i\}$ we obtain $\phi(A) \subseteq V_A(B)$ and $\phi | V_A(B) = I_{V_A(B)}$. Furthermore, $\phi(aa') = \sum_i x_i aa' y_i = (\sum x_i a y_i) a' = \phi(a) a'$ for $a' \varepsilon V_A(B)$. Similarly $\phi(a'a) = a'\phi(a)$. Hence, ϕ is $V_A(B)$ - $V_A(B)$ homomorphism. Since $V_A(B)$ is $V_A(B)$ - projective, $A = V_A(B) \oplus \ker \phi$.

PROPOSITION 4. *Let* Λ *be a central separable* C*-algebra and* Γ *a separable subalgebra* $(\Gamma \supseteq C)$. *Then* Γ *is a direct summand of* Λ *as a two-sided* Γ*-module.*

Proof: We know from {6}, Theorem 2 that $\Gamma = V_\Lambda(V_\Lambda(\Gamma))$ and $V_\Lambda(\Gamma)$ is C-separable. Hence the proposition follows from Lemma 2.

From now we assume that the subset S of H is a finite group G.

PROPOSITION 5. *Let* Λ *be a central separable* C*-algebra and* G *a finite subgroup of the group of* C*-automorphism of* Λ*, let* $\Gamma = \sum_{\sigma \varepsilon G} J_\sigma$. *Then the following statements are equivalent:*

1) $\Gamma = \sum_{\sigma \varepsilon G} \oplus J_\sigma$ *and* $|G|$ *is a unit in* C.

2) $\Gamma = \sum_{\sigma \varepsilon G} \oplus J_\sigma$ *and* Γ *is* C*-separable.*

3) Λ *is a strongly Galois extension of* Λ^G *and* Λ^G *is* C*-separable.*

Proof: 1) $\longleftrightarrow$ 2) It is clear from ({4} Lemma 4) by localization of C.

2) $\longrightarrow$ 3) Since Γ is C-separable, Γ is a direct summand of Λ as Γ-module by Proposition 4. Hence, G is strongly distinct by Theorem 3.

3) $\longrightarrow$ 1) $|G|$ is unit in C by {5} , Proposition 5,and the rest is clear.

LEMMA 3. $J_\sigma J_\tau = J_{\tau\sigma}$ *for any* σ, τ .

Proof: Let m be a maximal ideal in C. Then $(J_\sigma)_m = \mathrm{Hom}_{\Lambda_m} \ell(\Lambda_m, \Lambda_m \sigma) = C_m u_\sigma$, where $\sigma(y) = u_\sigma^{-1} y u_\sigma$. Hence, $(J_\sigma J_\tau)_m = C_m u_\sigma u_\tau$ and $u_{\tau\sigma}(u_\sigma u_\tau)^{-1}$,

$u_\sigma u_\tau u_{\tau\sigma}^{-1}$ belong to C_m. Therefore, $J_\sigma J_\tau = J_{\tau\sigma}$.

PROPOSITION 6. *Let* G *be a finite subgroup of* H *and assume* Λ *is a strongly Galois extension of* Λ^G; *let* Ω *be a separable* C-*algebra between* Λ *and* Λ^G. *Then the following statements are equivalent:*

1) $\Omega = \Lambda^H$ *for some subgroup* H *of* G.

2) $V_\Lambda(\Omega) = \sum_{\sigma\varepsilon S} J_\sigma$ *for some subset* S *of* G.

3) There exist elements $\{x_i \varepsilon \Omega, y_i \varepsilon \Lambda\}$ *such that* $\sum x_i y_i = I$ *and* $\sum \rho(x_i)y_i = 0$ *for* $\rho|\Omega \neq I_\Omega$, $\rho \varepsilon G$. *({8}, Proposition 3.5).*

Proof: Since $V_\Lambda(\Omega) \subseteq \sum_{\sigma\varepsilon G} \oplus J_\sigma$, S is a subgroup from Lemma 3. Furthermore, $\mathrm{Hom}_{\Omega_r}(\Lambda,\Lambda) = \Lambda_\ell \otimes_C V_\Lambda(\Omega)_r$ by {6} , Theorem 2, since Ω is C-separable and therefore 1) and 2) are equivalent.

1) $\longrightarrow$ 3) Let $G = \bigcup_i \rho_i H$. Then $\Gamma = \sum_{\sigma\varepsilon G} \oplus J_\sigma = \sum_i \Gamma_H J_{\rho_i}$ and Γ is a direct summand of Λ as Γ-module, where $\Gamma_H = \sum_{\sigma\varepsilon H} \oplus J_\sigma$. Let p be a projection of Λ onto $\Gamma_H J_i = J_H$. Then $p \varepsilon \mathrm{Hom}_{\Gamma H_\ell}(\Lambda,\Lambda) = (\Lambda^H)_\ell \otimes \Lambda_r$. Hence, there exist $\{x_i \varepsilon \Lambda^H , y_i \varepsilon \Lambda\}$ such that $\sum x_{i\ell} \otimes y_{ir} = p$.

3) $\longrightarrow$ 2) Put $H = \{\sigma | \varepsilon G, \sigma|\Omega = I_\Omega\}$. Then $V_\Lambda(\Omega) \supseteq \sum_{\sigma\varepsilon H} J_\sigma$. Let $y \varepsilon V_\Lambda(\Omega)$ and $y = x_{\rho_1} + x_{\rho_2} + \ldots$, where $x_{\rho_i} \varepsilon \Gamma_H J_{\rho_i}$. Then $y = yI = y\sum x_i y_i = \sum x_i y y_i = \sum_j \sum_i x_i x_{\rho_j} y_i = \sum x_{\rho_j} \sum \rho_j(x_i) y_i = x_{\rho_1}$. Hence $V_\Lambda(\Omega) = \sum_{\sigma\varepsilon H} J_\sigma$.

Finally, we shall give an example of a strongly Galois extension Λ of R, such that Λ is not a strongly Galois extension over its center with respect to its subgroup. However Λ is a strongly Galois extension over its center with respect to a suitable group.

Let G_2 be a cyclic group of order 2 and Q the field of rational numbers. Put $G = G_2 \times G_2$ and $K = Q(\sqrt{2})$. Then $L = K \otimes_Q K$ is a strongly Galois extension of Q with respect to G by {9} ,Proposition 1 . Let g and h be the inner-automorphisms of Q_2 induced by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

respectively. Then $(g) \times (h) \simeq G$ and Q_2 is a strongly Galois extension of Q with respect to G, since $\frac{1}{2}\{e_{11}, e_{11}, e_{22}, e_{22}, e_{21}, e_{12}, e_{12}, e_{21}\}$ is a family of Galois generators. Put $\Lambda = Q_2 \oplus L$. Then Λ is a strongly Galois extension of Q with group G if we define $g(a+b) = g(a)+g(b)$

for $g \in G$, $a \in Q$ and $b \in L$. It is clear that the fixing group of its center is equal to G. But if we define $g(a + b) = g(a) + b$, then Λ is a strongly Galois extension of its center with respect to G.

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