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A MEAN VALUE THEOREM AND DARBOUX'S PROPERTY FOR THE DERIVATIVE OF AN ADDITIVE SET FUNCTION WITH RESPECT TO A MEASURE ON E^n

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I) INTRODUCTION: During a recent investigation of existence and equality almos everywhere of the cross partial derivatives f_{xy} and f_{yx}, a somewhat different derivative Df for a function f(x,y) was used {4} . This derivative Df is also defined for a function f of n-variables. The purpose of this paper is to establish a mean value theorem and a Darboux property for the function Df, and to generalize these results to the derivative T'* as defined on pp. 268-271 of {8}. A similar result was obtained by L. Misik in {7}, but the technique is much more cumbersome. The method of proof is the same as that given in {5}. In the case n=2 the technique of proof is used to establish a theorem concerning the equality of the three derivatives. For simplicity the proofs and definitions will be given for n=2.

II) THE DERIVATIVE Df: Let $R = [a,b;c,d] = \{(x,y) \mid x\varepsilon[a,b], y\varepsilon[c,d]\}$. If f(x,y) is a function whose domain contains R, then the f-area of R is denoted by F(R) = f(b,d) - f(a,d) - f(b,c) + f(a,c). The ordinary area of R will be denoted by A(R). A rectangle R = [a,b;c,d] is said to be of order M, if $M \geq 1$ and $1/M \leq (d-c)/(b-a) \leq M$. One then defines the upper and lower derivatives of order M at a point (x,y) to be $\overline{\lim}$ and $\underline{\lim}$ respectively of ratios of f-areas to ordinary areas of rectangles or order M which contain (x,y) and whose areas converge to zero. Then f is said to have a derivative of order M, $D_M f(x,y) = D_M f(P)$, at P = (x,y) if the upper and lower derivatives of order M are equal. The function f is said to be two non-decreasing if the f-area of each sub-rectangle of R is non-negative. It follows {2}, that if f is of bounded variation in the sense of Hardy, then except for a set of measure zero, $D_N f = D_M f$ for each $N \geq M \geq 1$. The common value is denoted by Df. It also follows {4}, that f_{xy} and f_{yx} each exist except possibly on a set of measure zero.

III) THE MEAN VALUE THEOREM:

THEOREM 1: *If $D_M f$ exists at each point P of a closed rectangle R of order M and f is continuous at each point of R, then there exists a point Q ε R such that* $D_M f(Q) = F(R)/A(R)$.

Proof: Suppose $R_1 = R = [a,b;c,d]$ and divide R_1 into four rectangles using the lines $x = a+h/2$, $y = b+k/2$ where $h = b-a$ and $k = d-c$. Denote the rectangles, beginning in the lower left hand corner and proceeding counterclockwise, by R_{11}, R_{12}, R_{13}, R_{14}, and observe that each of the four rectangles is similar to R_1. It follows that

$$\sum_{i=1}^{4} F(R_{1i}) = F(R_1) \text{ and } \sum_{i=1}^{4} A(R_{1i}) = A(R_1)$$

and hence there must exist a j and a k such that

$$F(R_{1j}) \geq (1/4)F(R_1) \text{ and } F(R_{1k}) \leq (1/4)F(R_1).$$

We now proceed to find a rectangle R_2 of order M with sides parallel to the sides of R_1 such that $R_2 \subseteq R_1$, $F(R_2) = (1/4)F(R_1)$, and $A(R_2) = (1/4)A(R_1)$. If $F(R_{1i}) = (1/4)F(R_1)$ for some i, then choose $R_2 = R_{1i}$. Suppose equality does not hold for any i and consider the case j=3 and k=1. The other cases would follow in a similar manner. Let $\alpha = h/k$ and define the auxiliary function

$$g(t)=f(a+t+h/2,b+\alpha t+k/2)-f(a+t,b+\alpha t+k/2)-f(a+t+h/2,b+\alpha t)+f(a+t,b+\alpha t).$$

Then $g(0) = F(R_{11})$, $g(h/2) = F(R_{13})$, and g is a continuous function of t for $0 \leq t \leq h/2$. The ordinary intermediate value theorem for a function of one variable guarantees the existence of a $t_0 \varepsilon (0,h/2)$ such that $g(t_0) = (1/4)F(R_1)$. This value t_0 defines R_2 and we note that $F(R_2)/A(R_2) = F(R_1)/A(R_1)$. In the sequel we shall refer to the above selection process for determining R_2 as the sliding technique. We proceed inductively to define a nested sequence of closed rectangles $\{R_i\}$, each of order M with sides parallel to R_1, such that

i) $F(R_{i+1}) = (1/4)F(R_i)$ and

ii) $A(R_{i+1}) = (1/4)A(R_i)$.

By the nested interval theorem there exists exactly one point $Q \varepsilon \bigcap R_i$, and

$$D_M f(Q) = \lim_{i\to\infty} F(R_i)/A(R_i) = F(R_1)/A(R_1).$$

We shall say that the set function F has property I provided the auxiliary function g(t) has the intermediate value property along the lines x = constant, y = constant, and $y = \pm\alpha x$. We have the somewhat stronger result.

THEOREM 2. *If $D_M f$ exists at each point P of a closed rectangle R of order M and the set function F has property I, then there exists a point $Q \varepsilon R$ such that $D_M f(Q) = F(R)/A(R)$.*

The following example shows that theorem 2 is a stronger result.

EXAMPLE 1: Let $f(x,y)$ be defined as follows on the unit square:

$f(x,y) = 1$ if y is rational and

$f(x,y) = 0$ if y is irrational

Then $D_M f$ exists and is zero at each point of the unit square and the set function F has property I.

We now proceed to remove the condition that $D_M f$ exist along the boundary of R.

THEOREM 3. *If* R *is a rectangle of order* M, $D_M f$ *exists at each point* $P \in int(R)$, *and* F *has property* I, *then there exists a point* $Q \in int(R)$ *such that* $D_M f(Q) = F(R)/A(R)$.

Proof: We will use the same notation as in theorem 1 and modify the selection process to obtain, for some k, an $R_k \subset int(R)$. We consider the following two cases:

1) If $R_2 \neq R_{1i}$ for any i, then R_2 has at most one edge contained in bdry(R).

2) If $R_2 = R_{1i}$ for some i, say $i=1$, then the following argument allows us to choose R_3 with at most one edge contained in bdry(R).

Divide R_2 into R_{21}, R_{22}, R_{23}, R_{24}, and if $F(R_{21}) = F(R_{22}) = F(R_{23}) = F(R_{24}) = (1/4)F(R_2)$, choose $R_3 = R_{23} \subset int(R)$. If $F(R_{21}) \neq (1/4)F(R_2)$ for some i, then the sliding technique gives an R_3 with at most one edge contained in bdry(R). For case (1), suppose the bottom edge of R_2 is contained in bdry(R_1) and divide R_2 as before. If $F(R_{2i}) = (1/4)F(R_2)$ for $i=3$, choose $R_3 = R_{23}$ and if $F(R_{23}) \neq (1/4)F(R_2)$, then the sliding technique will again give an $R_3 \subset int(R)$.

IV) A DARBOUX PROPERTY:

THEOREM 4: *Let* $\mathcal{O}$ *be a connected open set in* E^2. *Suppose* $D_M f$ *exists at each point of* $\mathcal{O}$ *and that* F *has property* I. *Let* $P, Q \in \mathcal{O}$ *and suppose* $D_M f(P) = \alpha$, $D_M f(Q) = \beta$, $\alpha < \beta$, *and* $\lambda \in (\alpha,\beta)$. *If* $\overline{PQ}$ *is an arc which is contained in* $\mathcal{O}$ *with endpoints* P *and* Q, *then for each* $\varepsilon > 0$ *there exists a point* $S \in \mathcal{O}$ *such that the distance* $d(S,\overline{PQ})$ *from* S *to* $\overline{PQ}$ *is less than* ε *and* $D_M f(S) = \lambda$.

Proof: Construct a polygonal arc $\widehat{PQ}$ from P to Q consisting of horizontal and vertical straight line segments such that each point of $\widehat{PQ}$ is within $\min(\xi/2,\varepsilon/2)$ of $\overline{PQ}$, where $\xi = d(\overline{PQ},\text{bdry } \mathcal{O})$. Let $u =$

$= \min\left[\varepsilon/2, (\lambda-\alpha)/2\ ,\ (\beta-\lambda)/2\right]$. There exist rectangles R_1 and R_2 of order M centered at P and Q respectively with edges parallel to the coordinate axes and having the same base and height, such that $|F(R_1)/A(R_1)-\alpha| < u$, $|F(R_2)/A(R_2)-\beta| < u$, and $\mathrm{diam}(R_1) = \mathrm{diam}(R_2) \leq \min(\varepsilon/2, \xi/2)$.

Since F has property I the sliding technique allows us to obtain a rectangle R or order M, with sides parallel to the coordinate axes, centered at a point of $\hat{PQ}$ such that $\mathrm{diam}(R) = \mathrm{diam}(R_1)$ and $F(R)/A(R) = \lambda$. Theorem 1 implies the existence of a point $S \in R$ such that $D_M f(S) = \lambda$ and $d(S,\overline{PQ}) < \varepsilon$.

REMARK. Let u be Lebesgue measure on E^n and let T be any absolutely continuous measure with respect to u. Further suppose that the derivative $T'{}^*(x)$, as defined in {8}, exists at every point in an interval $R_n \subset E^n$. The above technique may be used to establish a mean value theorem and a Darboux property for this derivative. These results also hold if T is an additive set function defined on at least the closed intervals in E^n and has the intermediate value property along straight lines in the appropriate directions, u is a translational invariant measure which is finite on regular rectangles, and $T'{}^*(x)$ exists at every point in Int (R). A further generalization is given in Section VI of this paper.

V) A THEOREM ON THE EQUALITY OF THE DERIVATIVES f_{xy}, f_{yx}, and Df.

It is well known that if $f_{xy}(x,y)$ exists at each point of an open set $\mathcal{O}$ and R is a closed rectangle with $R \subset \mathcal{O}$, then there exists a point $P \in$ int (R) such that $f_{xy}(P) = F(R)/A(R)$. Example 1 shows that there are functions for which $f_{xy}(x,y)$ and Df exist on a rectangle and f_{yx} fails to exist at any point. Also, the example can be modified by defining $f(x,y) = 2$ whenever x and y are rational , $f(x,y) = 1$ if exactly one of x or y is rational, and zero otherwise to give a function such that Df exists on a rectangle while both f_{xy} and f_{yx} fail to exist at any point.

THEOREM 5. *If* f_{xy} *and* $D_M f$ *exist on an open set* $\mathcal{O}$ *and (a) the function* f_{xy} *is continuous or (b) the related set function* F *has the intermediate value property along straight lines in the appropriate directions and* $D_M f$ *is continuous, then* $f_{xy}(P) = D_M f(P)$ *for each* $P \in \mathcal{O}$.

Proof: Suppose f_{xy} is continuous. Let $\{R_i\}$ be a sequence of nested rectangles of order M contained in 0 and closing down on P. Then $D_M f(P) = \lim_{i\to\infty} F(R_i)/A(R_i)$. For each i there exists a point $P_i \in R_i$ such that $f_{xy}(P_i) = F(R_i)/A(R_i)$ and the continuity of f_{xy} gives the desired result.

Suppose that $D_M f$ is continuous and that the set function F has the intermediate value property and hence theorem 2 applies. Let $P = (x_o,y_o) \in \mathcal{O}$. If $\varepsilon > 0$ then there exists t_1 such that $0 < t_1 < \varepsilon$ and $|[f_x(x_o,y_o+t_1)-f_x(x_o,y_o-t_1)]/2t_1-f_{xy}(x_o,y_o)| < \varepsilon/3$. There exists $t_2 > 0$ such that $t_2 = nt_1$ for some integer n and so that $\left|\left[f(x_o+t_2,y_o+t_1)-f(x_o-t_2,y_o+t_1)\right]/\left[2t_2\right]-f_x(x_o,y_o+t_1)\right|/\left[2t_1\right] < \varepsilon/3$ and $\left|\left[f(x_o+t_2,y_o-t_1)-f(x_o-t_2,y_o-t_1)\right]/\left[2t_2\right]-f_x(x_o,y_o-t_1)\right|/\left[2t_1\right] < \varepsilon/3$. We can now divide rectangle $R = [x_o-t_2,x_o+t_2;y_o-t_1,y_o+t_1]$ into n squares and conclude from the sliding technique that there exists a square R' such that $R' \subset R$ and $F(R)/A(R) = F(R')/A(R')$. Hence there is a point $P' \in R' \subset R$ such that $D_M f(P') = F(R)/A(R)$ and hence $|D_M f(P') - f_{xy}(P_o)| < \varepsilon$ and $d(P',P) < \varepsilon$. Continuity implies the desired result.

VI) A FURTHER GENERALIZATION:

We shall say that the additive set function F has property C provided the auxiliary function g(t) as defined in theorem 1 is continuous along the lines x=constant, y=constant, and $y = \pm\alpha x$. Note that property C implies property I.

THEOREM 6. *Suppose that* S *and* T *are additive set functions defined on rectangles,* u *is a translational invariant measure, and* $S'*(p)$ *and* $T'*(p)$ *exist at each point* $p \in$ *Int* (R) *and* $T'*(p) \neq 0$ *for any* p. *Then there exists a point* $q \in$ *Int* (R_o) *so that*

$$\frac{S(R_o)}{T(R_o)} = \frac{S'*(q)}{T'*(q)}$$

Proof: Let $U(R) = S(R_o)T(R)-T(R_o)S(R)$. Then $U(R_o) = 0$ and U has property I. Hence there exists a point $q \in$ Int (R_o) so that $U'*(q) = 0 = S(R_o)T'*(q)-T(R_o)S'*(q)$. This holds without the condition that $T'*(p) \neq 0$ for $p \in R_o$. The result now follows.

The method of proof in the preceding theorem allows one to remove the condition of translational invariant u. Suppose S and T are additive set functions defined on rectangles and having property C. Define $dS(p)/dT$ to be the limit of the ratio of the S area to the T area of regular rectangles as the diameters of the rectangles tend to zero.

THEOREM 7. *If* dS/dT *exists at each point* p *in Int* (R_o), $T(R) \neq 0$ *for* $R \subset R_o$, *and* S *and* T *have property* C, *then there is a point* $q \in$ *Int* (R_o) *so that* $dS(q)/dT = S(R_o)/T(R_o)$.

Proof: Define U as in theorem 6. Then use the procedures of theorems 1 and 3 to define a nested sequence $\{R_i\}$ of rectangles closing down on $q \in \text{Int } R_o$ and such that $U(R_i) = 0$. Then $S(R_o)/T(R_o) = S(R_i)/T(R_i)$ and the result follows.

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