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ENRICHED SEMANTICS-STRUCTURE (META) ADJOINTNESS

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INTRODUCTION. In this paper, which has a purely theoretical aim and interest, we develop the rudiments of the general Structure-Semantics (meta) adjointness of Categorical Algebra. We do so by means of two different and parallel techniques, one using the concept of *Monads* (often called Triples, sometimes Standard Constructions, and some other times Triads), the other using the concept of *Theories*. We then relate (specifically) these techniques and prove them to be equivalent.

We do all this in the enriched context of a V-world, that is, our categories are V-categories and our functors are V-functors, where V is a given (fixed) closed (symmetrical monoidal closed) category

V-Monads have already been considered in many places in the literature, [1] [3] [6] [8] and probably more. In [8] a Semantics-Structure (meta) adjointness is established in which the Structure (meta) functor is only defined on V-functors which have a V-left adjoint. Here, in sections §1 and §2 we have reproduced parts of Chapter II of [3] , where we developed the Semantics-Structure (meta) adjointness by means of a technique relying heavily on the concept of Kan extensions. Structure is defined on the broader domain consisting of those V-functors for which the (right) Kan extensions of themselves along themselves exist . The Semantics-Structure (meta) adjointness is given by (essentially) a direct instance of the adjointness of this Kan extension.

V-Theories have not been considered yet in the literature. We introduce them here, and in doing so we have developed in detail the case in which the V-category involved is the base category V. We did so because of certain peculiarities (due to the presence of a V-codense cogenerator in V^{op}) which are not present in the more general case. These peculiarities allow us to stress the similarities with the first and original treatment of the subject (at least in its modern form), conceived by Lawere ([7]) in his work on Algebraic Theories in the category of sets. Here the concept of *cotensors* takes the role of *products*. A V-theory in V is a V-category with the same objects as V and in which any object is a cotensor of the unit object I. An algebra is then a cotensor preserving V-functor into V. We develop in sections §3 and §4 a Semantics-Structure (meta) adjointness in this context.

In section §5 we prove the equivalence referred to at the beginning of this introduction,and in doing so we take advantage of the (simple) equivalence between the structure (meta) functors to deduce the equivalence of the two semantics. In this way we avoid the need for the more complicated theorems of V-triplability and characterization of V-categories of algebras.

In section §6 we (briefly) indicate how to generalize these results to the general case of a V-theory in a V-category A, adopting in this case the V-versions of what have been considered as theories and algebras in [9] .

Throughout this paper (although it is not always necessary) we assume our base category V to be complete (all small inverse limits) and well powered. All the concepts and results (as well as the notation) of V-category theory used here can be found in [3]. All the logically illegitimate constructions, preceded here by the word (meta), become licit mathematical objects in any of the current foundations suited for category theory.

V-MONADS. Given a V-category A, recall that a *V-monad* in A is a V-endofunctor $A \xrightarrow{T} A$ together with a pair of V-natural transformations $TT \overset{\mu}{\Rightarrow} T$ and $id_A \overset{\eta}{\Rightarrow} T$, μ is associative and η is a left an right unit for μ in the sense that the following diagrams commute:

$$\begin{tikzcd} TTT \arrow[r, Rightarrow, "\mu T"] \arrow[d, Rightarrow, "T\mu"'] & TT \arrow[d, Rightarrow, "\mu"] \\ TT \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "id"'] & TT \arrow[d, Rightarrow, "\mu"] \\ & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, Rightarrow, "T\mu"] \arrow[dr, Rightarrow, "id"'] & TT \arrow[d, Rightarrow, "\mu"] \\ & T \end{tikzcd}$$

We write $\mathcal{T} = (T,\mu,\eta)$ and call μ the multiplication and η the unit. A *morphisms of monads* $\mathcal{T} \xrightarrow{\phi} \mathcal{T}'$ is a V-natural transformation $T \xRightarrow{\phi} T'$ such that the diagrams

$$\begin{tikzcd} TT \arrow[r, Rightarrow, "\phi\phi"] \arrow[d, Rightarrow, "\mu"'] & T'T' \arrow[d, Rightarrow, "\mu'"] \\ T \arrow[r, Rightarrow, "\phi"] & T' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[rr, Rightarrow, "\phi"] & & T' \\ & \mathrm{id}_A \arrow[ul, Rightarrow, "\eta"] \arrow[ur, Rightarrow, "\eta'"'] & \end{tikzcd} \quad \text{commute.}$$

V-monads in $\mathcal{A}$ with morphisms of monads between them form a (meta) category that we denote $\mathcal{M}o(\mathcal{A})$.

§1. SEMANTICS OF V-MONADS.

Given a V-monad $\mathcal{T} = (T,\mu,\eta)$, a $\mathcal{T}$-algebra is an object $A \in \mathcal{A}$ together with a $\mathcal{T}$-algebra structure, that is, a morphism $TA \xrightarrow{\alpha} A$, associative and for which ηA is a unit, in the sense that the diagrams:

$$\begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "\eta A"] \arrow[dr, "\mathrm{id}"'] & & TA \arrow[dl, "\alpha"] \\ & A & \end{tikzcd} \quad \text{commute.}$$

We write $\bar{A} = (A,\alpha)$ and call A the underlying object.

A morphism of algebras $\bar{A} \xrightarrow{f} \bar{B}$ is a map $A \xrightarrow{f} B$ in $\mathcal{A}$ such that the diagram

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd} \quad \text{commutes.}$$

$\mathcal{T}$-algebras and morphisms of algebras form a category $\mathcal{A}^{\mathcal{T}}$ provided with a functor $\mathcal{A}^{\mathcal{T}} \xrightarrow{U^{\mathcal{T}}} \mathcal{A}$, $U^{\mathcal{T}}\bar{A} = A$, $U^{\mathcal{T}}f = f$. $\mathcal{A}^{\mathcal{T}}$ is a V-category and $U^{\mathcal{T}}$ a V-functor by defining $\mathcal{A}^{\mathcal{T}}(\bar{A},\bar{B}) \xrightarrow{U^{\mathcal{T}}} \mathcal{A}(A,B)$ to be a V-equalizer of the pair of maps:

$$\begin{tikzcd}
A(A,B) \arrow[rr, "A(\alpha,\square)"] \arrow[dr, "T"'] & & A(TA,B) \\
& A(TA,TB) \arrow[ur, "A(\square,\beta)"'] &
\end{tikzcd}$$

U^T is obviously V-faithful and we call it the forgetful functor. The following proposition establishes the intuitive fact that V-functors $C \overset{\bar{S}}{\to} A^T$ are the same thing as V-functors $C \overset{\bar{S}}{\to} A$ together with a V-natural T-algebra structure $TS \overset{s}{\Rightarrow} S$.

PROPOSITION 1.1. *Given a V-functor $C \overset{S}{\to} A$, S admits a lifting into the T-algebras, that is, a V-functor $C \overset{\bar{S}}{\to} A^T$ such that $U^T\bar{S} = S$, if and only if there is an action of T on S, that is, a V-natural transformation $TS \overset{s}{\Rightarrow} S$ such that the diagrams:*

$$\begin{tikzcd}
TTS \arrow[r, Rightarrow, "Ts"] \arrow[d, Rightarrow, "\mu S"] & TS \arrow[d, Rightarrow, "s"] \\
TS \arrow[r, Rightarrow] & S
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
S \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "id"] & TS \arrow[d, Rightarrow, "s"] \\
& S
\end{tikzcd}
\quad \textit{commute.}$$

Proof. It is clear that in both cases we have the same data, i.e., a family of arrows $TSC \xrightarrow{sC} SC$, $C \in C$, and that the equations of T-algebra for each one of the sC are exactly the equations of T-action for s. Consider now the diagram:

$$\begin{tikzcd}
C(C,D) \arrow[r, "\bar{S}"] \arrow[dr, "S"'] & A^T(\bar{S}C,\bar{S}D) \arrow[d, "U^T"] \arrow[dl, phantom, "(1)"] & \\
& A(SC,SD) \arrow[r, "T"] \arrow[d, "A(sC,\square)"'] & A(TSC,TSD) \arrow[dl, "A(\square,sD)"] \arrow[dl, phantom, "(2)"'] \\
& A(TSC,SD) &
\end{tikzcd}$$

S equalizes the two maps of diagram (2) (that is, s is V-natural) if and only if there is a map $\bar{S}$ making diagram (1) commutative (that is, if there is a V-functor structure for the function $C \rightsquigarrow TSC \xrightarrow{s_C} SC$). This completes the proof. <u>Q.E.D</u>.

REMARK. Since $U^{\mathcal{T}}$, being V-faithful, reflects V-naturality, it follows that V-natural transformations $\bar{S} \overset{\phi}{\Rightarrow} \bar{H}$ are the same thing that V-natural transformations $S \overset{\phi}{\Rightarrow} H$ such that the diagram

$$\begin{tikzcd} TS \arrow[r, Rightarrow, "T\phi"] \arrow[d, Rightarrow, "s"'] \arrow[dr, phantom, "(1)"] & TH \arrow[d, Rightarrow, "h"] \\ S \arrow[r, Rightarrow, "\phi"] & H \end{tikzcd}$$ commutes.

The identity V-functor $A^{\mathcal{T}} \xrightarrow{id} A^{\mathcal{T}}$ is the lifting of $U^{\mathcal{T}}$, and so there is an action $TU^{\mathcal{T}} \overset{u}{\Rightarrow} U^{\mathcal{T}}$, $u\bar{A} = \alpha$.

Also, since $TT \overset{\mu}{\Rightarrow} T$ is an action of $\mathcal{T}$ on T, there is a lifting of T into the $\mathcal{T}$-algebras $A \xrightarrow{F^{\mathcal{T}}} A^{\mathcal{T}}$, $U^{\mathcal{T}}F^{\mathcal{T}} = T$, $F^{\mathcal{T}}A = (TTA \xrightarrow{\mu A} TA)$. It is clear that $uF^{\mathcal{T}} = \mu$. One of the equations in the definition of an action is exactly diagram (1) above for u, and so there is a V-natural transformation $F^{\mathcal{T}}U^{\mathcal{T}} \overset{\varepsilon}{\Rightarrow} id$, $U^{\mathcal{T}}\varepsilon = u$, that, together with $id \overset{\eta}{\Rightarrow} U^{\mathcal{T}}F^{\mathcal{T}}$, establishes the fact that $F^{\mathcal{T}}$ is V-left adjoint to $U^{\mathcal{T}}$.

The triangular equation $$\begin{tikzcd} & U^{\mathcal{T}}F^{\mathcal{T}}U^{\mathcal{T}} \arrow[dr, Rightarrow, "U^{\mathcal{T}}\varepsilon"] & \\ U^{\mathcal{T}} \arrow[ur, Rightarrow, "\eta U^{\mathcal{T}}"] \arrow[rr, Rightarrow, "id"] & & U^{\mathcal{T}} \end{tikzcd}$$ is the other

equation in the definition of action, and

$$\begin{tikzcd} & F^{\mathcal{T}}U^{\mathcal{T}}F^{\mathcal{T}} \arrow[dr, Rightarrow, "\varepsilon F^{\mathcal{T}}"] & \\ F^{\mathcal{T}} \arrow[ur, Rightarrow, "F^{\mathcal{T}}\eta"] \arrow[rr, Rightarrow, "id"] & & F^{\mathcal{T}} \end{tikzcd}$$ talked downstairs is $$\begin{tikzcd} & TT \arrow[dr, Rightarrow, "\mu"] & \\ T \arrow[ur, Rightarrow, "T\eta"] \arrow[rr, Rightarrow, "id"] & & T \end{tikzcd}$$

So we have just proven the following:

PROPOSITION 1.2. *The V-functor* U^T *has a V-left adjoint* F^T *and the V-monad* $(U^T F^T, U^T \varepsilon F^T, \eta)$ *is equal to* T. Q.E.D.

We call the V-functor F^T the *free functor* and a T-algebra of the form $F^T A$ a *free algebra*.

Given a morphism of monads $T' \xrightarrow{\phi} T$ it is trivial to see that $T'U^T \xRightarrow{\phi U^T} TU^T \xRightarrow{u} U^T$ is an action of T' on U^T, and so, there is a V-functor, denoted A^ϕ, which makes the triangle:

$$\begin{tikzcd} A^T \arrow[rr, "A^\phi"] \arrow[dr, "U^T"'] & & A^{T'} \arrow[dl, "U^{T'}"] \\ & A & \end{tikzcd}$$

commutative.

Given a composite $\psi \cdot \phi$, the V-functors $A^{\psi\cdot\phi}$ and $A^\phi \cdot A^\psi$ both correspond to the same action, and so, they are equal. The assignment of $A^T \xrightarrow{U^T} A$ to a V-monad T and of A^ϕ to a morphism of V-monads ϕ is then a contravariant (meta) functor between $\mathcal{M}o(A)$ and the (meta) comma category (V-Cat,A):

$$\mathcal{M}o(A)^{op} \xrightarrow{\mathcal{G}_m} (\text{V-Cat}, A) \;;$$

the *semantics* (meta) functor.

If $T \xrightarrow{\phi} T'$ is a morphism of V-Monads and $T'S \xRightarrow{s} S$ is an action of T' on S; the composite $TS \xRightarrow{\phi S} T'S \xRightarrow{s} S$ is an action of T on S, and it is not difficult to check the following:

PROPOSITION 1.3. *The one to one and onto correspondence*

$$\frac{S \to \mathcal{G}_m(T)}{TS \Rightarrow S} \qquad \textit{(Proposition 1.1)}$$

is natural in T *with respect to morphisms of V-monads. (Where the above arrow is understood to be a map in* (V-Cat,A) *and the above double arrow an action of* T *on* S*).*

§2. THE V-MONAD STRUCTURE

Given a V-functor $C \xrightarrow{S} A$, the right Kan extension of S along itself, $A \xrightarrow{\mathrm{Ran}_S(S)} A$, if it exists, has a structure of V-monad given by:

$$r_0 \ \frac{S \xRightarrow{id} S}{id \xRightarrow{\eta} \mathrm{Ran}_S(S)} \qquad \text{and}$$

$$r_0 \ \frac{\mathrm{Ran}_S(S)\ \mathrm{Ran}_S(S)\ S \xRightarrow{\mathrm{Ran}_S(S)\varepsilon\ r_0} \dfrac{\mathrm{Ran}_S(S) \xRightarrow{id} \mathrm{Ran}_S(S)}{\mathrm{Ran}_S(S)\ S \xRightarrow{\varepsilon} S}}{\mathrm{Ran}_S(S)\ \mathrm{Ran}_S(S) \xRightarrow{\mu} \mathrm{Ran}_S(S)}$$

(where r_0 is the one to one and onto correspondence which defines the right Kan extension).

We write $T_S = (\mathrm{Ran}_S(S), \mu, \eta)$ and call it the *codensity V-monad.* If it exists, we say that S *admits* a codensity V-monad. We say that S is *strongly tractable* if, furthermore, $\mathrm{Ran}_S(S)$ is preserved by the representables $A \xrightarrow{A(A,-)} V$. (Cf. [3], Proposition I.4.3: If A is cotensored, a right Kan extension with codomain A is preserved by the representables if and only if it is point-wise, that is, if and only if the Kan formula to compute it as a point-wise end of cotensors in A *can be used*).

A complete proof of the fact that the unit and multiplication defined above for $\mathrm{Ran}_S(S)$ actually define a V-monad as well as of the next two propositions is to be found in [3].

PROPOSITION 2.1. *Given any other* V-*monad* T *in* A, *actions of* T *on* S *and morphisms of* V-*monads* $T \to T_S$ *correspond to each other under* r_0*:*

$$r_0 \ \frac{TS \Longrightarrow S}{T \Longrightarrow \mathrm{Ran}_S(S)}$$

<u>Q.E.D.</u>

PROPOSITION 2.2. *If a* V-*functor* $B \xrightarrow{G} A$ *has a* V-*left adjoint* $A \xrightarrow{F} B$,

$\mathrm{id} \overset{\eta}{\Rightarrow} GF$, $FG \overset{\varepsilon}{\Rightarrow} \mathrm{id}$, *then it is strongly tractable and the codeneity V-monad is* $(GF, G\varepsilon F, \eta)$. *Furthermore,* $\mathrm{Ran}_G(G)$ *is preserved by any V-functor with domain* A. Q.E.D.

THEOREM I. *Given a V-functor* $C \overset{S}{\to} A$ *which admits a codensity V-monad, for every V-monad* $T \in \mathcal{M}(A)$, *there is, naturally in T, a one to one and onto correspondence between morphisms of V-monads* $T \to T_S$ *and V-functors* $C \to A^T$ *making the triangle*

$$\begin{tikzcd} C \arrow[rr] \arrow[dr, "S"'] & & A^T \arrow[dl, "U^T"] \\ & A & \end{tikzcd}$$

commutative, that is, maps $S \to \mathcal{G}_m(T)$ *in* (V-Cat,A). *As usual, we indicate this by* $\dfrac{S \to \mathcal{G}_m(T)}{T \to T_S}$

Proof. Immediate from Propositions 1.3 and 2.1. Q.E.D.

Let ${}_s\mathcal{G}_r$(V-Cat,A) be the full (meta) sub-category of (V-Cat,A) whose objects are the V-functors admitting a codensity V-monad. From propositions 1.2 and 2.2 we know that the semantics (meta) functor $\mathcal{G}_m$ takes its values in ${}_s\mathcal{G}_r$(V-Cat,A). The assignment of $T_S \in \mathcal{M}(A)$ to a V-functor $C \overset{S}{\to} A$ becomes then, by Theorem I, a contravariant (meta) functor, denoted $\breve{\mathcal{G}}_m$, in such a way that the one to one and onto corresponce (in Theorem I) is also natural in S . $\breve{\mathcal{G}}_m$ is then a left adjoint to semantics, and it is called *structure*.

$$\mathcal{M}(A)^{op} \underset{\breve{\mathcal{G}}_m}{\overset{\mathcal{G}_m}{\rightleftarrows}} {}_s\mathcal{G}_r(\text{V-Cat},A)$$

Given a V-functor $C \overset{S}{\to} A$ in ${}_s\mathcal{G}_r$(V-Cat,A), the codensity V-monad $T_S = \breve{\mathcal{G}}_m(S)$ is the structure V-monad of S.

Notice that the (meta) adjunction:

$$\frac{S \longrightarrow \mathcal{G}_m(T)}{T \longrightarrow \breve{\mathcal{G}}_m(S)}$$

is, essentially, just the one to one and onto correspondence which defines the right Kan extensions $\mathrm{Ran}_S(S)$.

It is immediate from Propositions 1.2 and 2.2 that the arrow $T \to \check{\mathcal{G}}_m \mathcal{G}_m(T)$ in $\mathcal{M}(A)$, $T \to T_{U^T}$, is the equality. That is, the codensity V-monad of U^T is T.

The arrow $S \to \mathcal{G}_m \check{\mathcal{G}}_m(S)$ in ${}_s\mathcal{G}_r(V\text{-Cat}, A)$

$$\begin{tikzcd} C \arrow[r, "\bar{S}"] \arrow[dr, "S"'] & A^{T_S} \arrow[d, "U^{T_S}"] \\ & A \end{tikzcd}$$

is given by the action $\mathrm{Ran}_S(S)\ S \xrightarrow{r_0(\mathrm{id})} S$.

The V-functor $\bar{S}$ is called the *semantical* comparison V-functor of S. When S has a V-left adjoint we have:

PROPOSITION 2.2. *Given any V-functor* $B \xrightarrow{G} A$ *with a V-left adjoint* $A \xrightarrow{F} B$, (ε, η): $F \dashv_V G$ *, the semantic comparison V-functor of* G, $B \xrightarrow{\bar{G}} A^{T_G}$ *is given by the action* $GFG \xRightarrow{G\varepsilon} G$ *(that is,* $G\varepsilon = r_0(\mathrm{id})$*)* *, and is unique making the following two triangles commutative:*

$$\begin{tikzcd} B \arrow[r, "\bar{G}"] \arrow[dr, "G"'] & A^{T_G} \arrow[d, "U^{T_G}"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\bar{G}"] & A^{T_G} \\ & A \arrow[ul, "F"] \arrow[u, "F^{T_G}"'] \end{tikzcd}$$

(a simple proof of this fact is given in [3], Proposition II.1.6).

Q.E.D.

V-THEORIES. By a *V-theory in V* we mean a pair (T, T), where T is a V-category whose objects are the objects of V (that we will write V^t when we think of them as belonging to T) and where $V(W,V) \xrightarrow{T} T(V^t, W^t)$ is a V-functor structure making the identity on objects a cotensor preserving V-functor $V^{op} \xrightarrow{T} T$.

We have then for each $V^t \in T$, $V^t = \bar{T}(V, I^t)$ (where $\bar{T}(V, I^t)$ is the cotensor (in T) of V with I^t), and hence, the V-objects of morphisms into I^t determine the whole V-structure of T. Specifically we have $\mathcal{T}(W^t,V^t) \overset{g}{\approx} \mathcal{V}(V,\mathcal{T}(W^t,I^t))$ (cotensoring isomorphism).

By a morphism of theories $(T,\mathcal{T}) \xrightarrow{\phi} (T',\mathcal{T}')$ we will understand a cotensor preserving V-functor $\mathcal{T} \xrightarrow{\phi} \mathcal{T}'$ sending I^t into $I^{t'}$; or, equivalently, any V-functor $\mathcal{T} \xrightarrow{\phi} \mathcal{T}'$ making the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\phi"] & & \mathcal{T}' \\ & \mathcal{V}^{op} \arrow[ul, "T"] \arrow[ur, "T'"'] & \end{tikzcd}$$

commutative.

V-theories in $\mathcal{V}$ with morphisms of theories between them form a (meta) category that we denote $\mathfrak{T}(\mathcal{V})$.

§3. SEMANTICS OF V-THEORIES.

Given a V-theory $(T,\mathcal{T})$, a $\mathcal{T}$-algebra is a cotensor preserving V-functor $\mathcal{T} \xrightarrow{\alpha} \mathcal{V}$. Since $V^t = \bar{\mathcal{T}}(V,I^t)$, we have $\alpha V^t = \mathcal{V}(V,\alpha(I^t))$ and so α on objects is completely characterized by its value at I^t. Also; the composite $\mathcal{V}^{op} \xrightarrow{\alpha\cdot T} \mathcal{V}$ is cotensor preserving, and hence, since I is a V-codense cogenerator of $\mathcal{V}^{op}$, it is representable: $\alpha\cdot T = \mathcal{V}(-,\alpha(I^t))$ (cf. [3], Theorem III.2.3). (α is cotensor preserving if and only if $\alpha\cdot T$ is cotensor preserving if and only if $\alpha\cdot T$ is representable).

We can then redefine a $\mathcal{T}$-algebra as being an object $A \in \mathcal{V}$ together with a $\mathcal{T}$-algebra structure, that is, maps $\mathcal{T}(V^t,W^t) \xrightarrow{\alpha} \mathcal{V}(\mathcal{V}(V,A),\mathcal{V}(W,A))$ giving a structure of V-functor $\mathcal{T} \xrightarrow{\alpha} \mathcal{V}$ to the function on objects $V^t \rightsquigarrow \mathcal{V}(V,A)$, and making the diagram:

$$(1) \quad \begin{tikzcd} \mathcal{T}(V^t,W^t) \arrow[rr, "\alpha"] & \arrow[d, phantom, "(1)"] & \mathcal{V}(\mathcal{V}(V,A),\mathcal{V}(W,A)) \\ & \mathcal{V}(W,V) \arrow[ul, "T"] \arrow[ur, "\mathcal{V}(-,A)"'] & \end{tikzcd}$$

commutative.

We write $\bar{A} = (A,\alpha)$ and call A the underlying object.

A morphism of algebras $\bar{A} \xrightarrow{f} \bar{B}$ is a V-natural transformation $\alpha \xRightarrow{f} \beta$. It is completely determined by its value at I^t and hence we can redefine a morphism of algebras as being a map $A \xrightarrow{f} B$ making the diagrams:

$$\begin{tikzcd}
\mathcal{T}(V^t,W^t) \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathcal{V}(\mathcal{V}(V^t,A),\mathcal{V}(W^t,A)) \arrow[d, "{\mathcal{V}(\Box,\mathcal{V}(\Box,f))}"] \\
\mathcal{V}(\mathcal{V}(V^t,B),\mathcal{V}(W^t,B)) \arrow[r, "{\mathcal{V}(\mathcal{V}(\Box,f),\Box)}"] & \mathcal{V}(\mathcal{V}(V^t,A),\mathcal{V}(W^t,B))
\end{tikzcd}$$

commutative.

$\mathcal{T}$-algebras and morphisms of algebras form a category $\mathcal{V}^{(\mathcal{T})}$ provided with a functor $\mathcal{V}^{(\mathcal{T})} \xrightarrow{U^{\mathcal{T}}} \mathcal{V}$, $U^{\mathcal{T}}\bar{A} = A$, $U^{\mathcal{T}}f = f$. $\mathcal{V}^{(\mathcal{T})}$ is a V-category and $U^{\mathcal{T}}$ a V-functor by defining $\mathcal{V}^{(\mathcal{T})}(\bar{A},\bar{B}) \xrightarrow{U^{\mathcal{T}}} \mathcal{V}(A,B)$ to be the I^t-projection of the (large) end:

$$\mathcal{V}^{(\mathcal{T})}(\bar{A},\bar{B}) = \int_{V^t} \mathcal{V}(\alpha V^t,\beta V^t) \xrightarrow{U^{\mathcal{T}}} \mathcal{V}(A,B) .$$

That the above end exists can be seen as follows:

Consider the diagram:

$$\begin{tikzcd}
 & & E \arrow[d] & & \\
 & & \mathcal{V}(A,B) \arrow[dll, "{\mathcal{V}(W,-)}"'] \arrow[drr, "{\mathcal{V}(V,-)}"] & & \\
\mathcal{V}(\mathcal{V}(W,A),\mathcal{V}(W,B)) \arrow[d, equal] \arrow[ddddrr, "f_2"'] & & (1)_{V,W} & & \mathcal{V}(\mathcal{V}(V,A),\mathcal{V}(V,B)) \arrow[d, equal] \arrow[ddddll, "f_3"] \\
\mathcal{V}(\alpha W^t,\beta W^t) \arrow[drr, "f_0"] & & & & \mathcal{V}(\alpha V^t,\beta V^t) \arrow[dll, "f_1"'] \\
(a) & & \mathcal{V}(\mathcal{T}(V^t,W^t),\mathcal{V}(\alpha V^t,\beta W^t)) \arrow[d, equal] & & (b) \\
 & & \mathcal{V}(\mathcal{T}(V^t,W^t),\mathcal{V}(\mathcal{V}(V,A),\mathcal{V}(W,B))) \arrow[d, "{\mathcal{V}(T,\Box)}"] & & \\
 & & \mathcal{V}(\mathcal{V}(W,V),\mathcal{V}(\mathcal{V}(V,A),\mathcal{V}(W,B))) & &
\end{tikzcd}$$

where the arrows f_0 f_1 f_2 and f_3 are the maps which correspond by adjointness to:

$$T(V^t,W^t) \xrightarrow{V(-,\beta W^t)\cdot\alpha} V(V(\alpha W^t,\beta W^t),V(\alpha V^t,\beta W^t))$$

$$T(V^t,W^t) \xrightarrow{\cdot V(\alpha V^t,-)\cdot\beta} V(V(\alpha V^t,\beta V^t),V(\alpha V^t,\beta W^t))$$

$$V(W,V) \xrightarrow{V(-,V(W,B))\cdot V(-,A)} V(V(V(W,A),V(W,B)),V(V(V,A),V(W,B)))$$

$$V(W,V) \xrightarrow{V(V(V,A),-)\cdot V(-,B)} V(V(V(V,A),V(V,B)),V(V(V,A),V(W,B)))$$

and where E is the intersection of all the equalizers of the two maps in diagrams $(1)_{V,W}$.

From diagram (1) (page 13) and the above definitions it is not difficult to see that diagrams (a) and (b) commute. This, together with the equation $V(A,B) = \int_V V(V(V,A),V(V,B))$ (cf. [3]) (V-Yoneda Lemma) easily implies that $E = \int_{V^t} V(\alpha V^t,\beta V^t)$.

U^T is the V-functor "evaluation at I^t", and from the above construction it is obvious that it is V-faithful. We call it the forgetful functor.

PROPOSITION 3.1. *The T-algebras* $T \xrightarrow{T(V^t,-)} V$ *are the values of a V-functor* $V \xrightarrow{F^T} V^{(T)}$, *V-left adjoint to* U^T.

Proof.

$$V^{(T)}(T(V^t,-),\alpha) \approx \alpha V^t = V(V,A)$$

where the above (V-natural in α) isomorphism is given by the V-Yoneda lemma. Q.E.D

We call the V-functor F^T the *free functor* and a T-algebra of the form $F^T V = (T(V^t,I^t),T(V^t,-))$ a *free algebra*.

Given a morphism of theories $(\mathrm{T}',\mathcal{T}') \overset{\phi}{\to} (\mathrm{T},\mathcal{T})$, it is clear that for any $\mathcal{T}$-algebra $\mathcal{T} \overset{\alpha}{\to} \mathcal{V}$, the composite $\mathcal{T}' \overset{\phi}{\to} \mathcal{T} \overset{\alpha}{\to} \mathcal{V}$ is a $\mathcal{T}'$-algebra. From the universal property of ends and the fact that $\mathrm{U}^{\mathcal{T}'}$ is V-faithful it is easy to see that this function between the objects of $\mathcal{V}^{(\mathcal{T})}$ and those of $\mathcal{V}^{(\mathcal{T}')}$ has a (unique) structure of V-functor , $\mathcal{V}^{\phi}$, making the diagram

$$\begin{tikzcd} \mathcal{V}^{(\mathcal{T})} \arrow[rr, "\mathcal{V}^{\phi}"] \arrow[dr, "\mathrm{U}^{\mathcal{T}}"'] & & \mathcal{V}^{(\mathcal{T}')} \arrow[dl, "\mathrm{U}^{\mathcal{T}'}"] \\ & \mathcal{V} & \end{tikzcd}$$

commutative. Again, it is completely straightforward to check the equation $\mathcal{V}^{\psi\cdot\phi} = \mathcal{V}^{\phi}\cdot\mathcal{V}^{\psi}$, and so, the assignment of $\mathcal{V}^{(\mathcal{T})} \xrightarrow{\mathrm{U}^{\mathcal{T}}} \mathcal{V}$ to a V-theory $(\mathrm{T},\mathcal{T})$ and of $\mathcal{V}^{\phi}$ to a morphism of V-theories is a contravariant (meta) functor between $\mathfrak{T}(\mathcal{V})$ and the (meta) comma category $(\text{V-Cat},\mathcal{V})$:

$$\mathfrak{T}(\mathcal{V})^{\mathrm{op}} \xrightarrow{\mathfrak{S}_{\mathrm{t}}} (\text{V-Cat},\mathcal{V})$$

the *semantics* (meta) functor.

§4. THE V-THEORY STRUCTURE.

Given a V-functor $\mathrm{C} \overset{\mathrm{S}}{\to} \mathcal{V}$; we will say that it is *tractable* if for any pair of objects $\mathrm{V},\mathrm{W} \in \mathcal{V}$, the end

$$\int_{\mathrm{C}} \mathcal{V}(\mathcal{V}(\mathrm{V},\mathrm{SC}),\mathcal{V}(\mathrm{W},\mathrm{SC})) \qquad \text{exists in } \mathcal{V}.$$

That is, if for any pair of objects $\mathrm{V},\mathrm{W} \in \mathcal{V}$, the class of V-natural transformations between $\mathcal{V}(\mathrm{V},\mathrm{S}(-))$ and $\mathcal{V}(\mathrm{W},\mathrm{S}(-))$ is a set, and furthermore, it is the underlying set of an object of $\mathcal{V}$, namely, the end displayed above.

There is no difficulty in checking that the objects of $\mathcal{V}$ together with the above end between them form a V-category, $\mathcal{T}_S$, the *clone of operations* of S;

$$\mathcal{T}_S(V^t,W^t) = \int_C \mathcal{V}(\mathcal{V}(V,SC),\mathcal{V}(W,SC)) \ .$$

The collection of maps (which is a V-natural family):

$$\mathcal{V}(W,V) \xrightarrow{\mathcal{V}(-,SC)} \mathcal{V}(\mathcal{V}(V,SC),\mathcal{V}(W,SC))$$

lifts into the end, providing a structure of (contravariant) V-functor to the identity map between objects:

$$\mathcal{V}^{op} \xrightarrow{T_S} \mathcal{T}_S \quad .$$

T_S has a V-left adjoint [putting W = I in the definition of tractable, it follows that $\int_C \mathcal{V}(\mathcal{V}(V,SC),SC) = Ran_S(S)(V)$ (see Proposition 5.1) exists, then, for any other W,

$\mathcal{V}(W, \int_C \mathcal{V}(\mathcal{V}(V,SC),SC)) = \mathcal{T}_S(V^t,W^t) = \mathcal{T}_S(V^t,T_S(W))$] and therefore it preserves cotensors. We have then that the pair $(T_S,\mathcal{T}_S)$ is a V-theory in $\mathcal{V}$, V-theory which we call "the *structure* of $C \overset{S}{\to} \mathcal{V}$ ".

PROPOSITION 4.1. *If a* V-*functor* $\mathcal{B} \overset{G}{\to} \mathcal{V}$ *has a* V-*left adjoint* $\mathcal{V} \overset{F}{\to} \mathcal{B}$, *then it is tractable and*

$$\mathcal{T}_G(V^t,W^t) \approx \mathcal{B}(FW,FV) \approx \mathcal{V}(W,GFV) \quad .$$

Proof. $\int_B \mathcal{V}(\mathcal{V}(V,GB),\mathcal{V}(W,GB)) \approx \int_B \mathcal{V}(\mathcal{B}(FV,B),\mathcal{B}(FW,B)) \approx \mathcal{B}(FW,FV)$

The second isomorphisms given by the V-Yoneda Lemma. Q.E.D.

THEOREM II. *Given a tractable* V-*functor* $C \overset{S}{\to} \mathcal{V}$, *there is a* V-*functor* $C \overset{\bar{S}}{\to} \mathcal{V}^{(\mathcal{T}_S)}$ *making the triangle*

$$\begin{tikzcd} C \arrow[r, "\bar{S}"] \arrow[dr, "S"'] & V^{(T_S)} \arrow[d, "U^{T_S}"] \\ & V \end{tikzcd} \quad (1) \qquad \textit{commutative}$$

nd such that given any other V-*theory* (T,*T*) *together with a* V-*func*
or $C \overset{G}{\to} V^{(T)}$ *making the triangle*

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[dr, "S"'] & V^{(T)} \arrow[d, "U^{T}"] \\ & V \end{tikzcd} \quad (2)$$

commutative, there is a unique morphism of theories $T \overset{\phi}{\to} T_S$ *making the triangle*

$$\begin{tikzcd} C \arrow[r, "\bar{S}"] \arrow[dr, "G"'] & V^{(T_S)} \arrow[d, "V^{\phi}"] \\ & V^{(T)} \end{tikzcd} \quad (3) \qquad \textit{commutative.}$$

Proof. For any $C \in C$ define $\bar{S}C \in V^{(T_S)}$, $\bar{S}C = (SC,\alpha_C)$ where
$T_S(V^t,W^t) \xrightarrow{\alpha_C} V(V(V,SC),V(W,SC))$ is the C-projection of the end.
By definition of T_S (page 17) ,

$$\begin{tikzcd} T_S(V^t,W^t) \arrow[rr, "\alpha_C"] & & V(V(V,SC),V(W,SC)) \\ & V(W,V) \arrow[ul, "T_S"] \arrow[ur, "V(-,SC)"'] & \end{tikzcd}$$

commutes, and so (SC,α_C) is a T_S-algebra. The collection of maps (which is a V-natural family)
$C(C,C') \overset{S}{\to} V(SC,SC') \xrightarrow{V(V,-)} V(V(V,SC),V(V,SC')) = V(\alpha_C(V^t),\alpha_{C'}(V^t))$
lift into the end $V^{(T_S)}(\bar{S}C,\bar{S}C')$, providing a structure of V-functor to $\bar{S}$ which (in particular) makes triangle (1) commutative.

Given $C \xrightarrow{G} V^{(T)}$, $GC = (SC,\gamma_C)$, then, there is a unique $T(V^t,W^t) \xrightarrow{\phi} T_S(V^t,W^t)$ such that for all $C \in C$, the diagram:

$$\begin{tikzcd} T(V^t,W^t) \arrow[r, "\phi"] \arrow[dr, "\gamma_C"'] \arrow[dr, phantom, shift left=3, "(4)"] & T_S(V^t,W^t) \arrow[d, "\alpha_C"] \\ & V(V(V,SC),V(W,SC)) \end{tikzcd} \qquad \text{commutes}$$

(recall that α_C was (by definition) the projection of the end). But the commutative diagrams (4) are exactly the equation $V^{\phi}\cdot\bar{S} = G$, that is, commutativity of triangle (3). <u>Q.E.D</u>.

Let $\mathfrak{T}_r(V\text{-Cat},V)$ be the full (meta) sub-category of $(V\text{-Cat},V)$ whose objects are the tractable V-functors. From Propositions 3.1 and 4.1 we know that the semantics (meta) functor $\mathfrak{G}_t$ takes its values in $\mathfrak{T}_r(V\text{-Cat},V)$. The assignment of $(T_S,T_S) \in \mathfrak{T}(V)$ to a V-functor $C \xrightarrow{S} V$ becomes then, by Theorem II, a contravariant (meta) functor, denoted $\check{\mathfrak{G}}_t$, left adjoint to semantics.

$$\mathfrak{T}(V)^{op} \underset{\check{\mathfrak{G}}_t}{\overset{\mathfrak{G}_t}{\rightleftarrows}} \mathfrak{T}_r(V\text{-Cat},V)$$

From the V-Yoneda Lemma and Propositions 3.1 and 4.1 it is clear that the arrow $(T,T) \to \check{\mathfrak{G}}_t\mathfrak{G}_t(T)$ in $\mathfrak{T}(V)$, $T \to T_{U^T}$, is the equality (or rather, an isomorphism). That is, the clone of operations of U^T is T.

The arrow $S \to \mathfrak{G}_t\check{\mathfrak{G}}_t(S)$ in $\mathfrak{T}_r(V\text{-Cat},V)$, $C \xrightarrow{\bar{S}} V^{(T_S)}$ has been constructed in Theorem II and it is called the semantical comparison V-functor.

§5. EQUIVALENCE BETWEEN THE V-MONAD AND THE V-THEORY TECHNIQUE OF PRODUCING A SEMANTICS-STRUCTURE (META) - ADJOINTNESS.

First let us check that the domain (meta) categories of the two structure (meta) functors coincide, that is, that they are both the same full (meta) sub-category of $(V\text{-Cat}, \mathcal{V})$.

PROPOSITION 5.1. *Given a* V-*functor* $C \overset{S}{\to} \mathcal{V}$, *then:* S *admits a codensity* V-*monad if and only if* S *is strongly tractable if and only if* S *is tractable.*

Proof. The first two statements are clearly equivalent since any right Kan extension with codomain $\mathcal{V}$ is pointwise. That tractable implies strongly tractable is easily seen by putting $W^t = I$ in the definition of tractable (page 16). The resulting end is just the Kan formula for pointwise computing of $\mathrm{Ran}_S(S)$. Vice-versa, assuming that $\mathrm{Ran}_S(S)$ exists, since the representables preserve it, for every $V \in \mathcal{V}$, $\mathrm{Ran}_S(\mathcal{V}(V,S(\,-\,)))$ exists, and, being with codomain $\mathcal{V}$, it is pointwise. Then, the pontwise Kan formula shows that S is tractable. <u>Q.E.D.</u>

In order to relate the domain (meta) categories of the two semantics (meta) functors it is in order to define the *Kleisly* V-category associated to a V-monad in $\mathcal{V}$ (cf [8]).

Recall that given a V-monad $T = (T,\mu,\eta)$, the objects of $\mathcal{V}$ with the following V-structure between them

$$K_T(W^t,V^t) \overset{(1)}{\underset{\text{def}}{=}} \mathcal{V}(W,TV)$$

constitute a V-category, K_T , the Kleisly V-category of T. The maps $\mathcal{V}(W,V) \xrightarrow{\mathcal{V}(\square,\eta V)} K_T(W^t,V^t)$ give a structure of V-functor $\mathcal{V} \overset{f^T}{\to} K_T$ to the identity between objects, which, just by definition has a V-right adjoint $K_T \overset{u^T}{\to} \mathcal{V}$ sending V^t into TV. The adjunction isomorphism is given by the *equality* (1). The V-monad associated to the adjoint pair f^T, $u^T = K_T(I^t,-)$ is clearly T again.

The pair $((f^T)^{op}, K_T^{op})$ is obviously a V-theory in $\mathcal{V}$, and there is no difficulty in seeing that the passage $\mathcal{M}o(\mathcal{V}) \overset{K\ell}{\to} \mathcal{T}(\mathcal{V})$,

$K\ell(T) = ((f^T)^{op}, K_T^{op})$ is a (meta) functor [the V-functor corresponding to a morphism of V-monads $T \overset{\phi}{\to} T'$ is given by

$K_T(W^t,V^t) \xrightarrow{V(W,\phi V)} K_{T'}(W^t,V^t)$] .

On the other hand, given any V-theory $(T,\mathcal{T})$, the V-functor $\mathcal{T} \xrightarrow{\mathcal{T}(-,I^t)} V^{op}$ is a V-left adjoint to $V^{op} \xrightarrow{\bar{T}} \mathcal{T}$ (the adjunction given by the cotensoring isomorphisms $\mathcal{T}(W^t,V^t) = \mathcal{T}(W^t,\bar{T}(V,I^t)) \overset{\sigma}{\approx}$ $\overset{\sigma}{\approx} V(V,\mathcal{T}(W^t,I^t))$. Hence, we rediscover T by means of the formula $T = \bar{T}(-,I^t)$.

The following formal manipulation,
$V(V,\mathcal{T}(\bar{T}(W,I^t),I^t)) \overset{\sigma}{\approx} \mathcal{T}(\bar{T}(W,I^t),\bar{T}(V,I^t)) = \mathcal{T}(W^t,V^t)$, proves that the Kleisly V-category associated to the V-monad determined by the V-adjoint pair σ: $\bar{T}(-,I^t) \dashv_V \mathcal{T}(-,I^t)$ is the dual of $\mathcal{T}$, while the commutativity of the diagram below

$$\begin{tikzcd}
V(W,V) \arrow[r, "V(\square{,}\eta V)"] \arrow[dr, "\bar{T}(-{,}I^t)"'] & V(W,\mathcal{T}(\bar{T}(V,I^t),I^t) \arrow[d, phantom, "\wr\ \sigma"] \\
& \mathcal{T}(\bar{T}(V,I^t),\bar{T}(W,I^t))
\end{tikzcd}$$

[where $\eta V = \sigma_0(\bar{T}(V,I^t) \xrightarrow{id} \bar{T}(V,I^t))$, $V \xrightarrow{\eta V} \mathcal{T}(\bar{T}(V,I^t),I^t)$ (ηV could properly be called "the Gelfand transformation")] shows that the de finition of $f^{\mathcal{T}}$ produces in this case the V-functor T. Therefore we wholly recover the starting V-theory $(T,\mathcal{T})$. This implies that the assignment of the V-monad $\mathcal{T}(\bar{T}(-,I^t),I^t)$ to a given V-theory $(T,\mathcal{T})$ is actually a (meta) functor $\mathcal{G}(V) \xrightarrow{\ell K} \mathcal{M}(V)$ which together with the (me ta) functor $K\ell$ establishes an equivalence of (meta) categories bet ween $\mathcal{G}(V)$ and $\mathcal{M}(V)$. (Notice that this V-monad sends an object $W \in V$ into the V-object of W-ary operations).

THEOREM III. *There is an equivalence of (meta) categories*

$$\mathcal{G}(V) \underset{K\ell}{\overset{\ell K}{\rightleftarrows}} \mathcal{M}(V)$$

between the (meta) categories of V-*Theories in* V *and of* V-*Monads in* V *such that the following diagram:*

$$\begin{tikzcd}
\mathcal{T}(V)^{op} \arrow[d, shift right, "\ell K"'] \arrow[drr, shift left, "\mathcal{G}_t"] & & \\
& & \mathcal{T}_r = {}_s\mathcal{T}_r(V\text{-Cat}, V) \arrow[ull, shift left, "\breve{\mathcal{G}}_t"] \arrow[dll, shift left, "\breve{\mathcal{G}}_m"] \\
\mathcal{M}(V)^{op} \arrow[uu, shift right, "K\ell"'] \arrow[urr, shift left, "\mathcal{G}_m"] & &
\end{tikzcd}$$

commutes up to natural isomorphisms.

1) $\ell K \cdot \breve{\mathcal{G}}_t \approx \breve{\mathcal{G}}_m \quad , \quad K\ell \cdot \breve{\mathcal{G}}_m \approx \breve{\mathcal{G}}_t$

2) $\mathcal{G}_m \cdot \ell K \approx \mathcal{G}_t \quad , \quad \mathcal{G}_t \cdot K\ell \approx \mathcal{G}_m$

where the (meta) functors $\mathcal{G}_?$ $\breve{\mathcal{G}}_?$ *are the respective semantics-structure (meta) adjointness.*

Proof. The (meta) functors ℓK and $K\ell$ have been defined and proven to be an equivalence in the considerations made before the statement of the Theorem .

Since the semantics (meta) functors are adjoints to the structure (meta) functors (Theorems I and II), it will be enough to prove equations 1) That is: Given a tractable (equivalently, strongly tractable, Proposition 5.1) V-functor $C \overset{S}{\to} V$:

a) The codensity V-monad of S is the V-monad associated by ℓK to the clone of operations of S. This is clear just by the definitions involved (the assignment $V^t \rightsquigarrow Ran_S(S)V$ was seen to be a V-left adjoint to $V^{op} \xrightarrow{T_S} T_S$, see page 17).

b) The clone of operations of S is the dual of the Kleisly V-category associated to the codensity V-monad of S. Again, this is clear just by the definitions involved

$$(T_S(V^t, W^t) = \int_C V(V(V,SC), V(W,SC)) = Ran_S(V(W,S(-))(V) =$$

$$= V(W, Ran_S(S)(V)) = K_{T_S}(W^t, V^t)).$$

These observations complete the proof of the theorem. Q.E.D.

Notice that equations 2) in the theorem just proven mean that given any V-monad, the V-category of algebras is V-isomorphic to the V-category of algebras over the dual of its Kleisly V-category and, vice-versa, given any V-theory, the V-category of algebras is V-isomorphic to the V-category of algebras over the associated V-monad.

§6. REMARKS ABOUT V-THEORIES IN A GENERAL V-CATEGORY A.

We have observed that for any V-theory in V, $V^{op} \xrightarrow{T} T$, the V-functor T has a V-left adjoint. This is ultimately due to the fact that $I \in V$ is a V-codense cogenerator of V^{op}: Also, for the same reason, given any T-algebra $T \xrightarrow{\alpha} V$, the composite $V^{op} \xrightarrow{T} T \xrightarrow{\alpha} V$ is a representable V-functor (cf.[3], Theorem III. 2.3).

We can define then a V-theory in A as a V-functor $A^{op} \xrightarrow{T} T$, bijection in objects *and* having a V-left adjoint. Similarly, a T-algebra as a V-functor $T \xrightarrow{\alpha} V$ *such* that $A^{op} \xrightarrow{T} T \xrightarrow{\alpha} V$ is representable (cf.[9]). All of section §3 of this paper can then be carried over with no great difficulty. In particular, the T-algebras form a V-category and the forgetful V-functor (sending α into the represening object of $\alpha \cdot T$) has a V-left adjoint (which sends $A \in A$ into $T(A^t,-)$, which is a T-algebra since the composite $T(A^t,-)\cdot T$ is represented by $\check{T} A^t \in A$ ($\check{T}$ the V-left adjoint to T)). We obtain in this way a Semantics (meta) functor which takes its values in the (meta) sub-category of (V-Cat,A) of *strongly* tractable V-functors. On the other hand, any strongly tractable V-functor $C \xrightarrow{S} A$ is tractable (but not vice-versa)(exactly the same proof given in Proposition 5.1 applies to this general case), and hence we can apply word by word the Structure(meta)functor construction developed in section §4 to strongly tractable V-functors $C \xrightarrow{S} A$. In this case,the clone of operations of S, $A^{op} \xrightarrow{T_S} T_S$ is such that T_S has a V-left adjoint (sending $A^t \in T_S$ into $Ran_S(S)(A)$) and hence it is a V-theory in A according to our definition above. We obtain in this way a Semantics-Structure (meta) adjointness between V-theories in A (having a V-left adjoint) and strongly tractable V-functors into A, which, is completely equivalent to the Semantics-Structure (meta) adjointness developed in sections §1 and §2. Theorem III with A in place of V holds exactly.

If we do not require a V-theory in A to be such that $A^{op} \xrightarrow{T} T$ has a V-left adjoint, then some new kind of phenomena appears which makes the situation different than in the case of V-theories in V.

A T-algebra is defined in the same way, i.e., any V-functor $T \xrightarrow{\alpha} V$ such that $\alpha \cdot T$ is representable. T algebras form a V-category with a forgetful V-functor which now in general will not have a V-left adjoint (the V-functors $T \xrightarrow{T(A^t,-)} V$ are not T-algebras since $T(A^t,-)\cdot T$ is not representable any more). This forgetful V-functor, however, is still tractable (but not strongly tractable), and we obtain a Semantics (meta) functor which takes its values in the (meta) sub-category of (V-Cat,A) of tractable V-functors. The structure (meta) functor construction (§4) applies exactly (in this case, the clone of operations $A^{op} \xrightarrow{T_S} T_S$ will not have a V-left adjoint) and we obtain a Semantics-Structure (meta) adjointness between V-theories in A and tractable V-functors into A which contains the previous ones.

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