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A POLYGENIC EXTENSION OF THE POLYNOMIALS OF BERNOULLI AND EULER

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SUMMARY. The polygenic ϕ - polynomials, β - polynomials, Bernoulli polynomials, η - polynomials and Euler polynomials are defined. Several results concerning these polynomials are obtained. The analogues of complementary argument theorem and Euler-MacLaurin theorem for polynomials are derived.

1. THE POLYGENIC ϕ - POLYNOMIALS. Consider the following equation:

|1.1| $$f_{p,q}(t,\bar{t})e^{azt+b(z\bar{t}+\bar{z}t)+c\bar{z}\bar{t}+g(t,\bar{t})}$$

$$= \sum_{0\leq m+n\leq\infty} \frac{(at+b\bar{t})^m\,(bt+c\bar{t})^n}{m!n!}\,\phi^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}\quad ,$$

where z, $\bar{z}$, t and $\bar{t}$ are four independent complex variables; a, b, c are three complex constants such that $b^2 - ac \neq 0$; $g(t,\bar{t})$ is an analytic polygenic function in the two independent complex variables t and $\bar{t}$.

It is remarked that $m \neq 0$, $p \neq 0$, iff $a \neq 0$, or $b \neq 0$, and $n \neq 0$, $q \neq 0$, iff $b \neq 0$, or $c \neq 0$.

Let $f_{p,q}(t,t)$ be an analytic function, with region of convergence given by $|t - t_o| < r_1$, and $|\bar{t} - \bar{t}_o| < s_1$. Further, let $e^{g(t,\bar{t})}$ have a region of convergence about the center $(t_o,\bar{t}_o)$ given by $|t - t_o| < r_2$, $|\bar{t} - \bar{t}_o| < s_2$. If $r=\min(r_1,r_2)$ and $s=\min(s_1,s_2)$, then it is seen that

$$f_{p,q}(t,\bar{t})e^{azt+b(z\bar{t}+\bar{z}t)+c\bar{z}\bar{t}+g(t,\bar{t})}\quad ,$$

has a region of convergence given by

|1.2| $$|t - t_o| < r\quad ,\quad |\bar{t} - \bar{t}_o| < s\quad .$$

It is recalled that, by the principle of permanence the coefficients of $(at + b\bar{t})^m\ (bt + c\bar{t})^n$ in the equation |1.1| can be compared in the region of convergence given by |1.2| .

In equation |1.1| , use the substitution

|1.3| $$u = at + b\bar{t}\quad ,\quad v = bt + c\bar{t}\quad ,$$

then equation |1.1| becomes

$$|1.4| \quad f_{p,q}(t,\bar{t})e^{azt+b(z\bar{t}+\bar{z}t)+c\bar{z}\bar{t}+g(t,\bar{t})}$$

$$= \sum_{0\leq m+n\leq\infty} \frac{u^m v^n}{m!n!} \phi^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\}$$

The expression $\phi^{p,q}_{m,n}\{(a+b)z \; ; \; (b+c)\bar{z}\}$, is defined to be the *polygenic* ϕ *- polynomial of total order* p+q *and total degree* m+n.

In equation |1.4| , let $z = 0$ and $\bar{z} = 0$, then

$$|1.5| \quad f_{p,q}(t,\bar{t})e^{g(t,\bar{t})}$$

$$= \sum_{0\leq m+n\leq\infty} \frac{u^m v^n}{m!n!} \phi^{p,q}_{m,n}$$

where $\phi^{p,q}_{m,n}$ are the ϕ *- numbers of total order* p+q and total degree m+n.

In equation |1.4| , let $z = z+w$ and $\bar{z} = \bar{z}+\bar{w}$, where w and $\bar{w}$ are two independent complex variables, and then equate the coefficients of $u^m v^n$. Here without loss of generality, it may be assumed that $m \geq n$. The result is symbolically written as

$$|1.6| \quad \phi^{p,q}_{m,n}\{(a+b)(z+w) \; ; \; (b+c)(\overline{z+w})\}$$

$$= (\phi^{p,q}\{(a+b)w \; ; \; (b+c)\bar{w}\} + z)^m \, (\phi^{p,q}\{(a+b)w \; ; \; (b+c)\bar{w}\} + \bar{z})^n ,$$

where $(\phi^{p,q}\{(a+b)w \; ; \; (b+c)\bar{w}\} + z)^m$ is expanded according to w, and $(\phi^{p,q}\{(a+b)w \; ; \; (b+c)\bar{w}\} + \bar{z})^n$ is expanded according to $\bar{w}$.

THEOREM 1.1. *The* ϕ *- polynomials for polygenic functions obey the following symbolic identity.*

$$|1.7| \quad \phi^{p,q}_{m,n}\{(a+b)z \; ; \; (b+c)\bar{z}\} = (\phi^{p,q} + z)^m \, (\phi^{p,q} + \bar{z})^n$$

This is obtained by letting $w = 0$ in |1.6|.

2. THE LAPLACEAN $\nabla^2 \phi^{p,q}_{m,n}$ OF A ϕ - POLYNOMIAL. Differentiate |1.7| with respect to z , then

$$|2.1| \quad \frac{\partial}{\partial z} \phi^{p,q}_{m,n}\{(a+b)z \; ; \; (b+c)\bar{z}\} = m(\phi^{p,q} + z)^{m-1}(\phi^{p,q} + \bar{z})^n \; .$$

Similarly ,

$$|2.2| \quad \frac{\partial}{\partial z}\,\phi^{p,q}_{m,n}\,\{(a+b)z\ ;\ (b+c)\bar{z}\} = n(\phi^{p,q} + z)^{m}\,(\phi^{p,q} + \bar{z})^{n-1}\ .$$

From the theory of polygenic functions, it is known that

$$\nabla^2 f(z) = 4\,\frac{\partial^2}{\partial z \partial \bar{z}}\ f(z)$$

Thus, the Laplacean of a ϕ - polynomial is given by

$$|2.3| \quad \nabla^2\ \phi^{p,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\} = 4mn(\phi^{p,q} + z)^{m-1}\,(\phi^{p,q} + \bar{z})^{n-1}\ .$$

THEOREM 2.1. *A ϕ - polynomial for polygenic functions is harmonic iff either* $m = 0$ *or* $n = 0$.

This follows directly from |2.3|.

From |2.1| and |2.2| , it is seen that

$$|2.4| \quad \int_d^3 \phi^{p,q}_{m,n}\,\{(a+b)z\ ;\ (b+c)\bar{z}\}dz = \frac{1}{m+1}\,\{(\phi^{p,q} + z)^{m+1}\,(\phi^{p,q} + \bar{z})^{n}$$

$$- (\phi^{p,q} + d)^{m+1}\,(\phi^{p,q} + \bar{z})\} \qquad ,$$

and

$$|2.5| \quad \int_{\bar{d}}^3 \phi^{p,q}_{m,n}\,\{(a+b)z\ ;\ (b+c)\bar{z}\}d\bar{z} = \frac{1}{n+1}\,\{(\phi^{p,q} + z)^{m}\,(\phi^{p,q} + \bar{z})^{n+1}$$

$$- (\phi^{p,q} + z)^{m}\,(\phi^{p,q} + \bar{d})^{n+1}\} \qquad .$$

By operating on both sides of the equation |1.4| by operators ${}^{z}\nabla$, ${}^{\bar{z}}\nabla$, ${}^{z}\Delta$, ${}^{\bar{z}}\Delta$, the following equations are established.

$$|2.6| \quad \frac{(u+1)}{2}\, f_{p,q}(t,\bar{t})\ e^{azt+b(z\bar{t}+\bar{z}t)+c\overline{zt}+g(t,\bar{t})}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^m v^n}{m!n!}\ {}^{z}\nabla\phi^{p,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\}$$

$$|2.7| \quad \frac{(v+1)}{2}\, f_{p,q}(t,\bar{t})\ e^{azt+b(\bar{z}t+z\bar{t})+c\overline{zt}+g(t,\bar{t})}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^m v^n}{m!n!}\ {}^{\bar{z}}\nabla\phi^{p,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\}$$

$|2.8|$ $$(u-1)\, f_{p,q}(t,\bar{t})\, e^{azt+b(\bar{z}t+z\bar{t})+c\overline{zt}+g(t,\bar{t})}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^n v^m}{m!n!}\, {}^{z}\Delta\phi^{p,q}_{m,n}\,\{(a+b)z\ ;\ (b+c)\bar{z}\} \qquad ,$$

$|2.9|$ $$(v-1)\, f_{p,q}(t,\bar{t})\, e^{azt+b(\bar{z}t+z\bar{t})+c\overline{zt}+g(t,\bar{t})}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^m v^n}{m!n!}\, {}^{\bar{z}}\Delta\phi^{p,q}_{m,n}\,\{(a+b)z\ ;\ (b+c)\bar{z}\} \qquad .$$

3. THE POLYGENIC β - POLYNOMIALS. In equation $|1.4|$, let

$|3.1|$ $$f_{p,q}(t,\bar{t}) = \frac{u^p v^q}{(e^u-1)^p(e^v-1)^q} \qquad ,$$

then $|1.4|$ becomes

$|3.2|$ $$\frac{u^p v^q}{(e^u-1)^p(e^v-1)^q}\, e^{azt+b(\bar{z}t+z\bar{t})+c\overline{zt}+g(t,\bar{t})}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^m v^n}{m!n!}\, \beta^{p,q}_{m,n}\,\{(a+b)z\ ;\ (b+c)\bar{z}\} \qquad ,$$

where $\beta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}$ is defined to be *the polygenic* β *- polynomial of total order* p+q *and total degree* m+n.

The β - polynomials can be shown to obey,

$|3.3|$ $${}^{z}\Delta\beta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = m\beta^{p-1,q}_{m-1,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} \quad ,$$

$|3.4|$ $${}^{\bar{z}}\Delta\beta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = n\beta^{p,q-1}_{m,n-1}\{(a+b)z\ ;\ (b+c)\bar{z}\} \quad , \text{and}$$

$|3.5|$ $${}^{\bar{z}}\Delta\ {}^{z}\Delta\ \beta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = mn\beta^{p-1,q-1}_{m-1,n-1}\{(a+b)z;(b+c)\bar{z}\} \ .$$

Equations $|3.3|$, $|3.4|$, $|3.5|$ may be symbolically written as

$|3.6|$ $$(\beta^{p,q} + \bar{z})^n\{(\beta^{p,q} + z + 1)^m - (\beta^{p,q} + z)^m\}$$

$$= m(\beta^{p-1,q} + z)^{m-1}\,(\beta^{p-1,q} + \bar{z})^n \qquad ,$$

$|3.7|$ $$(\beta^{p,q} + z)^m\{(\beta^{p,q} + \bar{z} + 1)^n - (\beta^{p,q} + \bar{z})^n\}$$

$$= n(\beta^{p-1,q} + z)^m \ (\beta^{p-1,q} + \bar{z})^{n-1}$$

and

|3.8| $$\{(\beta^{p,q} + z + 1)^m - (\beta^{p,q} + z)^m\}\{(\beta^{p,q} + \bar{z} + 1)^n - (\beta^{p,q} + \bar{z})^n\}$$

$$= m \cdot n(\beta^{p-1,q-1} + z)^{m-1} \ (\beta^{p-1,q-1} + \bar{z})^{n-1} \quad ,$$

respectively.

THEOREM 3.1. *Polygenic* β *- numbers obey the following recurrence relations.*

|3.9| $$\beta^{p,q}_{o,n} \{(\beta^{p,q} + 1)^m - \beta^{p,q}_{m,o}\} = m(\beta^{p-1,q}_{m-1,n}) \quad ,$$

|3.10| $$\beta^{p,q}_{m,o} \{(\beta^{p,q} + 1)^n - \beta^{p,q}_{0,n} = n(\beta^{p,q-1}_{m,n-1}) \quad ,$$

and

|3.11| $$\{(\beta^{p,q} + 1)^m - \beta^{p,q}_{m,0}\}\{(\beta^{p,q} + 1)^n - \beta^{p,q}_{o,n}\} = m \cdot n(\beta^{p-1,q-1}_{m-1,n-1}).$$

This is done by letting $z = 0$ in |3.6| , |3.7| , and |3.8| .

4. THE POLYGENIC BERNOULLI POLYNOMIALS. The polygenic Bernoulli polynomial $B^{p,q}_{m,n}\{(a+b)z ; (b+c)\bar{z}\}$, of total order p+q and total degree m+n , is obtained by letting $g(t,\bar{t}) \equiv 0$, in |3.2| . Thus

|4.1| $$\frac{u^p v^q}{(u-1)^p (v-1)^q} e^{azt+b(z\bar{t}+\bar{z}t)+c\bar{z}\bar{t}}$$

$$= \sum_{0 \le m+n \le \infty} \frac{u^m v^n}{m!n!} B^{p,q}_{m,n}\{(a+b)z ; (b+c)\bar{z}\} \quad ,$$

defines a Bernoulli polynomial.

As a Bernoulli polynomial is a ϕ polynomial and a β - polynomial , the following results are readily obtained.

|4.2| $$B^{p,q}_{m,n}\{(a+b)z ; (b+c)\bar{z}\} = (B^{p,q} + z)^m \ (B^{p,q} + \bar{z})^n \quad ,$$

where $B^{p,q}_{m,n}$ are *Bernoulli numbers* obtained by letting $z = \bar{z} = 0$ in |4.1| .

|4.3| $\frac{\partial}{\partial z} B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = m \, B^{p,q}_{m-1,n} \{(a+b)z \; ; \; (b+c)\bar{z}\}$,

|4.4| $\frac{\partial}{\partial \bar{z}} B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = n \, B^{p,q}_{m,n-1} \{(a+b)z \; ; \; (b+c)\bar{z}\}$,

|4.5| $\nabla^2 B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = 4 \, m \cdot n \, B^{p,q}_{m-1,n-1}\{(a+b)z ; (b+c)\bar{z}\}$,

|4.6| ${}^{z}\Delta \, B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = m \, B^{p-1,q}_{m-1,n} \{(a+b)z \; ; \; (b+c)\bar{z}\}$,

|4.7| ${}^{\bar{z}}\Delta \, B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = n \, B^{p,q-1}_{m,n-1} \{(a+b)z \; ; \; (b+c)\bar{z}\}$,

|4.8| ${}^{z}\Delta{}^{\bar{z}}\Delta B^{p,q}_{m,n}\{(a+b)z ; (b+c)\bar{z}\} = m \cdot n \, B^{p-1,q-1}_{m-1,n-1}\{(a+b)z ; (b+c)\bar{z}\}$,

|4.9| $(B^{p,q}_{o,n}) \, \{(B^{p,q} + 1)^m - (B^{p,q}_{m,o})\} = m \, (B^{p-1,q}_{m-1,n})$,

|4.10| $(B^{p,q}_{m,o}) \quad (B^{p,q} + 1)^n - (B^{p,q}_{o,n}) = n \, (B^{p,q-1}_{m,n-1})$.

A Bernoulli polynomial of total order zero and total degree m+n is defined by

|4.11| $B^{o}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = z^m \bar{z}^n$.

By repeated application of |4.6| , |4.7| , and |4.9| , the following result is established.

THEOREM 4.1. *If* $m \geq p$ *and* $n \geq q$, *then*

|4.12| ${}^{z}\Delta^{p} \, B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = m(m-1)\ldots(m-p+1)$

$B^{o,q}_{m-p,n} \{(a+b)z \; ; \; (b+c)\bar{z}\}$,

|4.13| ${}^{\bar{z}}\Delta^{q} \, B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = n(n-1)\ldots(n-q+1)$

$B^{p,q}_{m,n-q} \{(a+b)z \; ; \; (b+c)\bar{z}\}$,

and

|4.14| ${}^{z}\Delta^{p} \; {}^{\bar{z}}\Delta^{q} \, B^{p,q}_{m,n} \{(a+b)z \; ; \; (b+c)\bar{z}\} = \frac{m!n!}{(m-p)!\,(n-q)!} z^{m-p}\bar{z}^{n-q}$.

Consider now the integral

$$\int_z^{z+1} B^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}\,dz$$

$$= \frac{1}{m+1}\{B^{p,q}_{m+1,n}\{(a+b)(z+1);(b+c)\bar{z}\} - B^{p,q}_{m+1,n}\{(a+b)z;(b+c)\bar{z}\}]$$

$$= \frac{1}{m+1}\ {}^{z}\Delta\ B^{p,q}_{m+1,n}\ \{(a+b)z;(b+c)\bar{z}\}$$

$$= B^{p-1,q}_{m,n}\ \{(a+b)z;(b+c)\bar{z}\} \quad .$$

In particular let $z = \bar{z} = 0$, then

$$|4.15| \qquad \int_0^1 B^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}\ dz = B^{p-1,q}_{m,n} \quad .$$

Similarly,

$$|4.16| \qquad \int_0^1 B^{p,q}\{(a+b)z\ ;\ (b+c)\bar{z}\}\ d\bar{z} = B^{p,q-1}_{m,n} \quad .$$

5. THE COMPLEMENTARY ARGUMENT THEOREMS.

Consider $B^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}$, then

$$|5.1| \qquad \frac{u^p v^q}{(e^u-1)^p(e^v-1)^q}\ e^{a(p-z)t+b\{(p-z)\bar{t}+\bar{z}t\}+c\overline{zt}}$$

$$= \sum_{0\leq m+n\leq\infty} \frac{u^m v^n}{m!n!}\ B^{p,q}_{m,n}\ \{(a+b)(p-z)\ ;\ (b+c)\bar{z}\} \ .$$

Or,

$$\sum_{0\leq m+n\leq\infty} \frac{u^m v^n}{m!n!}\ B^{p,q}_{m,n}\ \{(a+b)\ (p-z)\ ;\ (b+c)\bar{z}\}$$

$$= \frac{(-u)^p v^q}{(e^{-u}-1)^p(e^v-1)^q}\ e^{-azt+b(-z\bar{t}+\bar{z}t)+c\overline{zt}}$$

$$= \sum_{0\leq m+n\leq\infty} \frac{(-u)^m\ v^n}{m!n!}\ B^{p,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\} \quad .$$

Thus the following result has been established.

THEOREM 5.1. *First Complementary Argument Theorem. A polygenic Bernoulli polynomial obeys,*

$$|5.2| \quad B^{p,q}_{m,n}\{(a+b)(p-z)\ ;\ (b+c)\bar{z}\} = (-1)^m\ B^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}\ .$$

The following two theorems are obtained similarly.

THEOREM 5.2. *Second Complementary Argument Theorem. The following identity is obeyed by a polygenic Bernoulli polynomial,*

$$|5.3| \quad B^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)(q-\bar{z})\} = (-1)^n\ B^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}\ .$$

And,

THEOREM 5.3. *Third Complementary Argument Theorem. For a polygenic Bernoulli polynomial, we have*

$$|5.4| \quad B^{p,q}_{m,n}\{(a+b)(p-z);(b+c)(q-\bar{z})\} = (-1)^{m+n}B^{p,q}_{m,n}\{(a+b)z;(b+c)\bar{z}\}\ .$$

In the next section polygenic η - polynomials are considered.

6. THE POLYGENIC η - POLYNOMIALS. In |1.4| , let

$$|6.1| \quad f^{p,q}_{m,n} = \frac{2^{p+q}}{(e^u+1)^p(e^u+1)^q}\ ,$$

then |1.4| becomes

$$|6.2| \quad \frac{2^{p+q}}{(e^u+1)^p(e^v+1)^q}\, e^{azt+b(z\bar{t}+\bar{z}t)+c\bar{z}\bar{t}+g(t,\bar{t})}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^m v^n}{m!n!}\ \eta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}\ ,$$

where $\eta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\}$, is defined to be the *polygenic* η - *polynomial of total order* p+q *and total degree* m+n.

By methods similar to those in section 3, the following equations are readily obtained.

|6.3| $\quad {}^{z}\nabla\eta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = \eta^{p-1,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\}$,

|6.4| $\quad {}^{\bar{z}}\nabla\eta^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = \eta^{p,q-1}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\}$,

|6.5| $\quad (\eta^{p,q}_{o,n})\ \{(\eta^{p,q}+1)^m + \eta^{p,q}_{m,o}\} = 2\eta^{p-1,q}_{m,n}$,

|6.6| $\quad (\eta^{p,q}_{m,o})\ \{(\eta^{p,q}+1)^n + \eta^{p,q}_{o,n}\} = 2\eta^{p,q-1}_{m,n}$,

and

|6.7| $\quad \{(\eta^{p,q}+1)^m + \eta^{p,q}_{m,o}\}\{(\eta^{p,q}+1)^n + \eta^{p,q}_{o,n}\} = 4\eta^{p-1,q-1}_{m,n}$,

where $\eta^{p,q}_{m,n}$ is , *the polygenic* η *- numbers of total order* p+q *and total degree* m+n , obtained by letting $z = \bar{z} = 0$ in |6.2| .

7. THE POLYGENIC EULER - POLYNOMIALS. In |6.2| let $g(t) \equiv 0$, then

|7.1| $$\frac{2^{p+q}}{(e^u+1)^p(e^v+1)^q}\ e^{azt+b(z\bar{t}+\bar{z}t)+c\overline{zt}}$$

$$= \sum_{0\le m+n\le\infty} \frac{u^m v^n}{m!n!}\ E^{p,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\}\ ,$$

where $E^{p,q}_{m,n}\ \{(a+b)z\ ;\ (b+c)\bar{z}\}$, is defined to be *a polygenic Euler polynomial of total order* p+q *and total degree* m+n.

The polygenic Euler number $E^{p,q}_{m,n}$, of total order p+q , and total degree m+n is defined by

|7.2| $\quad E^{p,q}_{m,n} = E^{p,q}_{m,n}\ \{(a+b)^p/2\ ;\ (b+c)^q/2\}$

Polygenic C *- numbers* $C^{p,q}_{m,n}$ are obtained by letting $z = \bar{z} = 0$ in |7.1| . Thus

|7.3| $\quad C^{p,q}_{m,n} = 2^{p+q}\ E^{p,q}_{m,n}\{0;0\}$.

The following results are readily obtained,

|7.4| $E^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = (1/2C^{p,q}+z)^m\ (1/2C^{p,q}+\bar{z})^n$,

|7.5| $\nabla^2 E^{p,q}_{m,n}\{(a+b)z;(b+c)\bar{z}\} = 4mnE^{p,q}_{m-1,n-1}\{(a+b)z;(b+c)\bar{z}\}$,

|7.6| $\nabla^z\nabla^{\bar{z}}E^{p,q}_{m,n}\{(a+b)z;(b+c)\bar{z}\} = E^{p-1,q-1}_{m,n}\{(a+b)z;(b+c)\bar{z}\}$.

Define

|7.7| $E^{o,o}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = z^m\ \bar{z}^n$

then

|7.8| $z^p_\nabla\ \bar{z}^q_\nabla\ E^{p,q}_{m,n}\{(a+b)z\ ;\ (b+c)\bar{z}\} = z^m\ \bar{z}^n$.

THEOREM 7.1. (I) *First Complementary Theorem.*

|7.9| $E^{p,q}_{m,n}\{(a+b)(p-z);(b+c)\bar{z}\} = (-1)^m E^{p,q}_{m,n}\{(a+b)z;(b+c)\bar{z}\}$,

(II) *Second Complementary Theorem.*

|7.10| $E^{p,q}_{m,n}\{(a+b)z;(b+c)(q-\bar{z})\} = (-1)^n E^{p,q}_{m,n}\{(a+b)z;(b+c)\bar{z}\}$,

(III) *Third Complementary Theorem.*

|7.11| $E^{p,q}_{m,n}\{(a+b)(p-z);(b+c)(q-\bar{z})\} = (-1)^{m+n}E^{p,q}_{m,n}\{(a+b)z;(b+c)\bar{z}\}$.

The proofs are similar to those in Section 5.

8. THE THEOREMS OF EULER-MACLAURIN FOR POLYGENIC POLYNOMIALS.

Define

|8.1| $B^{i,o}_{o,j} = B^{o,i}_{j,o} = 0$,

for all i and j, then it can be shown that

|8.2| $\frac{\partial}{\partial z} P(z,\bar{z}) = P\{\bar{z}+B^{1,0}\} - P\{z+w+B^{1,0}\ ;\ \bar{z}+B^{1,0}\}$,

where $P(z,\bar{z})$ is a polygenic polynomial of total degree m+n.

Hence

$$|8.3| \quad \frac{\partial}{\partial(z+w)} P(z+w;\bar{z}) = P\{z+w+B^{1,0}+1 \; ; \; \bar{z}+B^{1,0}\} - P\{z+w+B^{1,0};\bar{z}+B^{1,0}\}.$$

Now, if $m \geq n$, then by Taylor's theorem we have

$$P(z+B^{1,0} \; ; \; \bar{z}+B^{1,0}) = P(z;\bar{z}) + (B^{1,0} \frac{\partial}{\partial z} + B^{1,0} \frac{\partial}{\partial \bar{z}})^{(1)} P(z,\bar{z})$$

$$+ \ldots\ldots + \frac{(B^{1,0} \frac{\partial}{\partial z} + B^{1,0} \frac{\partial}{\partial \bar{z}})^{m}}{m!} P(z,\bar{z}) \quad ,$$

or

$$|8.4| \quad P(z+B^{1,0} \; ; \; \bar{z}+B^{1,0}) = P(z;\bar{z}) + B^{1,0}_{1,0} \frac{\partial}{\partial z} P(z,\bar{z}) + \ldots\ldots\ldots$$

$$+ B^{1,0}_{m,0} \frac{\partial^{m}}{\partial z^{m}} P(z,\bar{z}) \; .$$

Thus,

$$|8.5| \quad \frac{\partial}{\partial(z+w)} P(z+w \; ; \; \bar{z}) = {}^{z}\Delta P(z;\bar{z}) + B^{1,0}_{1,0} \{(a+b)w \; ; \; (b+c)\bar{w}\}$$

$${}^{z}\Delta\{\frac{\partial}{\partial z} P(z,\bar{z})\} + \ldots\ldots + \frac{1}{m!} B^{1,0}_{m,0}\{(a+b)w \; ; \; (b+c)\bar{w}\}^{z}\Delta\{\frac{\partial^{m}}{\partial z^{m}} P(z,\bar{z})\}.$$

The following result may now be established.

THEOREM 8.1 *The First Theorem of Euler-MacLaurin For Polygenic Polynomials.* *If* $P(z,\bar{z})$ *is a polygenic polynomial of total degree* $m+n$ *, where* $m \geq n$ *, then*

$$|8.6| \quad \frac{\partial}{\partial z} P(z,\bar{z}) = {}^{z}\Delta P(z,\bar{z}) + B^{1,0}_{1,0} \; {}^{z}\Delta\{\frac{\partial}{\partial z} P(z,\bar{z})\} + \ldots\ldots$$

$$+ \frac{1}{m!} B^{1,0}_{m,0} \; {}^{z}\Delta\{\frac{\partial^{m}}{\partial z^{m}} P(z;\bar{z})\} \quad .$$

This is done by letting $w = \bar{w} = 0$ in |8.5| .

THEOREM 8.2. *The Second Theorem of Euler-MacLaurin For Polygenic Polynomials. If* $P(z,\bar{z})$ *is a polygenic polynomial of total degree* $m+n$ *, with* $n \geq m$ *, then*

$$|8.7| \quad \frac{\partial}{\partial \bar{z}} P(z,\bar{z}) = {}^{\bar{z}}\Delta P(z,\bar{z}) + B^{0,1}_{0,1}\ {}^{\bar{z}}\Delta\{\frac{\partial}{\partial \bar{z}} P(z,\bar{z})\} + \cdots\cdots$$

$$+ \frac{1}{n!} B^{0,1}_{0,n}\ {}^{\bar{z}}\Delta\{\frac{\partial^n}{\partial \bar{z}^n} P(z,\bar{z})\} \quad .$$

The proof is similar to Theorem 8.1.

REFERENCES

{1} Milne Thomson, L.M. "The Calculus of Finite Differences", Cambridge University Press.

{2} Kasner, E. and DeCicco J., "Survey of Polygenic Functions", Scripta Matematica.

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