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NORMALITY AXIOMS

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1. Here we define and study some new separation axioms, which we call normality axioms. These are weaker than the corresponding regularity axioms, defined by Davis {1} and are connected with these and other separation axioms in a natural way. We also define and study a map which we shall call an *almost homeomorphism* , under which some of the normality and regularity axioms are preserved.

2. In the following definitions G , G_1 , G_2 etc. always denote nonempty open subsets; F , F_1 , F_2 etc. denote nonempty closed subsets; A , A_1 , A_2 etc. denote arbitrary nonempty subsets and x , x_1 , x_2 etc. denote arbitrary points of a given topological space (X,J). By $A_1 \longmapsto A_2$, we mean that there exist G_1 and G_2 such that $A_1 \subset G_1$, $A_2 \subset G_2$ and $A_1 \cap G_2 = \emptyset$, $A_2 \cap G_1 = \emptyset$. By $A_1 \longmapsto\!\!\!\!\!\!\longleftarrow A_2$ we mean that there exist G_1 and G_2 such that $A_1 \subset G_1$, $A_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$. By $A_1 \longmapsto\!\!\!\!\!\!\longleftarrow^{c} A_2$ we mean that there exist G_1 and G_2 such that $A_1 \subset G_1$, $A_2 \subset G_2$ and $\bar{G}_1 \cap \bar{G}_2 = \emptyset$.

DEFINITIONS. The space (X,J) will be said to satisfy the axiom

2.1 N_o iff $[x]'$ does not contain two nonempty disjoint closed sets.

2.2 N_1 iff $[\bar{x}_1] \cap [\bar{x}_2] = \emptyset$ implies $[\bar{x}_1] \longmapsto\!\!\!\!\!\!\longleftarrow [\bar{x}_2]$.

2.3 N_{1a} iff $[\bar{x}_1] \cap [\bar{x}_2] = \emptyset$ implies $[\bar{x}_1] \longmapsto\!\!\!\!\!\!\longleftarrow^{c} [\bar{x}_2]$.

2.4 N_2 iff $[\bar{x}] \cap F = \emptyset$ implies $|\bar{x}| \longmapsto\!\!\!\!\!\!\longleftarrow F$.

2.5 N_{2a} iff $[\bar{x}] \cap F = \emptyset$ implies that there exists a continuous function $f: (X,J) \longrightarrow [0,1]$ such that $f[\bar{x}] = [0]$, and $f[F] = [1]$.

REMARKS. We denote the axiom of normality by N_3 . The axiom R_{1a} has been defined in {2}*. R_3-axiom denotes $N_3 + R_o.R_{2a}$ is complete

* A topological space is said to satisfy the R_{1a}-axiom iff $[\bar{x}_1] \neq [\bar{x}_2]$ implies $[x_1] \longmapsto\!\!\!\!\!\!\longleftarrow^{c} [x_2]$.

regularity. Also, none of the normality axioms defined above implies R_o . This follows from the fact that N_3 does not imply R_o. {5, p. 100} .

3. THEOREM 3.1. *The following results hold for any topological space.*

(i) $N_3 \Longrightarrow N_{2a} \Longrightarrow N_2 \Longrightarrow N_{1a} \Longrightarrow N_1 \Longrightarrow N_o$.

(ii) $R_i \Longrightarrow N_i$, $i = 0,1,1a,2,2a,3$.

(iii) $N_i + T_1 = T_{i+1}$, $i=1,2,3$; $N_{1a} + T_1 = T_{2a}$, $N_{2a} + T_1 = T_{3a}$.

(iv) $N_i + R_o = R_i$, $i=1,1a,2,2a,3$.

(v) $N_o \not\Longrightarrow N_1 \not\Longrightarrow N_{1a} \not\Longrightarrow N_2 \not\Longrightarrow N_{2a} \not\Longrightarrow N_3$.

(vi) $R_3 \not\Longrightarrow T_o$.

Proof: (i) The proofs are obvious except for $N_1 \Longrightarrow N_o$. Suppose the space $(X,\mathcal{J})$ is not N_o . Then for some $x \varepsilon X$, $[x]'$ contains two nonempty disjoint closed sets F_1 and F_2 . Let $x_1 \varepsilon F_1, x_2 \varepsilon F_2$ then $[\bar{x}_1] \cap [\bar{x}_2] = \emptyset$ but $[\bar{x}_1]$ is not strongly separated from $[\bar{x}_2]$.

(ii) $R_o \Longrightarrow N_o$. If the space is not N_o then for some $x \varepsilon X$, $[x]'$ contains two nonempty disjoint closed sets F_1 and F_2 . Now $x \varepsilon \sim F_1$ but $[\bar{x}] \not\subset \sim F_1$. Hence the space is not R_o . The proofs for the other statements in this section are trivial.

(iii) Obvious.

(iv) Obvious.

(v) Obvious in view of (iii) and the fact that

$$T_1 \not\Longrightarrow T_2 \not\Longrightarrow T_{2a} \not\Longrightarrow T_3 \not\Longrightarrow T_{3a} \not\Longrightarrow T_4 .$$

(vi) Any indiscrete space containing more than one point is R_3 but not T_o .

REMARK 3.2. Y.C.Wu and S.M.Robinson {3} have given two axioms, which they call Strong T_o and Strong T_D , both of which are weaker then T_1-axiom and give the T_1-axiom in presence of N_3.

We give here an axiom, which we shall call the T_{c_o}-axiom, which is weaker than both the Strong T_o and the Strong T_D-axiom, is independent of the T_0-axiom and implies the T_1-axiom in presence of any one of the normality axioms (including the N_0-axiom).

DEFINITION 3.3. A topological space $(X,\mathcal{J})$ is said to satisfy the T_c-axiom iff for every $x \in X$, either $[x]' = \emptyset$ or $[x]'$ contains two nonempty disjoint closed sets.

The proof of the above assertions, which strengthen (iii), are easy.

4. We now give some theorems in which the N_1 and the N_2-axioms replace respectively the T_2-axiom and the axiom of regularity.

THEOREM 4.1. *A paracompact space is* N_2 *iff it is* N_1.

Proof: Let $[\bar{x}] \cap F = \emptyset$, where F is closed. For $y \in F$, $[\bar{y}] \cap [\bar{x}] = \emptyset$. Hence there exist open sets U_y and U_y^x such that $[\bar{y}] \subset U_y$, $[\bar{x}] \subset U_y^x$ and $U_y \cap U_y^x = \emptyset$. The family $\{\sim F\} \cup \{U_y : y \in F\}$ is an open cover of the space X and has a locally finite open refinement. The rest of the proof is similar to that of Lemma 2 in {6, p.154} .

THEOREM 4.2. *A paracompact space is* N_3 *iff it is* N_2.

Proof: Similar to that of Theorem 4.1.

THEOREM 4.3. *A Lindelöf space is* N_3 *iff it is* N_2.

Proof: Similar to the proof of the Theorem 7 in {6,p.139} .

THEOREM 4.4. *A space having σ-locally finite base is* N_3 *iff it is* N_2.

Proof: Similar to the proof of Lemma 1 in {6, p.168} .

5. It is well known that the N_3-axiom is preserved under closed and continuous mappings. We generalize this result partially.

DEFINITION 5.1. A closed and continuous mapping f of $(X,\mathcal{J})$ onto $(Y,\mathcal{U})$ is said to be an *almost homeomorphism* iff the inverse images of point closures are point closures. An almost homeomorphism becomes a homeomorphism if the domain space is T_1 .

THEOREM 5.2. *The normality axioms* N_0 , N_1 *and* N_2 *are preserved under almost homeomorphisms.*

Proof: For the N_o-axiom is trivial.

Now suppose $(X,\mathcal{J})$ is N_1. Let x_1, $x_2 \in Y$ be such that $[\bar{x}_1] \cap [\bar{x}_2] = \emptyset$ then $f^{-1}[\bar{x}_1]$ and $f^{-1}[\bar{x}_2]$ are disjoint point closures in $(X,\mathcal{J})$ and are therefore strongly separated by open sets U and V such that $f^{-1}[\bar{x}_1] \subset U$, $f^{-1}[\bar{x}_2] \subset V$. Then $\sim f\,[\sim U]$ and $\sim f\,[\sim V]$ are disjoint open neighborhoods of $[\bar{x}_1]$ and $[\bar{x}_2]$ respectively.

The proof for the N_2-axiom is similar.

THEOREM 5.3. *The regularity axiom* R_o *is preserved under almost homeomorphisms.*

Proof: Let $f: (X,\mathcal{J}) \longrightarrow (Y,\mathcal{U})$ be an almost homeomorphism and suppose $(X,\mathcal{J})$ is R_o. If x_1, $x_2 \in Y$ then $f^{-1}[\bar{x}_1]$ and $f^{-1}[\bar{x}_2]$ are point closures in $(X,\mathcal{J})$ and so either $f^{-1}[\bar{x}_1] = f^{-1}[\bar{x}_2]$ or $f^{-1}[\bar{x}_1] \cap f^{-1}[\bar{x}_2] = \emptyset$ This gives either $[\bar{x}_1] = [\bar{x}_2]$ or $[\bar{x}_1] \cap [\bar{x}_2] = \emptyset$.

COROLLARY. *The regularity axioms* R_1, R_2 *and* R_3 *are preserved under almost homeomorphisms.*

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