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NOTES ON COMARGINAL PROBABILITY MEASURES

by A. Diego and R. Panzone

1. INTRODUCTION.

By $S = (\Omega, A, B, C, P_1, P_2)$ we shall denote a system consisting of two σ-algebras B, C of subsets of Ω, the generated σ-algebra $A=\tau(B,C)$, and probabilities P_1, P_2 defined on B,C respectively, which are compatible in the sense that $P_1 = P_2 = P$ on the intersection σ-algebra $D = B \cap C$.

A probability Q on A such that its restrictions to B, C are P_1, P_2, respectively, is said to be *comarginal* with P_1, P_2, or briefly comarginal.

A probability Q on A, not neccesarily comarginal, is called *commutative* if on A-measurable Q-integrable functions:

$$(1) \qquad E_Q^B \cdot E_Q^C = E_Q^C \cdot E_Q^B = E_Q^D \qquad [Q] \quad ,$$

where E_Q^G denotes the conditional expectation operator with respect to the σ-algebra $G \subset A$ and to the measure Q.

The main problem we consider here is to search under what conditions a system S admits a comarginal and commutative measure Q. For such a measure we can assert its uniqueness. Owing to this fact and to the following example we shall call Q the *(generalized) product measure* on S.

Given the probability spaces (Ω_1, B', P_1') , (Ω_2, C', P_2') the system S formed by $\Omega = \Omega_1 \times \Omega_2$, $B = \Pi_1^{-1}(B')$, $C = \Pi_2^{-2}(C')$, $A = \tau(B,C)$, $P_i = P_i' \Pi_i$, (Π_i = projection on Ω_i), $i=1,2$, admits the comarginal and commutative measure $Q = P_1' \times P_2'$. The relation of commutation (1) is here Fubini's theorem.

In this example $D = \{\phi, \Omega\}$; the opposite extreme case, when $D = B$ (or $D = C$), gives us also an example very trivial for a product measure: $Q = P_2$ (or $Q = P_1$).

Intermediate cases can be given as follows:

Let $\Omega = \Omega_x \times \Omega_y \times \Omega_z$ and probabilities dx, dy, dz be given on σ-algebras B_x, B_y, B_z of Ω_x, Ω_y, Ω_z respectively. We define $B = \{\phi, \Omega_x\} \otimes B_y \otimes B_z$, $C = B_x \otimes \{\phi, \Omega_y\} \otimes B_z$ and the probabilities $dP_1 = dy\, dz$, $dP_2 = dx\, dz$. Here D is isomorphic to B_z and the system $(\Omega, A, B, C, P_1 P_2)$ has the product measure $dQ = dx\, dy\, dz$.

The preceding situation always appears in a Markov process. Assume

$\{x_i\}_{i=1,2..}$ is a Markov chain and $\mathcal{B} = \tau(x_1,\dots x_n)$, $\mathcal{C} = \tau(x_n,x_{n+1}\dots)$ and $\mathcal{D} = \tau(x_n)$. The product measure Q turns out to be the probability associated with the process, and the relation of commutation (1) can be rewritten as expressing the conditional independence of past ($\mathcal{B}$) and future ($\mathcal{C}$), given the present ($\mathcal{D}$). (cf. {1}, §14, or {4}, part A, ch. II).

For further examples we refer to {1} , §3.

2. COMARGINAL MEASURES AND BILINEAR FORMS.

We shall say that $T: \mathcal{B} \times \mathcal{C} \longrightarrow [0,+\infty)$ is a (positive) *bilinear form* on the system $S = (\Omega,A,\mathcal{B},\mathcal{C},P_1,P_2)$ if $T(B,C)$ is additive in each variable separately; and T will be said to be *compatible* (with P_1,P_2) if $T(B,\Omega) = P_1(B)$, $T(\Omega,C) = P_2(C)$, for any $B \in \mathcal{B}$, $C \in \mathcal{C}$.

To each finitely additive measure Q on the algebra $A_o = \mathcal{B} \vee \mathcal{C}$ comarginal with P_1, P_2 we associate the compatible bilinear form $T(B,C) = Q(BC)$, which verifies: $T(B,C) = 0$ if $BC = \emptyset$. Conversely:

THEOREM 1. *Every compatible bilinear form* T *on* C *such that* $T(B,C)=0$ *whenever* $BC = \emptyset$, *defines on* A_o *a unique finitely additive comarginal probability given by* $Q(B.C) = T(B,C)$.

Proof: We define $Q(BC) = T(B,C)$. If $BC = B'C'$, then $BC=(BB')(CC') = B_1C_1$. To prove that $T(B,C) = T(B',C')$ it is enough to prove that $T(B,C) = T(B_1,C_1)$. It follows from $T(B,C) = T(B_1,C) + T(B-B_1,C) = T(B_1,C)$, since $(B-B_1)C = \emptyset$, and from $T(B_1,C) = T(B_1,C_1)$. Hence Q is well defined on sets of the form BC , $B \in \mathcal{B}$, $C \in \mathcal{C}$.

Let $BC = \sum_\alpha B_\alpha C_\alpha$, where α runs on a finite family of indices. In order to prove that Q can be (uniquely) extended to A_o as a finitely additive measure it is enough to prove that $Q(BC) = \sum_\alpha Q(B_\alpha C_\alpha)$.

Let $\{B_i\}$ be the partition of B defined by the B_α's and $\{C_j\}$ that defined by the C_α's on C. We can assume from the beginning that $B_\alpha \subset B$, $C_\alpha \subset C$. Since $B_\alpha = \sum_m B_{\alpha,m}$, $C_\alpha = \sum_n C_{\alpha,n}$, denoting by $B_{\alpha,n}$ ($C_{\alpha,n}$) the sets of the mentioned partition of B (C) included in B_α (C_α), we have:

$$\sum_{i,j} B_iC_j = BC = \sum_\alpha B_\alpha C_\alpha = \sum_\alpha (\sum_m B_{\alpha,m})(\sum_n C_{\alpha,n}) = \sum_\alpha (\sum_{m,n} B_{\alpha,m}C_{\alpha,n})$$

This means that in the first and last sums appear the same non-void terms. Therefore, from the bilinearity of T we get

$$Q(BC) = T(B,C) = \sum_{i,j} T(B_i,C_j) = \sum_\alpha (\sum_{m,n} T(B_{\alpha,m},C_{\alpha,n})) = \sum_\alpha T(\sum_m B_{\alpha,m},\sum_n C_{\alpha,n}) =$$

$= \sum_\alpha T(B_\alpha, C_\alpha) = \sum_\alpha Q(B_\alpha . C_\alpha)$ Q.E.D.

REMARKS. 1) It is not true, in general, that Q is σ-additive. In fact, let Ω be the triangle on the plane defined by $x > 0$, $y > 0$, $x + y < 1$. $\mathcal{B}$ ($\mathcal{C}$) the Borel sets of Ω depending only of $x(y)$. For $B \in \mathcal{B}$ ($C \in \mathcal{C}$) we define $P_1(B) = m(B')$, ($P_2(C) = m(C')$), where $B'(C')$ denotes the projection of B (C) on the x (y) axis and m the Lebesgue measure on (0,1).

The bilinear form $T(B,C)=\int_0^1 1_{B'}(t)\, 1_{C'}(1-t)dt$ is compatible with P_1, P_2 but, as it is easy to see, Ω can be put as a countable sum of rectangles B.C for which $T(B,C) = 0$.

2) Let us observe that if T is bilinear and comarginal then T is also σ-bilinear; i.e. σ-additive in each variable. In fact,

$T(\sum_{j=0}^{\infty} B_j, C) = T(\sum_{i=1}^{n} B_j, C) + T(\sum_{n+1}^{\infty} B_j, C)$, then

$|T(\sum_{j=1}^{\infty} B_j, C) - \sum_{i=1}^{n} T(B_j, C)| \leq \sum_{n+1}^{\infty} T(B_j, \Omega) = \sum_{n+1}^{\infty} P_1(B_j) \longrightarrow 0$ if $n \longrightarrow \infty$.

Then the proof of the theorem remains true if we assume that α runs on a countable family of indices such that the B_α's and the C_α's define, respectively, countable partitions of the spaces Ω.

For example, we can assert the σ-additivity of Q if A is defined by a countable partition of Ω.

3) In theorem 1 we can assume $T(B_o, C_o) = 0$ if $B_o C_o = \emptyset$ for $B_o \in \mathcal{B}_o \subset \mathcal{B}$, $C_o \in \mathcal{C}_o \subset \mathcal{C}$, where $\mathcal{B}_o, \mathcal{C}_o$ are collections of sets with the *approximation property*: $P_1(B) = \sup_{\substack{B_o \subset B \\ B_o \in \mathcal{B}_o}} P_1(B_o)$, $P_2(C) = \sup_{\substack{C_o \subset C \\ C_o \in \mathcal{C}_o}} P_2(C_o)$, for $B \in \mathcal{B}$, $C \in \mathcal{C}$.

In fact, if $B_o \subset B$, $C_o \subset C$, and $B.C = \emptyset$: $T(B,C) = T(B-B_o+B_o, C-C_o+C_o) =$
$= T(B-B_o, C) + T(B_o, C-C_o) + T(B_o, C_o) \leq P_1(B-B_o) + P_2(C-C_o)$. The last member can be done arbitrarily small, hence $T(B,C) = 0$.

4) The σ-additivity of Q follows under the following hypothesis

1) $K_{\mathcal{B}} \subset \mathcal{B}$, $K_{\mathcal{C}} \subset \mathcal{C}$ are *semi-compact classes* (i.e. every countable family of $K_{\mathcal{B}}$ ($K_{\mathcal{C}}$) with an empty intersection has a finite subfamily which also intersects in the empty set) verifying the approximation property (as defined above).

2) $K_{\mathcal{B}}.K_{\mathcal{C}} = \{K.L \ ; \ K \in K_{\mathcal{B}}, \ L \in K_{\mathcal{C}}\}$ is a semicompact class of sets. In fact, the class L of finite unions of sets of $K_{\mathcal{B}}.K_{\mathcal{C}}$ enjoys the property of approximation in $\mathcal{B} \vee \mathcal{C}$, since, as it was shown above, for $K \subset B$, $L \subset C$ we have $Q(BC - KL) \leq P_1(B - K) + P_2(C - L)$, then $Q(BC - KL)$ can be done arbitrarily small. L being compact

and with the approximation property the σ-additivity of Q follows from a theorem of Alexandrov (cf {4}, pp. 47).

3. ∇-COMMUTATIVE SYSTEMS.

For any probability space (Ω,A,P) we define the measurable hull of $X \subset \Omega$ as a set $A \in A$ containing X except by a set of P-outer measure zero and with minimal P-measure. Of course, the measurable hull is defined except on a null set of A, and it provides a well defined element of the Boolean measure algebra: $A/[P]$. If $B \subset A$ is a σ-subalgebra of A the measurable hull of $X \in A$ with respect to B coincides with $\{E_P^B 1_X > 0\}$, $[P]$.

For a system $S = (\Omega,A,B,C,P_1,P_2)$ we shall designate $\nabla^1 X$, $\nabla^2 X$, ∇X the measurable hulls of $X \subset \Omega$ with respect to B,C,D and to the measures P_1, P_2 , P respectively.

If Q is a comarginal measure on S and E, F , G denote E_Q^B , E_Q^C , E_Q^D respectively, we can see that the condition $EF = FE = G[Q]$ (on bounded A- measurable functions) is equivalent to $Ef = Gf$ $[P_1]$ (on C-measurable functions) and also to $Ff = Gf$ $[P_2]$ (on B-measurable functions) (cf {1} , § 1).

For a comarginal measure Q on S, the condition $EF = FE = G$ implies $\nabla^1\nabla^2 X = \nabla^2\nabla^1 X = \nabla X$ $[Q]$ for $X \in A$. This condition is equivalent to $\nabla^2 B = \nabla B$ $[P_2]$, for any $B \in B$, and also to $\nabla^1 C = \nabla C$ $[P_1]$, for any $C \in C$. (cf. {1} , §5).

We note that any of these last conditions can be introduced even if we do not assume that a comarginal measure Q is known. Then we adopt the following definition:

We shall say that the system S *∇-commutes* if $\nabla^2 B = \nabla B$ $[P_2]$, $\forall B \in B$.

From the above considerations it follows:

In order that there exists a comarginal and commutative measure on $S = (\Omega,A,B,C,P_1,P_2)$ *it is necessary that S be a ∇-commutative system.*

An independent proof will be given in next theorem 2.

A ∇-commutative system is said to be *simple* if $D/[P]$ is the Boolean algebra $\{0,1\}$.

4. THE FINITELY ADDITIVE MEASURE ASSOCIATED TO A ∇-COMMUTATIVE SYSTEM.

Given the system $S = (\Omega, A, B, C, P_1, P_2)$ we observe that the conditional expectation operator G can be calculated on B (C)-measurable functions, with respect to P_1 (P_2) even if there is no comarginal measure. Hence we can define the compatible bilinear form:

$$(2) \qquad T(B,C) = \int G1_B . G1_C \, dP \; .$$

THEOREM 2. *In order that* $Q(BC) = T(B,C)$ *defines a finitely additive comarginal measure on* S *it is neccesary and sufficient that* S *be a* ∇-*commutative system.*

In this case, if Q *is a probability on* A *(i.e. if* Q *is* σ-*additive),* Q *is the unique commutative comarginal measure on* S. *(uniqueness of the product measure).*

Proof: By theorem 1, to prove that ∇-commutativity implies that Q is a finitely additive measure, we have to show that $B.C = \emptyset$ implies $T(B,C) = 0$. From $B.C = \emptyset$ we get $\nabla^2 B.C = \nabla B.C = \emptyset[P_2]$; i.e. $\{G1_B > 0\}.C = \emptyset[P_2]$. Then $T(B,C) = \int_C G1_B . dP_2 = 0$. Conversely, if Q is a finitely additive and comarginal measure, S ∇-commutes. In fact, $\nabla^2 B \subset \nabla B[P_2]$ and on the other hand from $P_2^*(B - \nabla^2 B) = 0$ we have $B - \nabla^2 B \subset C' \in C$ with $P_2(C') = 0$. Hence $Q(B - \nabla^2 B) \leq Q(C') = P_2(C') = 0$. Then,
$$Q(B - \nabla^2 B) = \int_{\complement \nabla^2 B} G1_B . dP_2 = 0 .$$
This means $\{G1_B > 0\}.\complement\nabla^2 B = \emptyset \; [P_2]$ which implies $\nabla B \subset \nabla^2 B[P_2]$.

If Q, as defined above, is a measure, and E, F the conditional expectations with respect to B, C :

$$(3) \qquad Q(B.C) = \int G1_B . G1_C \, dP = \int_C G1_B \, dQ = \int_C F1_B \, dQ$$

Then $G1_B = F1_B$, $\forall \; B \in B$. This implies the commutation of Q. Another comarginal commutative measure Q' must verify (3), but since

$$\int_C G1_B \, dQ' = \int_C F1_B dQ' = Q'(B.C) = Q(B.C) \; ,$$

Q and Q' coincide on $B \vee C$, and therefore on A. QED.

REMARK: The condition $\nabla^2 B = \nabla B \; [P_2]$, $B \in B$, defining a ∇-commutative system implies the symmetric one $\nabla^1 C = \nabla C \; [P_1]$, $C \in C$.

In fact, the first one implies that $\int G1_B . G1_C \, dP$ defines a finitely additive measure, and from this we derive $\nabla^1 C = \nabla C \; [P_1]$ as it was done with $\nabla^2 B = \nabla B \; [P_2]$ in the proof of theorem 2. Now, we obtain easily

$\nabla^2\nabla^1 X = \nabla^1\nabla^2 X = \nabla X$ [P] for each $X \in \mathcal{B} \vee \mathcal{C}$.

THEOREM 3. *i) If the ∇-commutative system S is simple (i.e. $\mathcal{D}/[P]\cdot =$ $= \{0,1\}$) then* $Q(B.C) = P_1(B).P_2(C)$. *In particular for a system S obtained from a cartesian product (as described in the introduction),* $Q = P_1' \times P_2'$.

ii) If, conversely, $Q(B.C) = P_1(B).P_2(C)$ *defines a finitely additive measure on* $\mathcal{B} \vee \mathcal{C}$*, S is a simple ∇-commutative system.*

Proof: i) It follows from theorem 2. and the fact that $G1_B = P_1(B).1_\Omega$, $G1_C = P_2(C).1_\Omega$.

ii) If $D \in \mathcal{D}$ we have $Q(D) = P(D) = Q(D.D) = P(D)^2$, then $P(D) = 0$ or 1. $G1_B$, $G1_C$ are computed like in i), then $\int G1_B.G1_C = P_1(B).P_2(C)$. From theorem 2) it follows that S is a ∇-commutative system.

We shall say that a system S is *complete* with respect to a comarginal probability P defined on $\mathcal{A}$ if every P-null set of $\mathcal{A}$ belongs to $\mathcal{D}$.

THEOREM 4. *If the system S is complete with respect to a comarginal probability* P *and* R *is a commutative probability on S equivalent to* P *then the product measure* Q *exists and it is equivalent to* P.

Proof: Assume $f = \frac{dR}{dP}$. By hypothesis $F_R = G_R$ on $\mathcal{B}$-measurable functions and $E_R = G_R$ on $\mathcal{C}$-measurable functions. We have (c.f. { 1 } §2):

$$E_R(h) = E(f.h)\Big/E(f) \quad ,$$

$$F_R(g) = F(f.g)\Big/F(f) \quad , \quad G_R(m) = G(f.m)\Big/G(f) \quad ,$$

where E, F, G denote here the conditional expectation operators with respect to the measure P and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ respectively.

In {1} , th. 2 , §10 , it is proved that the probability measures equivalent to R that also commute are characterized as those whose Radon-Nikodym derivatives with respect to R are of the form: g.h, where g (h) is a $\mathcal{B}(\mathcal{C})$-measurable function (both positive and finite [R]).

Let us consider the functions:

$$g = \frac{+\sqrt{Gf}}{Ef} \quad , \quad h = \frac{+\sqrt{Gf}}{Ff}$$

Then, $g.E(h.f) = Gf.E(f/Ff)\ /Ef = Gf.E_R(1/Ff) = Gf.G_R(1/Ff) = G(f/Ff) =$ $= GF(f/Ff) = 1$. Analogously $h.F(g.f) = 1$. From $\int_B ghf\ dP = \int_B gE(hf)\ dP =$

$= \int_B 1\, dP = P(B)$ and $\int_C ghf\, dP = P(C)$

we see that the probability Q defined by

$$Q(A) = \int_A gh.f\, dP = \int_A g\, h\, dR$$

is comarginal with P and since $\frac{dQ}{dR} = g.h$, it commutes and obviously $Q \sim R$. QED.

REMARK: If the system S is complete with respect to P any other comarginal measure R is absolutely continuous with respect to P, since $P(A) = 0$ implies $A \in \mathcal{D}$ and then $R(A) = P(A) = 0$. Then, if a product measure Q exists on S, we have $Q \ll P$. In spite of the fact that for $B \in \mathcal{B}$, $C \in \mathcal{C}$, $Q(B.C) = 0$ implies $P(B.C) = 0$ (since $\int G1_B G1_C dP = 0$ implies $P_2(\nabla B.C) = 0$ and then $P(B.C) = 0$) we have not $Q \sim P$, in general. Let us see the following example:

Let Ω be the product of X and Y, $X = Y = (0,1)$, $\mathcal{A}$ = the Borel sets of $X \times Y$, $\mathcal{B}$ ($\mathcal{C}$) the Borel sets independent of y (x), P the probability on $\mathcal{A}$ equal to $\frac{m\times m}{2}$ (m the Lebesgue measure on (0,1)) plus a measure of total mass 1/2 concentrated on the diagonal and uniformely distributed there.

The product measure $Q = m \times m$ is not equivalent to P.

If $D \in \mathcal{D}$ defines an atom of the σ-algebra $\mathcal{D}/[P]$ it can be seen that $S_D = (D, \mathcal{A} \cap D, \mathcal{B} \cap D, \mathcal{C} \cap D, P_1/P_1(D), P_2/P_2(D))$ is a simple ∇-commutative system, whenever S is a ∇-commutative system. Moreover, if Q is the (product) finitely additive measure on S defined above , $\frac{Q(B.C.D)}{Q(D)}$ is the product measure on S_D as it is easily seen. Since S_D is simple: $\frac{Q(B.C.D)}{Q(D)} = \frac{P_1(B.D)}{P(D)} \cdot \frac{P_2(C.D)}{P(D)}$, and then

$$Q(B.C.D) = P_1(B.D).P_2(C.D)/P(D) \tag{4}$$

THEOREM 5. *If in a ∇-commutative system S, $\mathcal{A}$ is defined by a countable partition of Ω, there exists the product measure* Q. *If* $\{D_i\}$ *is the partition defining $\mathcal{D}$, then* Q *is defined by:*

$$Q(B.C) = \sum_i \frac{P_1(BD_i).P_2(CD_i)}{P(D_i)} \tag{5}$$

Proof: The σ-additivity of Q follows from the second remark in §2. We have $Q(B.C) = \sum_i Q(B.C.D_i)$, and applying (4) we obtain the equality (5).

5. THE BOOLEAN MEASURE STRUCTURE OF A ∇-COMMUTATIVE SYSTEM

From a given probability space $(\Omega,\mathcal{A},P)$ and a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$, we get the (measure) Boolean algebra $A = \mathcal{A}/[P]$ quotient of $\mathcal{A}$ mod. P-null sets and the subalgebra $B \simeq \mathcal{B}/[P]$. To the operation of taking the $\mathcal{B}$-measurable hull in $\mathcal{A}$ corresponds in A a so called *monadic operator* (c.f. {2}).

Let A be a Boolean algebra with a subalgebra B such that for each $a \in A$ there exists an element $b \in B$ which is the least element of B verifying $b \geq a$; we set $b = \nabla a$, and we call ∇ the monadic operator in A related to B.

A monadic operator verifies the following properties: $\nabla 0 = 0$, $\nabla(a \vee b) = \nabla a \vee \nabla b$, $\nabla\nabla a = \nabla a$, $\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$.

The algebraic system (A,B,∇) is called a *monadic algebra.* Let us consider a ∇-commutative system $S = (\Omega,\mathcal{A},\mathcal{B},\mathcal{C},P_1,P_2)$ with a product measure Q. By passing to the quotient mod. $[Q]$ we get the Boolean (measure) algebras A,B,C,D, which are the images of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ resp.; B,C,D are subalgebras of A and $D = B \cap C$. If we denote by $\mathcal{A}_o = \mathcal{B} \vee \mathcal{C}$ then $A_o = B \vee C \simeq \mathcal{A}_o/[Q]$, where $B \vee C$ is the Boolean algebra generated by B and C.

If ∇^1,∇^2,∇ designate the monadic operators in A corresponding to the measurable hull operations in $\mathcal{A}$ denoted before with the same symbols we have $\nabla^1\nabla^2 a = \nabla^2\nabla^1 a = \nabla a$ for any $a \in A$. The same hold if we restrict our-selves to elements $a \in A_o = B \vee C$. So we have an instance of what is called a biadic algebra (c.f. {2}).

The algebraic system $(A_o,B,C,\nabla^1,\nabla^2)$ is called a *biadic algebra* if (A_o,B,∇^1), (A_o,C,∇^2) are monadic algebras, $A_o = B \vee C$ and $\nabla^1\nabla^2 = \nabla^2\nabla^1$. It is easy to see that in this case $\nabla = \nabla^1\nabla^2$ defines the monadic operator related to $D = B \cap C$.

We can say that the underlying Boolean structure of a ∇-commutative system is a biadic algebra. We have seen this when a product measure Q is given in the system and it is easy to see that the same is true even if Q does not admit a σ-additive extension from $\mathcal{A}_o$ to $\mathcal{A}$.

In particular, if S is a simple ∇-commutative system, we obtain a simple biadic algebra $(A_o,B,C,\nabla^1,\nabla^2)$ i.e. it verifies $D = B \wedge C = \{0,1\}$. For such simple algebras we have $A_o = B \oplus C$, direct sum of B, C which means that $A_o = B \vee C$ and, for $b \in B$, $c \in C$, $b \wedge c = 0$ implies $b = 0$ or $c = 0$ (in fact, if $b \wedge c = 0$ then $0 = \nabla(b \wedge c) = \nabla^1\nabla^2(b \wedge c) = \nabla^1(\nabla^2 b \wedge c) = \nabla^1(\nabla^2\nabla^1 b \wedge c) = \nabla^1(\nabla^1\nabla^2 b \wedge c) = \nabla b \wedge \nabla^1 c = \nabla b \wedge \nabla c$, therefore $\nabla b = 0$ or $\nabla c = 0$, so $b = 0$ or $c = 0$) (The converse also holds: if $A_o = B \oplus C$, $(A_o, B, C, \nabla^1, \nabla^2)$ is a simple biadic algebra).

In simple ∇-commutative systems, for example the systems obtained from cartesian products, what really matters from the point of view of the theory of measure Boolean algebras are the algebras B, C and A_o. Explicitly, if A'_o, B', C' are obtained from another simple ∇-commutative system S', then if we have Boolean isomorphisms $B \simeq B'$ and $C \simeq C'$ we get $A_o \simeq A'_o$. This is due to the fact that A_o, A'_o are direct sums. In fact, the direct sum $A_o = B \oplus C$ has the property of extension of homomorphisms: if $B \longrightarrow \bar{A}$ and $C \longrightarrow \bar{A}$ are Boolean homomorphisms with range a Boolean algebra $\bar{A}$, there exists one and only one extension of them to a homomorphism: $A_o = B \oplus C \longrightarrow \bar{A}$. (c.f. {5}).

On the other hand, if (B,P_1) (C,P_2) are given Boolean measure algebras we can construct at least a simple ∇-commutative system S for which the associated biadic algebra is precisely $(B \oplus C, B, C, \nabla^1, \nabla^2)$. In fact, it is well known that the Stone space of $B \oplus C$ is the cartesian product of the Stone spaces of B and C, $S(B \oplus C) = S(B) \times S(C)$ (Precisely the algebra of clopens of $S(B) \times S(C)$ is used to define $B \oplus C$). We set on the clopens of $S(B)$ and $S(C)$ the measures P_1 and P_2 in the obvious way and we extend P_1, P_2 to the σ-algebras generated by clopens associated to elements of B, C respectively. The product of the probability spaces so obtained gives us the required system S.

For general ∇-commutative systems we can prove analogous results.

To a ∇-commutative system $S = (\Omega, A, B, C, P_1, P_2)$ we have associated a biadic algebra (A_o, B, C). Moreover B,C are measure Boolean algebras with the probabilities P_1, P_2 defined on B, C, respectively, coinciding in $D = B \cap C$. Let us call $M(S) = (A_o, B, C, P_1, P_2)$ this Boolean measure structure associated with S. We shall say that the ∇-commutative systems S, S' have the same Boolean measure structure, $M(S) \equiv M(S')$, if under a unique Boolean isomorphism $A_o \simeq A'_o$, $B \simeq B'$, $C \simeq C'$ and $D \simeq D'$; and the probabilities P_i, P'_i $(i=1,2)$ correspond under the isomorphism.

THEOREM 6. *1) Given two probability Boolean algebras* (B,P_1), (C,P_2) *and sub-σ-algebras* $D \subset B$, $D' \subset C$ *such that* $D \simeq D'$ *under a fixed isomorphism preserving the measures* $P_1|_D$, $P_2|_{D'}$, *there exists a* ∇-*commutative system* S *such that* $M(S) = (\bar{A}_o, \bar{B}, \bar{C}, \bar{P}_1, \bar{P}_2)$, *where* $\bar{B} \simeq B, \bar{C} \simeq C$ *are measure preserving isomorphisms (with respect to* P_i, $\bar{P}_i$ $i=1,2$*) which restricted to* $\bar{D} = \bar{B} \cap \bar{C}$ *are isomorphisms* $\bar{D} \simeq D$, $\bar{D} \simeq D'$ *commuting with the given one* $D \simeq D'$.

2) Such a ∇*-commutative system admits a product measure* Q.

3) If S' *is another* ∇*-commutative system verifying the properties of* S *in (1), then* $M(S) \equiv M(S')$.

The proof of 1) and 3) are based on an algebraic theorem concerning biadic algebras that we give next.

We shall write the complete proof of theorem 6 in §7.

THEOREM 7. *1) Given two monadic algebras* (B,D,∇^1) *and* (C,D',∇^2) *and a fixed isomorphism:* $D \simeq D'$ *there exists a biadic algebra* $(\bar{A},\bar{B},\bar{C})$ *such that* $\bar{B} \simeq B$, $\bar{C} \simeq C$ *are isomorphisms which restricted to* $\bar{D} = \bar{B} \cap \bar{C}$ *give isomorphisms* $\bar{D} \simeq D$, $\bar{D} \simeq D'$ *commuting with the given one:* $D \simeq D'$.

2) If there is another biadic algebra $(\tilde{A},\tilde{B},\tilde{C})$ *with the same properties, then the isomorphisms* $\bar{B} \simeq \tilde{B}$, $\bar{C} \simeq \tilde{C}$ *obtained through the isomorphisms of* $\bar{B}$, $\bar{C}$ *with* B *and* C, *have a unique common extension to an isomorphism* $\bar{A} \simeq \tilde{A}$.

Proof: By identifying the isomorphic algebras D, D' through the given isomorphism we can , without loss of generality, consider only the case that B and C are extensions of the same algebra D. So we have the monadic algebras (B,D,∇^1) and (C,D,∇^2).

A filter $F \subset B$ corresponds in $X = S(B)$ with the set X_F of ultrafilters of B that contains F, this is a closed subset that represents with the relative topology the quotient algebra B/F (c.f. {5}). In other words, $X_F = S(B/F)$, and in such a way that if $\hat{b}$ is the clopen set that represents $b \in B$, then $\hat{b} \cap X_F$ is the clopen set that represents the class in B/F containing b. Given an ultrafilter U in D, let us denote with (U) the filter generated in B by U. Then $\{X_{(U)}\}$ is a partition of X. In fact, if $m \in X$ corresponds to the ultrafilter M and $U = M \cap D$, then $m \in X_{(U)}$; if $m \in X_{(U)} \cap X_{(U')}$, then $M \cap D \supset U, U'$, which implies $U = U'$.

Moreover, the monadic operator ∇^1 corresponds with saturation with respect to the partition $\{X_{(U)}\}$; i.e. if $a \in B$, $\widehat{\nabla^1 a} = \text{sat}\,\hat{a} = \bigcup\{X_{(U)} \;;\; \hat{a} \cap X_{(U)} \neq \emptyset\}$.

We include the proof of this well-known fact (c.f. {2}) for the sake of completeness. If $d \in D$, then $d \in U$ iff $X_{(U)} \subset \hat{d}$, and it is equivalent to $X_{(U)} \cap \hat{d} \neq \emptyset$, as it is easy to see. In consequence, $\hat{a} \cap X_{(U)} \neq \emptyset$ iff $\widehat{\nabla^1 a} \supset X_{(U)}$, which is also equivalent to $\nabla^1 a \in U$. In fact, $\nabla^1 a \in U$ implies $\hat{a} \cap X_{(U)} \neq \emptyset$, since otherwise $X_{(U)} \subset \complement\hat{a} = \widehat{\complement a}$, i.e. $\complement a \in (U)$, then for some $d \in U$, $d \leq \complement a$, and hence $d \wedge \nabla^1 a = 0$, which contradicts the fact that U is a proper filter of D.

Let us observe that given a set X, a partition Π of X and a algebra B of subsets of X stable under the sat operation with respect to Π, if D is the algebra of saturated sets of B, we have a monadic algebra (B,D,sat.)

Let us suppose now that an extension A of the algebras B, C exists, such that (A,B,C) is a biadic algebra; it is easy to verify that the ∇ operators defined by B, C on A coincide on B, C with the previously given. We call $\nabla = \nabla^1\nabla^2 = \nabla^2\nabla^1$ to the monadic operator defined by D. Since $B \oplus C$ applies homomorphically onto A, preserving the identity mappings on B and C, $A = B\oplus C/F$ for a filter F in $B \oplus C$. Then the Stone space of A is a closed subset T of $S(B\oplus C) = X \times Y$.

Calling $\{Y_{(U)}\}$ the partition of Y associated to the ultrafilters U of D we have, after elimination of superfluous parentheses:

$$(*) \qquad T = \textstyle\sum_U X_U \times Y_U$$

In fact, if $M \in T$ corresponds to an ultrafilter $\bar{M}$ of A, and $M = (M',M'')$ ($M' \in X$, $M'' \in Y$), considering the homomorphism mentioned above it follows that $M' = B.\bar{M}$, $M'' = C.\bar{M}$. Therefore, $M'.D = M''.D = U$ is an ultrafilter of D. Therefore, $M \in X_U \times Y_U$. Conversely, if $(M',M'')\in X_U\times Y_U$, $M'.D = M''.D = U$. To see that $(M',M'') \in T$ it suffices to see that there is an ultrafilter $\bar{M}$ of A which is a simultaneous extension of M' and M''. It is enough to verify that if $b \in M'$, $c \in M''$ then $b \wedge c \neq 0$. But $\nabla b \wedge \nabla c = \nabla(b \wedge c) \in U$ and therefore is not zero, which implies $b \wedge c \neq 0$.

Let us consider now the following partitions of T:

1) $\{T_x\}$, $x \in X$, $T_x = (\{x\} \times Y) \cap T$ 2) $\{T_y\}$, $y \in Y$, $T_y = (X \times \{y\}) \cap T$. 3) $\{T_U\}$, U ultrafilter of D, $T_U = X_U \times Y_U$.

They define the sat operators corresponding to ∇^1, ∇^2, ∇ of the biadic algebra (A,B,C). This is immediate for ∇^1, ∇^2. For ∇ we observe that the monadic algebra (A,D) is represented by T and the partition associated with the ultrafilters U of D, which is precisely $\{X_U\times Y_U\}$ as it was shown in the proof of (*).

If we start with another extension A' of B and C such that (A',B,C) is a biadic algebra we get again the same set T representing S(A') because the second member of (*) depends only on (B,D) and (C,D). Then $A \simeq A'$ by a unique common extension of the identity isomorphisms of B and C. This proves 2) (except for isomorphic identifications).

The fact that the definitions of T, the partitions 1), 2), 3), and the clopens corresponding to elements of B and C depend only on (B,D) and (C,D) allow us to construct a biadic algebra of sets (A',B',C') such that $B \simeq B'$, $C \simeq C'$ and these isomorphisms when restricted to D give an isomorphism $D \simeq B' \cap C' = D'$. That will prove (except for isomorphic identifications) the first part of the theorem.

Let us define A' as the algebra of sets of the form $a' = \hat{a} \cap T$, where $\hat{a}$ is a clopen in $S(B) \times S(C) = X \times Y$. The algebras $B' = \{(\hat{b}\times Y) \cap T\}_{b\in B}$, $C' = \{(X\times\hat{c}) \cap T\}_{c\in C}$ generate A'.

We define for $a' \in A'$, $\nabla^1 a'$ as the saturated set with respect to $\{T_x\}$, and $\nabla^2 a'$ that obtained from $\{T_y\}$. We must show that $\nabla^1 a' \in B'$ to show that (A',B') is monadic, the same for ∇^2, and that ∇^1, ∇^2 commute. To this end, let us prove:

$$(6) \qquad \nabla^1 c' = (\widehat{\nabla c} \times Y) \cap T \qquad , \text{ for } c' \in C'.$$

$\nabla^1 c' = \nabla^1 (\sum^* X_U \times (\hat{c} \cap Y_U))$, where $\hat{c}$ is the projection of c' on Y, and the star means that the sum is extended to those U such that $\hat{c} \cap Y_U \neq \emptyset$. Therefore, $\nabla^1 c' = \sum^* (\nabla^1 (X_U \times (\hat{c} \cap Y_U)) = [(\sum^* X_U) \times Y] \cap T$. Then, since $\hat{c} \cap Y_U \neq \emptyset$ is equivalent to $\widehat{\nabla c} \supset Y_U$, i.e. to $\nabla c \in U$, we get $\sum^* X_U = \widehat{\nabla c}$, which proves the formula.

To show that $\nabla^1 a' \in B'$, for every $a' \in A'$, it suffices to see it for $a' = b' \cap c'$, $b' \in B'$, $c' \in C'$; $\nabla^1 (b' \cap c') = b' \cap \nabla^1 c' =$

$= [(\hat{b} \cap \widehat{\nabla c}) \times Y] \cap T \in B'$.

If $b' = c'$, then from (6) $b' = (\hat{d}_1 \times Y) \cap T = (X \times \hat{d}_2) \cap T$, which implies $\hat{d}_1 = \hat{d}_2$. Therefore, $D' = B' \cap C'$ is defined as those sets of A' such that are saturated with respect to $\{T_U\}$ and project on the same element of D. Therefore, (6) means that $\nabla^1 c' \in D'$, which is equivalent to the commutation of ∇^1 , ∇^2 (c.f. {1}).

6. AN APPROXIMATION PROCESS.

The generalized product measure Q, when it exists, and in general the finitely additive product measure Q, associated with a ∇-commutative system S, can be obtained as a limit of simpler measures in the way described in the next theorem. We need the following preliminary result:

PROPOSITION 1. *i) If* F *is a finite part of* (A,D,∇), *a monadic algebra, and* $a_1,\dots,a_r$, *the atoms of the Boolean subalgebra generated by* F, *then* $a_1,\dots,a_r$, $\nabla a_1,\dots,\nabla a_r$ *generate a subalgebra* A_o *which is the*

the least one containing F *and stable for* ∇, *(i.e.,* $\nabla A_o \subset A_o$*).*

ii) Assume A_o *is a finite subalgebra of the biadic algebra* $(A,B,C,\nabla^1,\nabla^2)$*, stable for* $\nabla = \nabla^1\nabla^2$, $D = B\cap C$. *If* $B_o = B\wedge A_o$, $C_o = C\cap A_o$ *then there exist operators* ∇_o^1, ∇_o^2 *, such that* $(A_o,B_o,C_o,\nabla_o^1,\nabla_o^2)$ *is a biadic algebra.*

iii) Suppose that $(A_\lambda)_{\lambda\varepsilon\Lambda}$ *is the family of all the finite subalgebras of* A *stable for* ∇*, ordered by inclusion, and generated by their elements belonging to* B *or* C. *Then:*

1) every $(A_\lambda, B_\lambda = A_\lambda \cap B, C_\lambda = A_\lambda \cap C, \nabla_\lambda^1, \nabla_\lambda^2)$ *is a biadic algebra,*

2) $(A_\lambda)_{\lambda\varepsilon\Lambda}$ *is filtering, if ordered by inclusion,*

3) $\bigcup_\lambda A_\lambda = A$.

Proof: i) It is evident that every subalgebra containing F and stable for ∇ must contain the a_i's and the ∇a_i's. Then, it suffices to prove that A_o is stable for ∇. Every atom of A_o is of the form $a = a_i \wedge \bigwedge_j \nabla a_j$ where j runs on some indices 1,...,r. Therefore, $\nabla a = \nabla a_i \wedge \bigwedge_j \nabla a_j \ \varepsilon\ A_o$.

To finish the proof it suffices to observe that every element of A_o is a union of atoms and that ∇ distributes over the union.

ii) ∇_o^i exists because A_o is finite. $a\ \varepsilon\ A_o$ implies $\nabla a = \nabla^2\nabla^1 a \leq \nabla_o^2\nabla_o^1 a \leq \nabla_o a = \nabla a$, where ∇_o denotes the operator relative to $D\cap A_o$. Then $\nabla_o^2\nabla_o^1 = \nabla_o$ and analogously, $\nabla_o^1\nabla_o^2 = \nabla_o$.

iii) It follows from i) and ii) and the observation that every $a\varepsilon B\vee C$ belongs to a finite subalgebra generated by a finite set $F\cup G$ with $F\subset B$, $G\subset C$.

Given the ∇-commutative system $S = (\Omega, A, B, C, P_1, P_2)$ let $M(S) = (A_o, B, C, P_1, P_2)$ be the associated Boolean measure structure. We apply proposition 1 to the biadic algebra (A_o,B,C) to get the filtering family $(A_\lambda)_\lambda$ described in iii). For each λ we select a representative $S_\alpha = (\Omega, A_\lambda, B_\lambda, C_\lambda, P_1, P_2)$ of $(A_\lambda, B_\lambda, C_\lambda, P_1, P_2)$, that means: B_λ, C_λ are finite subalgebras of B, C such that $M(S_\alpha) = (A_\lambda, B_\lambda, C_\lambda, P_1, P_2)$, $A_\lambda = B_\lambda \vee C_\lambda$. Here $\alpha = A_\lambda$, and we assume the α's ordered by inclusion of the A_λ's. Ussing theorem 5 we know that the measure Q_α associated to S_α is defined by

$$Q_\alpha(B.C) = \textstyle\sum_j P_1(BD_j)\ P_2(CD_j)/P(D_j)$$

where the sum is on the atoms of $\mathcal{D}_\lambda$ (it defines a comarginal commutative measure on S_α). Denote with G_α the conditional expectation operator relative to $\mathcal{D}_\lambda$ and the probability Q_α in the system S_α.Then

THEOREM 8. *In a* ∇*-commutative system* S, *it holds:*

i) For $B \in \mathcal{B}$, $C \in \mathcal{C}$, $G_\alpha 1_B \longrightarrow G1_B$ *and* $G_\alpha 1_C \longrightarrow G1_C$, *when* $\alpha\uparrow$, *uniformely a.e.* $[P]$.

ii) $Q(A) = \lim\limits_{\alpha} Q_\alpha(A)$, *for any* $A \in \mathcal{B} \vee \mathcal{C}$.

Proof: ii) follows from i) and

$$Q_\alpha(B.C) = \int G_\alpha 1_B . G_\alpha 1_C \, dP \longrightarrow \int G1_B . G1_C \, dP = Q(B.C)$$

i) For $B \in \mathcal{B}_\lambda$, $\alpha = A_\lambda$, we have $G_\alpha 1_B = \sum_j (P_1(BD_j)/P(D_j))\, 1_{D_j}$, where the sum is on the atoms of $\mathcal{D}_\lambda$, since $Q_\alpha(BD_j) = \int_{D_j} G_\alpha 1_B \, dP =$

$= P(BD_j)/P(D_j)\cdot Q_\alpha(D_j) = P_1(BD_j)$ $(P_1 = Q_\alpha$ on $\mathcal{B}_\lambda)$.

Given $G1_B$, let us divide the set of real numbers on intervals $I_i = [m_i, M_i)$ of length $\varepsilon > 0$. Only for a finite number of them $D_i = (G1_B)^{-1}(I_i) \neq \emptyset$ $[P]$. Consider any finite system S_α such that $D_i \in \mathcal{D}_\lambda$ for all those D_i. Call $\{D_{ij}\}$ the family of atoms of $\mathcal{D}_\lambda$ contained in D_i.

Since, $$m_i P(D_{ij}) \le \int_{D_{ij}} G\, 1_B \, dP \le M_i P(D_{ij}) \qquad ,$$

we obtain: $m_i P(D_{ij}) \le P_1(BD_{ij}) \le M_i P(D_{ij})$. Therefore on D_{ij}:

$|G\, 1_B - P_1(BD_{ij})/P(D_{ij})| \le \varepsilon$, a.e $[P]$. Then

$$|\textstyle\sum_i \sum_j P_1(BD_{ij})/P(D_{ij})\, 1_{D_{ij}} - G\, 1_B| \le \varepsilon \qquad \text{a.e } [P] \qquad \text{QED.}$$

REMARK: If a comarginal probability P exists on A we have $G_\alpha 1_B = E_P(1_B | \mathcal{D}_\lambda)$. Using a result from martingale theory due to Helms {3}, we know that the G_α's form a uniformly integrable martingale converging in L^1 to $G\, 1_B$, which implies ii).

7. PRODUCT MEASURE

For special cases of ∇-commutative systems we can assert that a product measure exists.

One of these cases, the discrete case, was considered in theorem 5. Using remark 4) of §2 we have also:

THEOREM 9. *If the* ∇*-commutative system* $S = (\Omega, A, \mathcal{B}, \mathcal{C}, P_1, P_2)$ *is such* that there are semi-compact classes $K_B \subset B$, $K_C \subset C$ with the property of approximation and $K_B.K_C = \{K.L \; ; \; K \in K_B \; , \; L \in K_C\}$ is also semi-compact , then $Q(B.C) = \int G1_B.G1_C dP$ defines, when extended to A, a probality, i.e. the product measure on S.

Another case in which we can assert the existence of product measure is referred in theorem 6, the proof of which we give now:

Proof of theorem 6. We will suppose, like in the proof of theorem 7, that D is identified to D' through the given isomorphism. By theorem 7 we have an extension algebra A_o of B, C such that (A_o, B, C) is biadic and $B \wedge C = D$. Then we have in the Stone space T of A_o the algebras of clopens $\hat{A}_o$, $\hat{B}$, $\hat{C}$ relative to A_o, B , C. We set $B = \tau(\hat{B})$, $C = \tau(\hat{C})$ and $A = \tau(\hat{A}_o) = \tau(B,C)$. We define in $\hat{B}$, $\hat{C}$ the measures P_1, P_2 given in B, C in the canonical way and extend them to B, C.

Let us prove that $S = (T,A,B,C,P_1,P_2)$ is a ∇-commutative system. If $\hat{b} \in \hat{B}$ and $\hat{c} \in \hat{C}$ are such that $\hat{b}.\hat{c} = \emptyset$, i.e. $b \wedge c = 0$ in A ; we have $0 = \nabla^2(b \wedge c) = \nabla^2 b \wedge c = \nabla^2\nabla^1 b \wedge c = \nabla b \wedge c$. Then $\hat{\nabla b}.\hat{c} = \emptyset$, where $\hat{\nabla b} \in \hat{B} \cap \hat{C}$ contains $\hat{b}$. Hence $\{G1_{\hat{b}} > 0\} \subset \hat{\nabla b}$ $[P]$, and then

$$\int G1_{\hat{b}}.G1_{\hat{c}}\; dP = \int G1_{\hat{b}}.1_{\hat{c}}\; dP_2 \leq P_2(\hat{\nabla b}.\hat{c}) = 0 \quad ,$$

where G is defined on $B(C)$-measurable functions is the expectation operator relative on $D = B \cap C$.

But $\hat{B} \subset B$ and $\hat{C} \subset C$ have the approximation property. Using remark 3 of §2 we can assert that $Q(B.C) = \int G1_B.G1_C\; dP$ is a finitely additive comarginal measure on S, and from theorem 2, S is a ∇-commutative system.

Q being finitely additive on $B \vee C \supset \hat{A}_o$ is a fortiori σ-additive on the algebra of clopens $\hat{A}_o$ and then can be extended uniquely to $A = \tau(\hat{A}_o)$.

This proves 1) and 2) of the theorem (except for identifications) 3) follows immediately from theorem 7.

Finally we shall prove:

THEOREM 10. *If* $S = (\Omega,A,B,C,P_1,P_2)$ *is a* ∇*-commutative system with the property:*

(P) Ω *is the only set of* B *containing a set* $C \in C$ *with* $P_2(C) > 0$, *then* $Q(B.C) = P_1(B).P_2(C)$ *can be extended to a probability on* A. *((P) implies that* S *is simple).*

Proof: Let us define the "section" in x of a "rectangle" B.C by $(B.C)_x = C$ if $x \in B$ and $\emptyset$ otherwise.

In order to prove the σ-additivity of $Q = P_1.P_2$ it is enough to prove that $\sum_\alpha B_\alpha.C_\alpha = \Omega$ implies $\sum_\alpha P_1(B_\alpha).P_2(C_\alpha) = 1$. For each pair $\alpha \neq \beta$ of indices the set $N_{\alpha,\beta} = \{x \; ; \; (B_\alpha C_\alpha)_x.(B_\beta C_\beta)_x \neq \emptyset \; [P_2]\}$ is contained in a set of $\mathcal{B}$ of P_1-measure zero. In fact, from $B_\alpha C_\alpha B_\beta C_\beta = \emptyset$ we have either $P_1(B_\alpha.B_\beta) = 0$ or $P_2(C_\alpha.C_\beta) = 0$. In the first case, it follows from $N_{\alpha\beta} \subset B_\alpha B_\beta$, in the second one, since $(B_\alpha C_\alpha)_x.(B_\beta C_\beta)_x \subset C_\alpha.C_\beta$ for every x, we have $N_{\alpha\beta} = \emptyset$.

Then except for a set B_o of $\mathcal{B}$ of P_1-measure zero $(\supset \bigcup_{\alpha\neq\beta} N_{\alpha\beta})$

(7) $P_2(\bigcup_\alpha (B_\alpha.C_\alpha)_x) = \sum_\alpha P_2((B_\alpha.C_\alpha)_x) = \sum_\alpha P_2(C_\alpha).1_{B_\alpha}(x)$.

Let b_x be the set of the partition of Ω defined by the sets B_α such that $b_x \ni x$

(8) $$b_x \subset \bigcup_\alpha (B_\alpha.C_\alpha)_x$$

In fact, let $y \in b_x$ and suppose $y \in B_\alpha.C_\alpha$, then $B_\alpha \supset b_x \ni x$ and this implies $(B_\alpha.C_\alpha)_x = C_\alpha$. Since $y \in C_\alpha$ we have $y \in \bigcup_\alpha (B_\alpha.C_\alpha)_x$. This proves (8).

From $\mathcal{B} \ni b_x \subset \bigcup_\alpha (B_\alpha.C_\alpha)_x \in \mathcal{C}$, and the assumed property (P) we have $P_2(\bigcup_\alpha (B_\alpha.C_\alpha)_x) = 1$ for every x.

Hence, by virtue of (7) we have $\forall x \notin B_o$: $\sum_\alpha P_2(C_\alpha)1_{B_\alpha}(x) = 1$. i.e. $\sum_\alpha P_2(C_\alpha).1_{B_\alpha} = 1_\Omega \; [P_1]$. By integration with respect to P_1 :

$$\sum_\alpha P_2(C_\alpha).P_1(B_\alpha) = 1 \qquad \text{QED.}$$

Remark: Condition (P) is equivalent to: (P*) if $C \in \mathcal{C}$ contains $B \neq \emptyset$, $B \in \mathcal{B}$, then $C = \Omega$ a.e. $[P_2]$.

REFERENCES

{1} DIEGO, A. and PANZONE, R., On the algebras associated to commuting conditional expectation operators, II, Rev. UMA, number 1, (1968).

{2} HALMOS, P., Algebraic logic, I and II, Comp. Math. 12, (1955), 217-249 and Fund. Math., XLIII, (3), 255-325.

{3} HELMS, L.L., Mean convergence of martingales, Trans. An. Math. Soc., 87, (1958), 439-446.

{4} MEYER, P.A., Probability and Potentials, (1966).

{5} SIKORSKI, R. , Boolean Algebras, Springer-Verlag (1960).

Universidad Nacional del Sur
Bahía Blanca, Argentina.