Revista de la
Unión Matemática Argentina
Volumen 25, 1970.

UNIQUENESS OF DISTRIBUTIONS

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1. INTRODUCTION. Let f be a distribution on an open set O in R^n and $\phi(x)$ an infinitely differentiable function with support in $|x| \leq 1$ and integral equal to 1. Let $\phi_t(x) = t^{-n}\phi(x/t)$ and consider the function

$$(1) \qquad F(x,t) = (f*\phi_t)(x) \quad , \qquad t>0 \quad ,$$

i.e. the convolution of f and ϕ_t, which is well defined for x at distance larger than t from the complement O' of O. We are interested in the extent to which $\lim_{t\to 0} F(x,t)$ determines f. If O coincides with R^n we may replace the condition on the support of ϕ by one less restrictive but sufficient to insure the existence of $F(x,t)$. For example, if f is the Fourier transform of a function g and the integral of $|g|$ over a sphere of radius r does not grow faster than a fixed power of r, we can take ϕ to be the Fourier transform of $e^{-|x|}$ and $F(x,t)$ becomes the Abel means of the Fourier integral of g. Thus in this case our problem becomes that of uniqueness of Abel summable Fourier integrals. We shall consider two modes of approach of the point (x,t) to the hyperplane $t = 0$, namely, non-tangential and normal. Results on non-tangential limits are relatively simple and require no conditions on f. Furthermore, they have interesting applications to the theory of partial differential equations. On the other hand, results on normal limits require some restrictions on f. The ones we present here are closely related to the work of V. Shapiro (see [3]).

2. STATEMENT OF RESULTS.

We shall always assume that f and ϕ are real an we shall associate with them the functions

$$\overline{F}(x) = \overline{\lim}\, F(y,t) \ , \ F_*(x) = \underline{\lim}\, F(y,t) \ , \ t>0 \ , \ t \to 0 \ , \ |x-y| \leq t$$

* This research was partly supported by NSF GP - 23563

where the upper limit $\bar{F}$ is taken through *all* positive values of t and the lower limit F_* only through a fixed but otherwise arbitrary sequence tending to zero. These are non-tangential limits. We also consider the normal limits

$$f_*(x) = \underline{\lim}\, F(x,t) \;,\; \underline{f}(x) = \underline{\lim}\, F(x,t) \;,\; t>0 \;,\; t \to 0$$

where, again, $\underline{f}$ is taken through all positive values of t and f_* through an arbitrary but fixed sequence tending to zero.

THEOREM 1. *Suppose that $\phi(x) \geqslant 0$ has support in $|x| \leqslant 1$ and integral equal to 1. Suppose that h is locally integrable in O and ν is a measure which is finite on compact subsets of O and such that $\bar{F}(x) \geqslant h(x)$ almost everywhere in O and either $F_*(x) > -\infty$ or*

(2) $$|F(y,t)| - F(y,t) = o[(\nu_* \phi_t)(y)] \quad , \; |x-y| \leqslant t$$

as t tends to zero through the sequence defining F_*, for all x in O. Then f-h coincides with a measure in O.

Here, and throughout this paper, "measure" means a non-negative measure, and to avoid tedious repetitions we shall always assume that a measure in an open set is finite on its compact subsets.

THEOREM 2. *If in the preceding theorem we have $\phi(0) > 0$, $h(x) = 0$ almost everywhere and $\underline{f}(x) \leqslant 0$ everywhere in O, then $f = 0$.*

THEOREM 3. *The assertions of the preceding theorems remain valid if condition (2) is replaced by the following: the set of points where $F_*(x) = -\infty$ is a countable union of sets E such that if χ_t denotes the characteristic function of the set of points at distance less than t from E then*

$$t^{-\alpha} \int \chi_t(x)\, dx \quad , \quad n \geqslant \alpha > 0$$

remains bounded as t tends to zero, and for $|x-y| \leqslant t$ and $x \in E$, $F(y,t)t^{\alpha}$ tends to zero uniformly as t tends to zero through the sequence defining F_*. The number α may depend on the set E.

This result has interesting applications to the theory of linear partial differential equations. They are extensions to general equations or systems of familiar facts about analytic functions such as the theorem of Besicovitch (see [2] , chapter V, th. 5.3) or the theorem of Looman-Menchoff. Although the result stated below, when specialized to the case of the Cauchy-Riemann equations, neither implies nor is implied by the theorem of Looman-Menchoff, it is of the same general character. For some recent results in the same direction see also [4]. In order to state our results we must recall some generalized notions of differentiability introduced in [1]. If h is a locally integrable function in an open set O we shall say that h belongs to $T^1_\alpha(x_0)$, $\alpha \geqslant -n$, if there exists a polynomial P of degree less than or equal to α, $P = 0$ if $\alpha < 0$, such that

$$t^{-n-\alpha} \| \chi_t(x-x_0) \, [h(x) - P(x-x_0)] \|_1$$

where $\chi_t(x)$ is the characteristic function of the sphere of radius t with center at the origin, remains bounded as $t \to 0$. If on the other hand this expression tends to zero as $t \to 0$, we say that h belongs to $t^1_\alpha(x_0)$. If h belongs to $t^1_\alpha(x_0)$ for all x_0 in a set E with the coefficients of the corresponding polynomials bounded in E and the preceding expression tending to zero uniformly, we say that h belongs to $t^1_\alpha(E)$. When h belongs to $t^1_\alpha(x_0)$ the coefficients of P are uniquely determined and one defines the generalized derivatives of h of orders less than or equal to α at x_0 as the corresponding derivatives of P at the origin. Thus if L is a differential operator of order less than or equal to α, $(Lh)(x_0)$ can be defined accordingly. These notions can be generalized in the obvious way to the case of vector-valued functions.

THEOREM 4. *Let* L *be a system of linear partial differential operators of order m with coefficients of terms of order k in C^{m-k}. Suppose the vector-valued function h belongs to $t^1_m(x)$ for almost all x in an open set O and satisfies the equation $L\,h = 0$ there. Suppose that at the remaining points x either h belongs to $T^1_m(x)$ or else x belongs to a countable union of sets* E *such that $h \in t^1_{m-\alpha}(E)$, $0 < \alpha = \alpha_E \leqslant n$, and, if $\chi_t(x)$ denotes the characteristic function of the set of points at distance less than t from* E*,*

$$t^{\alpha} \int \chi_t(x) \; dx$$

is finite and remains bounded as $t \to 0$. Then h is a weak solution of the system $L\,h = 0$.

We may complete this statement with the observation that if L is elliptic determined or overdetermined and has infinitely differentia ble coefficients then h coincides almost everywhere with an infinitely differentiable function.

We pass now to the results on normal limits. They are contained in the following

THEOREM 5. *Let f be a distribution with compact support and $\phi(x)$ an infinitely differentiable function such that $\phi(x) = \eta(|x|)$ where $\eta(t)$ is a non-decreasing function of t such that $\eta^{(k)}(t) = O(t^{-n-k-\varepsilon})$ as $t \to \infty$. Suppose that the Fourier transform $\hat{f}$ of f satisfies the following condition*

$$a) \qquad \int_{|z|<r} |\hat{f}(z)|\, dz = o(r^2) \quad as \quad r \to \infty$$

Let the function h be locally integrable in the open set O and suppose that $\overline{F}(x) \geqslant h(x)$ almost everywhere and $\underline{f}(x) > -\infty$ everywhere in O. Then $f-h$ coincides with a measure in O.

The conclusion remains valid if the condition that $\underline{f}(x) > -\infty$ is replaced by the following weaker one. There exist a closed non-dense subset C of O, a measure ν in O and a non-increasing function $\lambda(t) > 1$ in $(0,1)$ with $\lambda(t) \to \infty$ as $t \to 0$, such that, if χ denote the characteristic function of the sphere $|x| \leqslant 1$, then

$$\underline{\lim}_{t\to 0}\ t^2\ (\nu * \chi_t)(x) > 0$$

at every x of C, and if x is a point of $O-C$ such that $\underline{f}(x) = -\infty$ then

$$(3) \qquad \begin{aligned} &\int_0^1 t\ \lambda(t)\ [|F(x,t)| - F(x,t)]\ dt < \infty \quad and \\ &|F(x,t)| - F(x,t) = o\,[(\nu * \phi_t)(x)] \quad as \quad t \to 0 \end{aligned}$$

THEOREM 6. *Let f and ϕ be as in the preceding theorem. Suppose that the Fourier transform $\hat{f}$ of f satisfies the condition that*

$$b) \qquad \hat{f}(z)(1+|z|^2)^{-r} \in L^q\ , \ with\ rq < 1\ and\ \ 1 < q \leqslant 2\ , \ or$$

$$r = q = 1\ .$$

Let h be a locally integrable function and ν a measure in an open *set 0. Suppose that* $\overline{F}(x) \geq h(x)$ *almost everywhere in 0 and either* $\underline{f}(x) > -\infty$ *or*

$$|F(x,t)| - F(x,t) = o[(\nu_* \phi_t)(x)]$$

as $t \to 0$ *for all x in 0. Then f-h coincides with a measure in 0.*

THEOREM 7. *Suppose that under the assumptions of either of the two preceding theorems we have* $h(x) = 0$ *and* $\underline{f}(x) \leq 0$ *everywhere in 0. Then* $f = 0$ *in 0.*

We note that condition *a)* in theorem 5 is closely related to the condition on the coefficients in the theorem on uniqueness of Abel summable Fourier series of Verblunsky-Shapiro (see [3]). We shall see that, as in the case of Fourier series, it cannot be replaced by the weaker condition $O(r^2)$. This will be shown in the last section where we also give an example illustrating the limits of possible improvements of theorem 6 by exhibiting an f such that $\hat{f}(z) = O(|z|^{-(n-3)/2})$ as $|z| \to \infty$ and that $F(x,t) \to 0$ as $t \to 0$ for all x.

3. We start with some lemmas which will be used in the proof of our results.

LEMMA 1. *Let* F_* *be defined by letting t tend to zero through a sequence S. Suppose that* $F_*(x) > -\infty$ *for all x in an open set 0. Let C be closed and such that* $C \cap 0$ *is non-empty. Then there exists an open subset* 0_1 *of 0 with* $C \cap 0_1$ *non-empty and such that* $F(y,t) \geq -N > -\infty$ *in the set* $|y-x| \leq t$, $x \in C \cap 0_1$, $t \in S$.

Proof. Let $\underline{F}(x) = \inf F(y,t)$, $|y-x| \leq t$, $t \in S$. Then $\underline{F}$ is upper semicontinuous and everywhere finite in 0. Consider the sets $\{\underline{F}(x) \geq -k$, $x \in C \cap 0\}$, $k = 1,2,\ldots$. They are relatively closed in $C \cap 0$ and their union is $C \cap 0$. Since $C \cap 0$ is of the second category in itself, one of these sets contains a non-empty relatively open subset $C \cap 0_1$ of $C \cap 0$. This proves the lemma.

LEMMA 2. *Suppose that* $\underline{f}(x) > -\infty$ *for all x in an open set 0. Let C be closed and such that* $C \cap 0$ *is non-empty. Then there exists an open subset* 0_1 *of 0 with* $C \cap 0_1$ *non-empty and such that* $F(x,t) \geq -N >$

$> -\infty$ *for* x *in* $C \cap O_1$.

The proof of this is almost identical to that of the preceding lemma and is left to the reader.

LEMMA 3. *Let* f *be a distribution in an open set* O *and* $\phi \geqslant 0$ *have support in* $|x| < 1$. *Suppose that* $F(x,t) \geqslant -N > -\infty$ *for* $x \in O$, $t \in S$ *and* $t < \delta$, *and let* $\overline{F}(x) \geqslant 0$ *almost everywhere in* O. *Then* f *coincides with a measure in* O.

Proof. Let $\phi(x) = \tilde{\phi}(-x)$ and $\zeta(x) \geqslant 0$ be infinitely differentiable and supported in O. Then

$$\int F(x,t)\zeta(x)\, dx = \int (f_*\phi_t)(x)\zeta(x)\, dx = f[(\zeta_*\tilde{\phi}_t)]$$

Since $F(x,t) \geqslant -N$ for $x \in O$ and $t \in S$, and since $\zeta_*\tilde{\phi}_t$ and all its derivatives converge uniformly as $t \to 0$, if we let $t \to 0$ through S we will have

$$f(\zeta) = \lim f[(\zeta_*\tilde{\phi}_t)] \geqslant \underline{\lim} \int F(x,t)\zeta(x)\, dx \geqslant -N \int \zeta(x)\, dx$$

Thus the distribution f+N is such that $(f+N)(\zeta) \geqslant 0$ for every $\zeta \geqslant 0$ with support in O and therefore coincides with a measure μ_1 in O. Let now g be the Radon-Nikodym derivative of μ_1 with respect to Lebesgue measure and ν its singular part. Then

$$\int \phi_t(x-y)\, d\nu(y) + \int g(y)\phi_t(x-y)\, dy = F(x,t) + N$$

Now, at almost all x_0 the derivative $\lim_{t\to 0} \nu(S_t)/|S_t|$, where S_t is the sphere with center at x_0 and radius t, exists and is equal to zero, and at such points the first integral above tends to zero as $x \to x_0$ and $t \to 0$ with $|x-x_0| < t$. On the other hand, at every Lebesgue point x_0 of g the second integral above tends to $g(x_0)$ as $x \to x_0$ and $t \to 0$ with $|x-x_0| < t$. Thus we have

$$g(x) = \overline{F}(x) + N \geqslant N$$

almost everywhere in O. This shows that $\mu_1 - Ndx$ is still non-negative whence it follows that f coincides with the measure $\mu = \mu_1 - Ndx$ in O.

LEMMA 4. *Let* f *and* ϕ *be as in theorem 5. Suppose that* $F(x,t) \geqslant -N$

for x in an open set 0 and $t \in S$ and let $\overline{F}(x) \geqslant 0$ almost everywhere in 0. Then f coincides with a measure in 0.

Proof. The argument used in proving the preceding lemma applies to the present case with only minor changes. We first observe that on account of the properties of ϕ, if g is a distribution with compact support then $g * \phi_t \to 0$ as $t \to 0$ in the complement of the support of g. Then, as in the previous lemma, we show that $f + N = \mu_1$ in 0. Given an open subset 0_1 of 0 and an infinitely differentiable function η which is equal to 1 in 0_1 and vanishes outside 0 we will have that

$$(\eta\mu_1 * \phi_t)(x) - F(x,t) \to 0$$

as $t \to 0$ for all x in 0_1, and arguing with $\eta\mu_1$ as we did above with μ_1 it will follow that $\eta\mu_1 - N dx$ is non-negative in 0_1. Since 0_1 is an arbitrary open set regularly contained in 0, the desired result will follow.

LEMMA 5. *Let f and ϕ be as in theorem 5. Suppose that f+N coincides with a measure in an open set 0 where $\overline{F}(x) \geqslant 0$ holds almost everywhere. Then f itself coincides with a measure in 0.*

This was shown in the second part of the proof of the preceding Lemma.

Proof of theorem 1. At first we shall assume that $h = 0$ and $\nu = 0$, i.e. that $\overline{F}(x) \geqslant 0$ almost everywhere and $F_*(x) > -\infty$ everywhere in 0. Then according to lemma 1 every open subset of 0 contains a neighborhood where $F(x,t)$ is bounded below uniformly for $t \in S$ and where consequently according to lemma 3, f coincides with a measure. Now, if f coincides with a measure in every set of a family of open sets, it coincides with a measure in its union. Thus there exists a maximal open subset 0_1 of 0 where f coincides with a measure. Now, suppose that 0_1 is a proper subset of 0. Then according to lemma 1 there exists an open set 0_2, $0_1 \subset 0_2 \subset 0$, containing 0_1 properly and a number N such that for $y \in 0_2 - 0_1$, $|x-y| \leqslant t$, $t \in S$ we have $F(x,t) \geqslant -N$. Let now 0_3 be the set of points of 0_2 at distance greater than ε from its complement. If ε is sufficiently small, then 0_3 contains points not in 0_1. Consider now $F(x,t)$ with $x \in 0_3$ and $t \in S$, $t < \varepsilon$. If x is at distance greater than t from the complement of 0_1, since ϕ_t is non-negative and has support in $|x| < t$

and since f coincides with a measure in O_1 we have

$$F(x,t) = (f_*\phi_t)(x) \geqslant 0$$

If the distance between x and the complement of O_1 is less than or equal to t, then there exists a y in O_2-O_1 with $|x-y| \leqslant t$ and we have $F(x,t) \geqslant -N$. Consequently $F(x,t)$ is bounded below in O_3 for $t \in S$ and $t < \varepsilon$ and according to lemma 3, f coincides with a measure in O_3. But O_3 is not contained in O_1, and thus O_1 is not maximal, a contradiction. Hence O_1 coincides with O, and the theorem is established in this special case.

In the general case, given a large integer N and $\varepsilon > 0$ we let g be the distribution defined by $g = f-h_N + \varepsilon\nu$, where $h_N(x) = h(x)$ if $h(x) \leqslant N$ and $h_N(x) = N$ if $h(x) > N$. Then, as readily seen, we have $\overline{G}(x) \geqslant 0$ almost everywhere in O and $G_*(x) > -\infty$ everywhere in O. Hence g coincides with a measure in O and for every testing function η, $\eta \geqslant 0$, we have

$$g(\eta) = f(\eta) - \int h_N\eta \, dx + \varepsilon \int \eta \, d\nu \geqslant 0 \quad .$$

Letting $N \to +\infty$ and $\varepsilon \to 0$, we find that

$$f(\eta) - \int h \, \eta \, dx \geqslant 0$$

whence the desired conclusion follows.

Proof of theorem 2. Since $h(x) \geqslant 0$ almost everywhere in O , according to theorem 1 our distribution f coincides with a measure μ in O. Suppose that $\mu \neq 0$. Then there exists at least one point x_0 such that, if S_t denotes the sphere with center at x_0 and radius t, $\underline{\lim}\, t^{-n} \mu(S_t) > 0$ as $t \to 0$. Now, if $\phi(x) > \varepsilon$ for $|x| < \varepsilon$, since $\phi \geqslant 0$, we have $F(x_0,t) \geqslant t^{-n} \varepsilon\mu(S_{t\varepsilon})$, and consequently $\underline{f}(x_0) > 0$, a contradiction. Hence we must have $\mu = 0$, and our assertion is established.

Proof of theorem 3. We start with the observation that if F_* is redefined by making t tend to zero through an arbitrary subsequence of the one originally used to define F_*, the hypotheses of the theorem will still be satisfied and the conclusion will be proved if we show that a proper choice of the subsequence will imply the existence of a measure ν satisfying (2). Let

$$|F(x,t)|\,t^{\alpha_m} \leqslant \varepsilon_m(t) \to 0$$

where $|x-y| \leqslant t$, $x \in E_m$ and t tends to zero through the sequence defining F_*. Let us select a subsequence t_k of this such that $\sum_k \varepsilon_m^{1/2}(t_k) \leqslant M_m < \infty$ for all m. If $\chi_t^m(x)$ denotes the characteristic function of the set of points at distance less than $2t$ from E_m , let

$$t^{-\alpha_m} \int \chi_t^m(x)\, dx \leqslant N_m \quad , \quad t \leqslant 1 \quad .$$

Consider now the function

$$g(x) = \sum_{k,m.} \varepsilon_m^{1/2}(t_k)\; 2^{-m} \chi_{t_k}^m(x)\; t_k^{-\alpha_m}\; M_m^{-1} N_m^{-1}$$

Then if $x \in E_m$, $|x-y| \leqslant t = t_k$ we have

$$(g_* \phi_t)(y) \geqslant \varepsilon_m^{1/2}(t_k)\; 2^{-m}\; t_k^{-\alpha_m}\; M_m^{-1}\; N_m^{-1} \int \chi_{t_k}^m(z)\; \phi_{t_k}(y-z)\; dz \;\geqslant$$

$$\geqslant M_m^{-1}\; N_m^{-1}\; |F(y,t_k)|\; \varepsilon_m^{-1/2}(t_k)$$

which shows that for $x \in \cup\, E_m$, $|x-y| \leqslant t_k$ we have

$$F(y,t_k) = o\,[(g_* \phi_{t_k})(y)]$$

as $t_k \to 0$. Since g is clearly an integrable function, its indefinite integral gives the desired measure ν.

Proof of theorem 4. For simplicity we shall restrict ourselves to the case of a single differential operator of the form

$$Lh = \sum_\beta \left(\frac{\partial}{\partial x}\right)^\beta (a_\beta h)$$

Furthermore, and without loss of generality, we may assume that the coefficients a_β belong to $C^{m-|\beta|}$ in the closure of $\mathcal{O}$. Let now Q be a polynomial and $R = h - Q$. Consider the distribution $f = Lh$ and let ϕ be as in theorem 1. Then

$$F(x,t) = (f_* \phi_t)(x) = (Lh_* \phi_t)(x) = (LQ_* \phi_t)(x) + (LR_* \phi_t)(x) \quad .$$

The first term in the last expression is dominated by a multiple of a bound for the coefficients of Q, and for the second we have

$$|(LR_*\phi_t)(x)| = \left| \int \sum a_\beta(y)\ R(y)\ (\tfrac{\partial}{\partial x})^\beta\ \phi_t(x-y)\ dy\ \right| \leqslant$$

$$\leqslant c\ t^{-n-m}\int_{|x-y|<2t} |R(y)|\,dy$$

where c is a constant. Thus if h belongs to $t^1_{m-\alpha}(E)$, $\alpha > 0$, and $x_0 \in E$, by setting $Q(x) = P(x-x_0)$, we see that for $|x-x_0| \leqslant t$

$$|F(x,t)| \leqslant c + o(t^{-\alpha})$$

as $t \to 0$, and consequently $F(x,t)\ t^\alpha \to 0$ as $t \to 0$ with $|x-x_0| \leqslant t$ uniformly for $x_0 \in E$. If on the other hand $h \in T^1_m(x_0)$ or $t^1_m(x_0)$, we see that $F(x,t)$ remains bounded or tends to zero as $t \to 0$ with $|x-x_0| \leqslant t$. Thus, according to theorem 3, f coincides with a measure in O. Since the same conclusion holds for $-f$, it follows that $f = 0$ in O and thus h is a weak solution of the system $Lh = 0$ in O.

The proof of theorems 5 and 6 will require a few more lemmas.

LEMMA 6. *If f is a distribution with compact support* C, *there exists a g with compact support such that $\Delta g - f$ is infinitely differentiable and vanishes in a neighborhood of the support of f.*

Proof. Let $\zeta(x)$ be an infinitely differentiable spherically symmetric function with support in $|x| < 1$ and integral equal to 1. Let $l(x)$ be the fundamental solution of Laplace's equation and $h = l - (l_*\zeta)$. Then $h = \delta - \zeta$, where δ denotes Dirac's delta function, and since $l(x)$ is harmonic in $|x| > 0$ and $\zeta(x)$ is spherically symmetric and has support in $|x| < 1$, the mean value property of harmonic functions implies that $h(x) = 0$ for $|x| > 1$. Consider now the function $g_1 = (f_*\zeta)_*l$. Evidently, g_1 is infinitely differentiable and $g_1 = (f_*\zeta)$ has compact support. Now let $g = (h_*f) + \psi g_1$, where $\psi \in C^\infty_0$ and $\psi = 1$ in a neighborhood of C

$$\Delta g = \Delta\ (h_*f) + \Delta\ (\psi g_1) = (f_*\Delta\ h) + \Delta\ g_1 + \Delta[(1-\psi)g_1] =$$

$$= f - \Delta[(1-\psi)g_1]\ .$$

Thus, $\Delta g - f = \Delta|(1-\psi)g_1|$ vanishes in the complement of the support of $(1-\psi)$, i.e. in a neighborhood of C. Furthermore, since ψ and g_1 are infinitely differentiable, the same holds for $\Delta g - f$.

LEMMA 7. *Let f and ϕ be as in theorem 5. Then $(f*\phi_t)(x) \to 0$ as $t \to 0$, uniformly in the complement of any neighborhood of the support of f.*

Proof. According to our assumptions the derivatives of order k of $\phi(x)$ are of the order $|x|^{-n-k-\varepsilon}$ as $|x| \to \infty$. Thus, as $t \to 0$, ϕ_t and all its derivatives tend uniformly to zero in $|x| > \delta > 0$. This clearly implies the assertion of the lemma.

LEMMA 8. *Let $\eta(t)$ be as in theorem 5 and let $\psi(x) = -\eta'(|x|)|x|^{-1}$. Then if g is a distribution with compact support $t^{-2}(g*\psi_t)(x) \to 0$ as $t \to 0$, uniformly outside any neighborhood of the support of g.*

Proof. Again, according to our assumptions, ψ and its derivatives of order k are of the orders $|x|^{-n-2-\varepsilon}$ and $|x|^{-n-k-2-\varepsilon}$ respectively as $|x| \to \infty$. This implies that $t^{-2}\psi_t(x)$ and all its derivatives converge to zero uniformly in $|x| > \phi > 0$, whence the desired conclusion follows.

LEMMA 9. *Let ϕ be as in theorem 5 and ψ as in lemma 8. Let g be a distribution with compact support, $f = \Delta g$,*

$$F(x,t) = (f*\phi_t)(x) \quad , \quad G(x,t) = (g*\psi_t)(x) \quad .$$

Then

$$G(x,t_2) - G(x,t_1) = \int_{t_1}^{t_2} sF(x,s)\, ds \quad .$$

Proof. One merely has to verify that

$$\frac{d}{dt}\psi_t = t\,\Delta\,\phi_t$$

which implies that

$$\frac{d}{dt}G(x,t) = \frac{d}{dt}(g*\psi_t) = (g*\frac{d}{dt}\psi_t) = t(g*\Delta\,\phi_t) =$$

$$= t(\Delta\, g*\phi_t) = t(f*\phi_t) = tF(x,t)$$

LEMMA 10. *Let g be a distribution with compact support. Suppose that g coincides with a function $\bar{g}$ in an open set O. Let $\bar{g}$ be upper semicontinuous in O and* $\lim_{t\to 0} (\bar{g}*\psi_t)(x) = \bar{g}(x) \geq -\infty$ *for all x in O, where ψ is the function in lemma 8 normalized so as to have integral equal to* 1. *Then if $f = \Delta g$ and $\underline{f}(x) > 0$ in O, $\bar{g}$ is subharmonic in O.*

Proof. We must prove that if $x_0 \in O$ and S is a sphere contained in O with center at x_0 then

$$\bar{g}(x_0) \leq |S|^{-1} \int_S \bar{g}(x)\, d\sigma$$

where $|S|$ is the surface area of S and $d\sigma$ is the area element. To show this we let B be the closed ball with boundary S and $\bar{g}_1 = \bar{g}$ in B and $\bar{g}_1 = 0$ in the complement B' of B. Let h be continuous in B harmonic in the interior of B and vanish in B'. Suppose that $\bar{g}_1 \leq h$ on S. Since $\bar{g}_1 - h$ is upper semicontinuous in B, it takes a maximum M at a point x_1 in B. Suppose that x_1 is in the interior of B, at distance $\varepsilon > 0$ from B'. Since $g = \bar{g} = \bar{g}_1$ in the interior of B, if $t_1 < t$ and t is sufficiently small, according to lemmas 8 and 9 and the fact that $\underline{f}(x_1) > 0$ we will have

$$(\bar{g}_1 * \psi_t)(x_1) - (\bar{g} * \psi_{t_1})(x_1) = G(x_1, t) - G(x_1, t_1) + o(t^2) =$$

$$= c \int_{t_1}^{t} s F(x_1, s)\, ds + o(t^2) \geq \frac{c}{4} \underline{f}(x_1)(t^2 - t_1^2) + o(t^2)$$

and letting $t_1 \to 0$ we obtain

$$(\bar{g}_1 * \psi_t)(x_1) - \bar{g}(x_1) \geq \frac{c}{4} \underline{f}(x_1) t^2 + o(t^2) .$$

On the other hand, since ψ is spherically symmetric and h is harmonic in the interior of B, we have

$$(h * \psi_t)(x_1) - h(x_1) = \int [h(y) - h(x_1)] \psi_t(x_1 - y)\, dy =$$

$$= \int_{|x_1 - y| > \varepsilon} [h(y) - h(x_1)]\, \psi_t(x_1 - y)\, dy = o(t^2)$$

Finally, since $\psi \geq 0$ and $\bar{g}_1 - h \leq M$ in B and vanishes in B', we have

$$(\bar{g}_1 - h) * \psi_t](x_1) \leq M \int_B \psi_t(x_1 - y)\, dy = M - M \int_{B'} \psi_t(x_1 - y)\, dy = M + o(t^2).$$

Combining these estimates we obtain

$$M \geq [(\overline{g}_1 - h)*\psi_t](x_1) + o(t^2) \geq \overline{g}(x_1) - h(x_1) + \frac{c}{4}\underline{f}(x_1)t^2 + o(t^2) =$$

$$= M + \frac{c}{4}\underline{f}(x_1)t^2 + o(t^2)$$

that is

$$0 \geq \frac{c}{4}\underline{f}(x_1)t^2 + o(t^2)$$

which is impossible, since $\underline{f}(x_1) > 0$. Thus the maximum occurs on S, and since $\overline{g}_1 \leq h$ on S, we have $\overline{g}_1 - h \leq 0$ in B. Thus

$$\overline{g}(x_0) = \overline{g}_1(x_0) = |S|^{-1}\int_S h(x)d\sigma.$$

Since this holds for any h with $h \geq \overline{g}$ on S, the desired conclusion follows.

LEMMA 11. *Let g be a distribution with compact support and ψ the function in lemma 8 normalized so as to have integral equal to* 1. *Suppose that as $t \to 0$, $(g*\psi_t)(x) \to \overline{g}(x) \geq -\infty$ for all x in an open set O, where the function $\overline{g}(x)$ is upper semicontinuous and locally integrable in O. Let $f = \Delta g$ and $\overline{F}(x) \geq 0$ almost everywhere and $\underline{f}(x) > -\infty$ everywhere in O. Then if g coincides with $\overline{g}$ in O, f coincides with a measure in O.*

Proof. We use the well known fact that an upper semicontinuous locally integrable function $\overline{g}$ is subharmonic in an open set O if and only if $\Delta\overline{g}$ coincides with a measure there. Let O_1 be the largest open subset of O in which f coincides with a measure. Then, since $\phi \geq 0$, lemma 7 implies that $\underline{f}(x) \geq 0$ in O_1. If O_1 is a proper subset of O, then, according to lemma 2, there exists an open subset O_2 of O, containing O_1 as a proper subset, such that $\underline{f} > -N < 0$ in $O_2 - O_1$. Let $\chi(x)$ be the characteristic function of the set O_2 and let $g_1 = g + \frac{N}{2n}\chi(x)|x|^2$. Then from lemma 8 and the fact that $\Delta g_1 = f + N$ it follows that g_1 satisfies the conditions of lemma 10 in O_2 and therefore it coincides with a subharmonic function and $\Delta g_1 = f + N$ coincides with a measure in O_2. But then, since $\overline{F}(x) \geq 0$ almost everywhere in O, by lemma 5 we conclude that f coincides with a measure in O_2. This contradicts the assumed maximality of O_1. Thus O_1 cannot be a proper subset of O and the proof of the lemma is complete.

LEMMA 12. *Let g be a distribution with compact support and let* $f = \Delta g$ *satisfy condition a) in theorem 5. Let* $\zeta(x) \geqslant 0$ *be spherically symmetric, infinitely differentiable, with support in* $|x| < 1$, *and such that* $\int \zeta\, dx = \int \psi\, dx$, *where* ψ *is as in lemma 8. Then*

$$|(g * \zeta_t)(x) - (g * \psi_t)(y)| \to 0$$

as $t \to 0$, *uniformly in x and y, provided that* $|x-y| \leqslant t$.

Proof. On account of the properties of ζ and ψ we have the inequalities

$$\hat{\zeta}(z) \leqslant c(1+|z|)^{-1}, \quad \hat{\psi}(z) \leqslant c(1+|z|)^{-1}$$

where c is a constant. Consequently, if $|x-y| \leqslant t$, then

$$|(g * \psi_t)(x) - (g * \psi_t)(y)| = \left| \int \hat{g}(z) [e^{-2\pi i(x.z)} - e^{-2\pi i(y.z)}] \hat{\psi}(tz) dz \right| \leqslant$$

$$\leqslant ct \int_{|z|t<1} |\hat{g}(z)||z|\, dz + ct^{-1} \int_{|z|t>1} |\hat{g}(z)||z|^{-1}\, dz$$

On the other hand, since $\hat{\zeta}$ and $\hat{\psi}$ have bounded first order derivatives and $\hat{\zeta}(0) = \hat{\psi}(0)$,

$$|(g * \zeta_t)(x) - (g * \psi_t)(x)| = \left| \int \hat{g}(z)\, e^{-2\pi i(x.z)} [\hat{\zeta}(tz) - \hat{\psi}(tz)]\, dz \right| \leqslant$$

$$\leqslant ct \int_{|z|t<1} |\hat{g}(z)||z|\, dz + ct^{-1} \int_{|z|t>1} |\hat{g}(z)||z|^{-1}\, dz$$

and it will suffice to show that the two last integrals tend to zero as $t \to 0$. To see this let

$$\int_{|z|<u} |\hat{g}(z)||z|^2 dz = u^2 \varepsilon(u)$$

Then *a)* implies that $\varepsilon(u) \to 0$ as $u \to \infty$ and

$$t \int_{|z|t<1} |\hat{g}(z)||z|\, dz = t \int_0^{t^{-1}} u^{-1}\, d\varepsilon(u)u^2 = \varepsilon(t^{-1}) + t \int_0^{t^{-1}} \varepsilon(u)\, du$$

which tends to zero as $t \to 0$. On the other hand

$$t^{-1}\int_{|z|t>1} |\hat{g}(z)|\,|z|^{-1}\,dz = t^{-1}\int_{t^{-1}}^{\infty} u^{-3}d\varepsilon(u)u^2 \leqslant 3t^{-1}\int_{t^{-1}}^{\infty} u^{-2}\varepsilon(u)\,du$$

and this also tends to zero as $t \to 0$, and the lemma is established.

LEMMA 13. *Under the conditions of the preceding lemma we also have*

$$|(g*\psi_t)(x) - (g*\psi_t)(y)| \to 0$$

as $t \to 0$, *uniformly in* x *and* y, *provided that* $|x-y| \leqslant t$.

This is an immediate consequence of the preceding lemma which was incidentally established in the course of its proof.

LEMMA 14. *Let* $h(x) \in L^p(R^n)$, $p > 1$, $n > 2$, *and let*

$$g(x) = \int h(y)|x-y|^{-n+\alpha}\,dy \quad , \; n > \alpha p > 2 \quad ,$$

if the integral is absolutely convergent, $g(x) = -\infty$ *otherwise. Then if*

$$k_1(x) = \int |h(y)|^p |x-y|^{-n+2}\,dy \leqslant \infty$$

the function $k(x) = g(x) - \varepsilon\, k_1(x)$ *is upper semicontinuous for every* ε, $\varepsilon > 0$.

Proof. Since, as readily seen, $|x|^{-n+\alpha}$ is integrable to the power $p/(p-1)$ in the complement of any neighborhood of the origin, the contribution to the integrals in the lemma from $|x| \geqslant N$ is continuous in $|x| < N$. Thus it will suffice to prove the lemma under the assumption that h has support in $|x| < N$. Let $q = p/(p-1)$ and $r = |(n-\alpha) - (n-2)/p|q$. Then $r < n$ and Hölder's inequality gives

$$\int |h(y)|\,|x-y|^{-n+\alpha}\,dy \leqslant k_1(x)^{1/p}\left[\int_{|y|<N} |x-y|^{-r}\,dy\right]^{1/q} .$$

This shows that if $k_1(x)$ is finite then the integral defining g is absolutely convergent. Let us denote now by $I_1(x,t)$ and $I_2(x,t)$

the integral defining g extended over $|x| < t$ and $|x| \geq t$ respectively, and define similarly $J_1(x,t)$ and $J_2(x,t)$ with the integral expressing $k_1(x)$. Then, from Hölder's inequality again, we obtain

$$I_1(x,t) \leqslant c\; J_1(x,t)^{1/p}\; t^{(n-r)/q}$$

Let now x_0 be a point such that $k_1(x_0) < \infty$. Then, if $|x_0 - x| = t/2$, we have $2|x-y| \geq |x_0-y|$ for $|x-y| \geqslant t$ and, consequently the integrands of $I_2(x,t)$ and $J_2(x,t)$ are dominated by a multiple of the integrands of $I_2(x_0,0)$ and $J_2(x_0,0)$ respectively. But, since $k_1(x_0)$ is finite, the last two integrals are absolutely convergent, and this implies that

$$I_2(x,t) - \varepsilon\; J_2(x,t) \to I_2(x_0,0) - \varepsilon\; J_2(x_0,0) - k(x_0)$$

as $x \to x_0$. On the other hand we have

$$I_1(x,t) - \varepsilon\; J_1(x,t) \leqslant c\; J_1(x,t)^{1/p}\; t^{(n-r)/q} - \varepsilon\; J_1(x,t) \leqslant$$

$$\leqslant \sup_s \left[c\; s^{1/p}\; t^{(n-r)/q} - \varepsilon\; s \right] \doteq (c/p\varepsilon)^{1/(p-1)}\; t^{(n-r)/p} \quad ,$$

whence it follows that

$$\overline{\lim_{t\to 0}}\; [\, I_1(x,t) - \varepsilon\; J_1(x,t)\,] \leqslant 0 \quad .$$

Combining this with our previous result we obtain

$$\overline{\lim}\; k(x) = \lim\; [I_2(x,t) - \varepsilon\; J_2(x,t)] + \overline{\lim}\; [I_1(x,t) - \varepsilon\; J_1(x,t)] \leqslant$$

$$\leqslant k(x_0)$$

as $x \to x_0$. Suppose now that $k_1(x_0) = \infty$. Then since, as readily seen, $k_1(x)$ is lower semicontinuous, we have $\lim k_1(x) = \infty$ as $x \to x_0$, and since

$$|g(x)| \leqslant c\; k_1(x)^{1/p}$$

it follows that

$$\lim\; k(x) = \lim\; [g(x) - \varepsilon\; k_1(x)] = -\infty = k(x_0) \quad , \quad x \to x_0$$

and the proof of the lemma is complete.

Proof of theorem 5. We start observing that, without loss of generality, we can make some additional simplifying assumptions. In the first place, lemma 6 shows that with only a harmless alteration of $\hat{f}$ we may assume that $f = \Delta g$, where g has compact support. Furthermore, by restricting our attention to subsets of O if necessary, we may also assume that h is integrable and ν totally finite in O. Evidently, it will suffice to prove our theorem in the case when h is bounded above, and subtracting from h an appropriate infinitely differentiable function with compact support we can further reduce the proof to the case $h \leqslant 0$.

In our proof we shall need some auxiliary functions and distributions we now introduce. We extend h and ν to all of R^n by setting $h(x) = 0$ outside O and $\nu = 0$ on every set not intersecting O. We choose an arbitrary positive number ε and applying lemma 6 we let g_2 be a distribution with compact support such that $\Delta g_2 - (\varepsilon\nu - h)$ is infinitely differentiable and has support at distance not less than 1 from O. We set $\Delta g_2 = f_2$, $f_1 = f + f_2$, $g_1 = g + g_2$ and, as before, we let $F(x,t) = (f * \phi_t)(x)$, and define similarly $F_1(x,t)$ and $F_2(x,t)$.

We shall first prove some properties of the functions F and F_1 . Let us begin by showing that $t^2 F(x,t) \to 0$ as $t \to 0$, uniformly in x. Let $\delta(s)$ be defined by

$$\int_{|z| \leqslant s} |\hat{f}(z)| \, dz = \delta(s) s^2$$

Then condition *a)* in our theorem implies that $\delta(s) \to 0$ as $s \to \infty$. Since $\phi(x)$ is integrable and has integrable derivatives of all orders, its Fourier transform $\hat{\phi}$ satisfies the inequality $|\hat{\phi}(z)| \leqslant c\,(1+|z|)^{-3}$ with some constant c. Consequently

$$t^2 |F(x,t)| = t^2 |(f * \phi_t)(x)| = t^2 \int |\hat{f}(z)| \, |\hat{\phi}(tz)| \, dz \leqslant$$

$$\leqslant ct^2 \int_0^\infty (1+st)^{-3} d\delta(s) s^2 = 3ct^3 \int_0^\infty \delta(s) s^2 \, (1+st)^{-4} ds =$$

$$= 3c \int_0^\infty \delta(s/t) s^2 (1+s)^{-4} \, ds$$

and the last integral evidently tends to zero as $t \to 0$.

Next we shall show that $\overline{F}_1(x) \geqslant 0$ almost everywhere and $\underline{f}_1(x) > -\infty$

everywhere in O. On account of the fact that $f_2 - (\varepsilon\nu - h)$ is an infinitely differentiable function with support disjoint from O, we have

$$F_2(x,t) = [(\varepsilon\nu - h)*\phi_t](x) + [(f_2 - \varepsilon\nu + h)*\phi_t](x) =$$

$$= [(\varepsilon\nu - h)*\phi_t](x) + o(1)$$

as $t \to 0$ for every x in O. Now, if x is a Lebesgue point of h and $y \to x$ and $t \to 0$ with $|y-x| \leqslant t$ we have $\lim\,(h*\phi_t)(y) = h(x)$ and consequently, since $(\nu*\phi_t)(y) \geqslant 0$,

$$\overline{F}_1(x) = \overline{\lim}\, F_1(y,t) \geqslant \overline{\lim}\, F(y,t) + \underline{\lim}\, F_2(y,t) \geqslant \overline{F}(x) - h(x) \geqslant 0$$

On the other hand, since $h \leqslant 0$, we have $F_2(x,t) \geqslant \varepsilon(\nu*\phi_t)(x) + o(1)$ as $t \to 0$. This evidently implies that $\underline{f}_1(x) \geqslant \underline{f}(x)$ everywhere in O. Now, if x is a point of O not in C and such that $\underline{f}\,(x) = -\infty$, then from condition (3) in our theorem it follows that $\underline{f}_1(x) \geqslant 0$. If x is a point of C then $\underline{\lim}_{t\to 0}\, t^2(\nu*\chi_t)(x) > 0$ and, as we saw above, $\lim_{t\to 0} t^2 F(x,t) = 0$. But since evidently $\phi_t(x) \geqslant c\ \chi_{\delta t}(x)$ for some positive δ and c, we have $\underline{\lim}_{t\to 0}\, t^2 F_2(x,t) > 0$, and consequently $\lim_{t\to 0} F_1(x,t) = +\infty$, that is, $\underline{f}_1(x) = +\infty$. Thus in all cases we have $\underline{f}_1(x) > -\infty$, as we wished to show.

Let us turn now to the distributions g. Let $\overline{\phi}(x)$ be like the function ϕ in our theorem, but having support in $|x| < 1$. Let ψ and $\overline{\psi}$ be related to ϕ and $\overline{\phi}$ respectively as in lemma 8 and normalized so as to have integrals equal to 1. Set $G(x,t) = (g*\overline{\psi}_t)(x)$ and define similarly G_1 and G_2. We shall show that $\lim_{t\to 0} G_1(x,t) = \overline{g}_1(x)$ exists everywhere in O, is upper semicontinuous and locally integrable and coincides with the distribution g_1 in O. Since $\overline{F}_1(x) \geqslant 0$ a.e. and $\underline{f}_1(x) > -\infty$ everywhere in O, it will follow by lemma 11 that f_1 coincides with a measure in O. Since $f_1 = f + \varepsilon\nu - h$ in O, and since ε can be taken arbitrarily small, this in turn will imply that $f-h$ coincides with a measure in O, which is the assertion of the theorem.

We begin with some observations. By lemma 9 we have

$$G(x,t_2) - G(x,t_1) = c\int_{t_1}^{t_2} s\,(f*\overline{\phi}_s)(x)\,ds \quad , \; c > 0 \;.$$

Then, since $\overline{\Phi} \geqslant 0$, if x is a point in an open set in which f coincides with a measure, and d is the distance from x to the complement of the set, $G(x,t)$ is a non-decreasing function of t for $0 < t \leqslant d$. A similar remark applies to G_1 and G_2 . Thus, since f_2 coincides with $\varepsilon\nu - h$ in a sufficiently large open set containing O, $G_2(x,t)$ is a non-decreasing function of t for $0 < t \leqslant 1$ and all x in O. About the function $G(x,t)$ we remark that, according to lemma 12, it can be replaced by $(g*\psi_t)(x)$ with an error which is a bounded function of x and tends uniformly to zero as $t \to 0$.

To show the existence of $\overline{g}_1$ we observe that $G_1 = G + G_2$. Since $G_2(x,t)$ decreases as $t \to 0$, it has a limit, finite or infinite for all x in O.

On the other hand, if x is a point of O not in C, by lemmas 12 and 9 we have

$$G(x,t) = (g*\psi_t)(x) + o(1) = (g*\psi_1)(x) - c\int_t^1 s\, F(x,s)\, ds + o(1) =$$

$$= (g*\psi_1)(x) + c\int_t^1 s\,[|F(x,s)| - F(x,s)]\, ds - c\int_t^1 s\,|F(x,s)|\,ds + o(1)$$

where $c > 0$, and since by condition (3) in our theorem the first integral in the last expression has a finite limit as $t \to 0$, we conclude that, as $t \to 0$, $G(x,t)$ also has a limit, finite or infinite. Combining this with our previous observation we conclude that $\overline{g}_1(x)$ exists for all x not in C. Finally, if x is a point of C, since obviously [illegible]$) \geqslant c\, \chi_{\delta t}(x)$ for some positive δ and c, we have

$$(f_2 * \overline{\Phi}_t)(x) \geqslant c\,(\nu * \chi_{\delta t})(x) \geqslant c_1 t^{-2} \quad , \quad 0 < t \leqslant 1 \quad , \quad c_1 > 0 \quad .$$

Consequently

$$G_1(x,t) = G(x,t) + G_2(x,t) = (g*\psi_t)(x) + G_2(x,t) + o(1) =$$

$$= (g*\psi_1)(x) + G_2(x,1) - c\int_t^1 s\, F(x,s)ds - c\int_t^1 s\,(f_2 * \overline{\Phi}_s)(x)ds + o(1)$$

$$\leqslant (g*\psi_1)(x) + G_2(x,1) - c\int_t^1 s^{-1}[F(x,s)s^2 + c_1]ds + o(1)$$

where, again, $c > 0$. Since, as we saw, $F(x,s)s^2 \to 0$ as $s \to 0$, the last expression tends to $-\infty$ as $t \to 0$, and consequently $\overline{g}_1(x) = -\infty$.

We proceed now to show that $\bar{g}_1$ has the required properties and coincides with g_1 in O. At first we shall assume that the set C is empty. Let O_1 be the largest subset of O with the property that $\bar{g}_1$ is upper semicontinuous, locally integrable and coincides with g_1 in O_1, and suppose that O_1 is a proper subset of O. Consider the function

$$\int_0^1 \lambda(s)\; s\,[|F(x,s)| - F(x,s)]\; ds$$

which evidently is lower semicontinuous in O. Since C is empty, on account of condition (3) in our theorem, this function is finite everywhere in O. Thus, as in the proof of lemma 1, we conclude that there exists an open set O_2, containing O_1 as a proper subset, such that the integral above is bounded, say by N, in $O_2 - O_1$.

Let now x be a point in O_1 at distance not larger than s from $O_2 - O_1$, and let $s < t \leqslant 1$. Then if y is a point of $O_2 - O_1$ and $|x-y| \leqslant s$, by lemma 12 we have

$$(4)\qquad \begin{aligned} &|G(x,s) - (g*\psi_s)(y)| \leqslant \theta(t) \\ &|G(x,t) - (g*\psi_t)(y)| \leqslant \theta(t) \end{aligned}$$

where $\theta(t)$ tends to zero as $t \to 0$. On the other hand, by lemma 9,

$$(g*\psi_t)(y) - (g*\psi_s)(y) = c\int_s^t u\; F(y,u)\; du \quad , \quad c > 0 \quad ,$$

and since $\lambda(t)$ is a decreasing function of t,

$$(g*\psi_s)(y) - (g*\psi_t)(y) \leqslant c\int_s^t u\,|\,|F(y,u)| - F(y,u)|\; du \leqslant$$

$$\leqslant c\;\lambda(t)^{-1}\int_s^t \lambda(u)\; u\,|\,|F(y,u)| - F(y,u)|\; du \leqslant c\; N\;\lambda(t)^{-1}$$

and combining these inequalities we find that

$$G(x,s) \leqslant G(x,t) + 2\theta(t) + c\; N\;\lambda(t)^{-1} \quad ,$$

and, since $G_2(x,t)$ is a non-decreasing function of t, this, in turn, implies that

(5) $$G_1(x,s) \leqslant G_1(x,t) + 2\theta(t) + c\,N\,\lambda(t)^{-1}$$

which holds for $d \leqslant s < t \leqslant 1$, d being the distance of x from $O_2 - O_1$. Now for $d > s$, since $\bar{g}_1 = g_1$ is upper semicontinuous in O_1 and therefore $f_1 = \Delta g_1$ coincides with a measure in O_1, $G_1(x,s)$ is a non-decreasing function of s, and thus (5) is seen to hold also for $0 < s < t \leqslant 1$ and all x in O_2. Now letting s tend to zero we obtain

(6) $$\bar{g}_1(x) \leqslant G_1(x,t) + 2\theta(t) + c\,N\,\lambda(t)^{-1}$$

Suppose now that $x \to x_0$, $x_0 \in O_2$. Then, since the righthand side of the preceding inequality is a continuous function of x we find that

$$\overline{\lim_{x \to x_0}}\ \bar{g}_1(x) \leqslant G_1(x_0,t) + 2\theta(t) + c\,N\,\lambda(t)^{-1}$$

and since $\theta(t) \to 0$ and $\lambda(t) \to \infty$ as $t \to 0$, letting t tend to zero we obtain

$$\overline{\lim_{x \to x_0}}\ \bar{g}_1(x) \leqslant \bar{g}_1(x_0)$$

which is the desired upper semicontinuity of $\bar{g}_1$ in O_2.

Let now $\zeta(x) \geqslant 0$ be infinitely differentiable with compact support contained in O_2. Since, as $t \to 0$, $\zeta_* \bar{\psi}_t$ converges uniformly with all its derivatives to ζ, we have

$$\lim_{t \to 0} \int G_1(x,t)\,\zeta(x)\,dx = \lim_{t \to 0} g_1(\zeta_* \bar{\phi}_t) = g_1(\zeta) \quad .$$

On the other hand, on account of (5), $G_1(x,t)$ is bounded above for $0 < t \leqslant 1$ and x in any compact subset of O_2. Thus, since $\lim_{t \to 0} G_1(x,t) = \bar{g}_1(x)$, by Fatou's lemma we have

$$\int \bar{g}_1(x)\,\zeta(x)\,dx \geqslant \lim_{t \to 0} \int G_1(x,t)\,\zeta(x)\,dx = g_1(\zeta)$$

and thus $\bar{g}_1$ is seen to be locally integrable in O_2. But then multiplying (6) by $\zeta(x)$, integrating and letting t tend to zero, we get the preceding inequality reversed. Thus we have equality and

$$\int \bar{g}_1(x)\,\zeta(x)\,dx = g_1(\zeta)$$

for all ζ. Thus $\bar{g}_1$ is upper semicontinuous, locally integrable and coincides with g_1 in O_2, which contains O_1 as a proper subset, in contradiction with the assumed maximality of O_1. Hence O_1 must coincide with O, and, as observed earlier, this proves the theorem in the case when C is empty.

Let us pass now to the case when C is non-empty. We shall show that $\bar{g}_1$ is upper semicontinuous, locally integrable and coincides with g_1 in O, and then the desired conclusion will follow as before. Let O_1 be again the largest open subset of O on which $\bar{g}_1$ has these properties. Since, as we know now, O_1 must contain O - C we have $O - O_1 \subset C$. Consider the function $\inf_t t^2(\nu * \chi_t)(x)$, $0 < t \leq 1$. Because $t^n \chi_t$ is the characteristic function of the closed sphere $|x| \leq t$, this function is upper semicontinuous and, according to our hypotheses, positive at every point of C. Consequently, as in the proof of lemma 1, it follows that there exists and open subset O_2 of O, containing O_1 as a proper subset, such that $\inf_t t^2(\nu * \chi_t)(x) \geq \delta > 0$ for all x in $O_2 - O_1$. Set now

$$\omega(x) = \int_0^1 (\nu * \chi_t)(x)\, dt$$

Evidently, this is an integrable function. If $d(x)$ denotes the distance between x and the set $O_2 - O_1$, $2d(x) \leq s \leq 1$ and y is a point in $O_2 - O_1$ such that $|x-y| \leq \frac{5}{4} d(x)$, then as readily seen

$$(\nu * \chi_s)(x) \geq c(\nu * \chi_{s/4})(y) \geq c\, s^{-2}$$

where the c are positive constants. Thus, for $d(x) \leq 1/2$, we have

$$\omega(x) \geq c \int_{2d(x)}^1 s^{-2}\, ds \geq c d(x)^{-1}$$

which shows that $\omega(x) = +\infty$ in $O_2 - O_1$, which therefore has Lebesgue measure zero, and that $d(x)^{-1}$ is integrable in a neighborhood of $O_2 - O_1$. Let now $d(x) \leq s < t \leq 1$. Then since, as we saw at the beginning of the proof, $t^2\, F(x,t)$ is bounded, by lemma 9 we have

$$|(g * \phi_t)(x) - (g * \phi_s)(x)| = c \left| \int_s^t u F(x,u) du \right| \leq c \int_{d(x)}^1 u^{-1} du = c \log d(x)^{-1}$$

and this, combined with (4) and the fact that $G_2(x,t)$ is a non-decreasing function of t gives

$$(7) \qquad G_1(x,s) \leqslant G_1(x,t) + 2\theta(t) + c \log d(x)^{-1}$$

which is analogous to (5), and which for the same reasons as in the case of (5), holds also for $0 < s < t \leqslant 1$ and $d(x) \leqslant 1$. Letting s tend to zero we obtain

$$(8) \qquad \bar{g}_1(x) \leqslant G_1(x,t) + 2\theta(t) + c \log d(x)^{-1} .$$

This shows that $\bar{g}_1(x) \to -\infty$ as $d(x) \to 0$, and since $\bar{g}_1(x) = -\infty$ for $x \in C$, it follows that $\bar{g}_1$ is upper semicontinuous in O_2. To prove that $\bar{g}_1$ is locally integrable and coincides with g_1 in O_2 we argue with (7) and (8) as we did in the preceding case with (5) and (6), keeping in mind that $\log d(x)^{-1}$ is integrable in a neighborhood of $O_2 - O_1$. This will contradict the assumed maximality of O_1, showing that O_1 must coincide with O, as we wished to show.

Proof of theorem 6. As in the case of the preceding theorem, we may assume that $f = \Delta g$, where g has compact support, that $h(x) \leqslant 0$ and is defined and integrable in all of R^n, and that ν is defined on all Borel subsets of R^n and is totally finite.

We shall assume first that $n > 2$ and $q > 1$. Let

$$\hat{l}(z) = \hat{g}(z)|z|^{2(1-s)} = -4\pi^2 \hat{f}(z)|z|^{-2s}$$

where s is such that $r \leqslant s < q^{-1}$ and $(n-2sq)/(n-2) < q$. Since g has compact support, $\hat{g}$ and $\hat{l}$ are continuous and bounded near the origin. Since $\hat{f}(z)(1+|z|^2)^{-r}$ is in L^q and $s \geqslant r$, $\hat{f}(z)|z|^{-2s}$ and $\hat{l}(z)$ are integrable to the q-th power in $|z| < 1$. Thus $\hat{l}(z)$ is in L^q and its inverse Fourier transform $l(x)$ is in L^p, $p = q/(q-1)$.

Let now $\alpha = 2(1-s)$ and c a constant such that the Fourier transform of $c\,|x|^{-n+\alpha}$ coincides with $|z|^{-\alpha}$. Let $\bar{g}(x)$ be defined by

$$\bar{g}(x) = c \int l(y)|x-y|^{-n+\alpha}\, dy$$

if the integral is absolutely convergent, or $\bar{g}(x) = -\infty$ otherwise. Since $(n-2sq)/(n-2) < q$ and $sq < 1$ one verifies readily that $n > p\alpha > 2$ and $p(n-\alpha) > n$, so that $|x|^{-n+\alpha}$ is integrable to the q-th power in $|x| > 1$ and the integral above is absolutely convergent for almost all x and $\bar{g}(x)$ is locally integrable. Furthermore, our distribution g coincides with the function $\bar{g}$. In fact, if $\hat{\zeta}(x)$ is an infinitely differentiable function with compact support, we have

$$\int \bar{g}(x)\ \hat{\zeta}(x)\ dx = c \int l(x) \int \hat{\zeta}(y)\,|x-y|^{-n+\alpha}\ dy\ dx$$

Since the Fourier transform of $|z|^{-\alpha}$ is $c|x|^{-n+\alpha}$, the inner integral above is the Fourier transform of $\zeta(z)|z|^{-\alpha}$, ζ here being the inverse Fourier transform of $\hat{\zeta}$. On the other hand the convolution $\hat{\zeta}(x) * |x|^{-n+\alpha}$ evidently belongs to L^q and therefore, by Plancherel's theorem, we have

$$\int \bar{g}(x)\ \hat{\zeta}(x)\ dx = c \int l(x) \int \hat{\zeta}(y)\,|x-y|^{-n+\alpha}\ dy\ dx =$$

$$= \int \hat{l}(z)\ \zeta(z)\,|z|^{-\alpha} dz = \int \hat{g}(z)\ \zeta(z)\ dz$$

which shows that the Fourier transform of $\bar{g}$ coincides with $\hat{g}$, that is, $\bar{g}$ coincides with g.

Let now ε be an arbitrary positive number and $\zeta(x)$ an infinitely differentiable function with compact support which equals 1 in O. Let $\bar{g}_1(x) \geqslant -\infty$ be defined by

$$\text{(9)}\qquad \begin{aligned} \bar{g}_1(x) &= \bar{g}(x) - c\ \zeta(x) \int \varepsilon\ |l(y)|^p\,|x-y|^{-n+2} dy\ + \\ &+ c\ \zeta(x) \int h(y)\,|x-y|^{-n+2}\ dy - c\ \zeta(x) \int \varepsilon\,|x-y|^{-n+2}\ d\nu(y) \end{aligned}$$

where $-c|x|^{-n+2}$, $c > 0$ is the fundamental solution of Laplace's equation. Evidently $\bar{g}_1$ is locally integrable and the distribution $f = \Delta\bar{g}_1$ coincides with

$$\Delta\ \bar{g} + \varepsilon|l|^p - h + \varepsilon\nu = f + \varepsilon|l|^p - h + \varepsilon\nu$$

in O. As we shall see, f_1 has the property that $\overline{F}_1(x) \geqslant 0$ almost everywhere and $\underline{f}_1(x) > -\infty$ everywhere in O, and $\bar{g}_1$ satisfies the conditions of lemma 11. Thus it will follow that f_1 coincides with a measure in O, and since this will hold regardless of the value of ε we will conclude that $f-h$ also coincides with a measure in O, and our theorem will be established.

On account of lemma 7 we have

$$F_1(x,t) \geq F(x,t) - (h_*\phi_t)(x) + \varepsilon(\nu_*\phi_t)(x) + o(1)$$

uniformly in any compact subset of O. From this it follows that $\underline{f}_1 \geq \underline{f}$. Now if at the point x we have $\underline{f}(x) = -\infty$, then, according to our hypotheses, $F(x,t) = o[(\nu_* \phi_t)(x)]$ and consequently $\underline{\lim}_{t\to o} F(x,t) + \varepsilon(\nu_*\phi_t)(x) \geq 0$, and since $h \leq 0$, it follows that $\underline{f}_1(x) \geq 0$, and thus we have $\underline{f}_1(x) > -\infty$ in all cases. On the other hand, if x is a Lebesgue point of the function h and $t \to 0$ and $y \to x$ with $|y-x| \leq t$, we have $(h_*\phi_t)(y) \to h(x)$ and therefore $\overline{F}_1(x) \geq \overline{F}(x) - h(x)$. Thus we have $\overline{F}_1(x) \geq 0$ almost everywhere in O.

Let us turn now to the function $\overline{g}_1$. Evidently, since $h \leq 0$, the last two terms on the right of (9) are upper semicontinuous functions of x, and according to lemma 14 the sum of the two first is upper semicontinuous in O. Thus $\overline{g}_1$ is upper semicontinuous in O, and there remains only to show that $\overline{g}_1$ has the property that $(\overline{g}_1 * \psi_t)(x) \to \overline{g}_1(x)$ as $t \to 0$ for all x in O. To see this let $0 > \beta > -n$ and consider the convolution $\psi_t * |x|^\beta$. Evidently $\sup_t \psi_t * |x|^\beta$ is finite everywhere but at the origin. Furthermore, as readily verified, it is homogeneous of degree β and spherically symmetric. Thus we have

$$\psi_t * |x|^\beta \leq \sup_t \psi_t * |x|^\beta = c\,|x|^\beta$$

Thus if μ is a signed measure and the convolution $\mu * |x|^\beta$ is absolutely convergent at the point x_o then by the dominated convergence theorem we have

$$\lim_{t\to o} [(\mu * |x|^\beta) * \psi_t](x_o) = \lim_{t\to o} [\mu * (|x|^\beta * \psi_t)](x_o) =$$
$$= (\mu * |x|^\beta)(x_o)$$

We use this observation to calculate the limit of $(\overline{g}_1 * \psi_t)$. For this purpose we convolve the righthand side of (9) with ψ_t. According to Lemma 8, in calculating the limit of the resulting expression at points of O we may drop the function $\zeta(x)$ and we obtain a sum of terms of the form $\mu * |x|^\beta$. Thus if at a point x all integrals of the righthand side of (9), including the one defining $\overline{g}(x)$, are absolutely convergent we have $\lim_{t\to o} (\overline{g}_1 * \psi_t)(x) = \overline{g}_1(x)$

If on the other hand one of the integrals is divergent, then $\bar{g}_1(x) = -\infty$, and the desired result follows from the upper semicontinuity of $\bar{g}_1$. This concludes the proof of the theorem in the case $n > 2$, $q > 1$.

In the case $n \leq 2$ or $q = r = 1$ the distribution g, $f = \Delta g$, coincides with a continuous function. In fact, if $q > 1$ we have

$$\int_{|z|>1} |\hat{g}(z)|\,dz \leq \left[\int_{|z|>1} |\hat{f}(z)|^q |z|^{-2rq} dz\right]^{1/q} \left[\int_{|z|>1} |z|^{-2(1-r)p} dz\right]^{1/p}$$

and since $2(1-r)p = 2(1-r)q/(q-1) = (2q-2rq)/(q-1) > 2$ the last integral above is convergent. But, because $\hat{g}$ is continuous, this implies that it is integrable and that g coincides with a continuous function. If on the other hand $q = r = 1$

$$\int_{|z|>1} |\hat{g}(z)|\ dz = \int_{|z|>1} |\hat{f}(z)|\,|z|^{-2}\ dz < \infty$$

and, again, g coincides with a continuous function, and the rest of the proof consists in applying lemma 11 to the function

$$\bar{g}_1(x) = \bar{g}(x) + \zeta(x) \int \varepsilon\ \Phi(x-y)\ d\nu - \zeta(x) \int h(y)\ \Phi(x-y)\ dy$$

where $\Phi(x)$ is the fundamental solution of Laplace's equation. The argument is identical with the one used above and need not be repeated here.

Proof theorem 7. Theorems 5 and 6 having been established, theorem 7 can be proved by using lemma 7 and the argument used in the proof of theorem 2. The details are left to the reader.

4.

In this last section we shall give two examples of distributions f for which $(f * \phi_t)$ tends everywhere to zero with t. The first is the analogue of the series $\sum n \sin nx$ which is known to be everywhere Abel summable to zero. In this case $\hat{f}$ barely violates condition *a)* of theorem 5 but is unbounded. In the second example, which is more complicated, $\hat{f}(z)$ is of the order $|z|^{-(n-3)/2}$ as $|z| \to \infty$, and thus is bounded for $n = 3$ and tends to zero as $|z| \to \infty$ if $n \geqslant 4$. We remind the reader that if $\hat{\phi}(z) = e^{-|z|}$, then $(f_* \phi_t)$ is the Abel means of the Fourier integral of $\hat{f}$. Thus the Fourier integrals of the functions $\hat{f}$ are Abel summable to zero everywhere.

Let $x = (u, \bar{x})$, $u \in \mathrm{R}$, $\bar{x} \in \mathrm{R}^{n-1}$, and $\omega(\bar{x})$ be infinitely differentiable and have compact support. Let f be the distribution defined by

$$f(\zeta) = \int \left(\frac{\partial}{\partial u} \zeta \right)(0, \bar{x}) \; \omega(\bar{x}) \; d\bar{x}$$

Then f has support in $u = 0$ and, setting $z = (v, \bar{z})$, we have

$$\hat{f}(z) = 2\pi i v \; \hat{\omega}(\bar{z})$$

whence it follows that

$$\int_{|z|<r} |\hat{f}(z)| \; dz \leqslant 2\pi \int_{-r}^{r} |v| \; dv \int |\hat{\omega}(\bar{z})| \; d\bar{z} \leqslant c \; r^2$$

Now, since f has support in $u = 0$, we have

$$(f_* \phi_t)(u, \bar{x}) \to 0 \quad , \quad u \neq 0 \quad , \quad t \to 0$$

and if $u = 0$, on account of the spherical symmetry of ϕ_t we have

$$\frac{\partial}{\partial v} \phi_t(u-v, \bar{x}-\bar{y}) = 0 \quad , \quad u = v = 0$$

and, consequently $(f_* \phi_t)(0, \bar{x}) = 0$. Thus $(f_* \phi_t) \to 0$ everywhere as $t \to 0$.

In our second example we shall assume that $n \geqslant 2$. Let S be the sphere $|x| = 1$ and let f be defined by

$$f(\zeta) = \int_S \left[\zeta + 2(n-1)^{-1} \frac{\partial}{\partial \nu} \zeta \right] d\sigma$$

where $d\sigma$ stands for the element of area of S and $\frac{\partial}{\partial \nu}$ denotes normal outer differentiation. To calculate $\hat{f}$ we merely replace ζ by

$e^{2\pi i(x.z)}$ in the preceding integral. Setting $|x| = r$, $|z| = \rho$, $(x.z) = r\rho \cos s$ we have $d\sigma = c\,(\sin s)^{n-2}\,ds$ and our integral becomes

$$c^{-1}\,\hat{f}(z) = \int_0^{\pi} e^{2\pi i\rho \cos s}\,(\sin s)^{n-2}\,ds\, +$$

$$+\, 4\pi i(n-1)^{-1}\rho \int_0^{\pi} e^{2\pi i\rho\cos s}\cos s\,(\sin s)^{n-2}ds$$

and replacing $\cos s$ by t and integrating these integrals by parts we see that $\hat{f}(z) = O(|z|^{-(n-3)/2})$ as $|z| \to \infty$.

Let now $\phi(x) = \eta(|x|)$, where $\eta(t)^{(k)} = O(t^{-n-k-\varepsilon})$ as $t \to \infty$ and let us calculate $\lim\,(f_*\phi_t)(x)$ as $t \to 0$. Since the support of f is S, by lemma 7, this limit is zero if $x \notin S$. If $x \in S$ we have

$$(11)\quad (f_*\phi_t)(x) = \int_S [t^{-n}\eta(|x-y|/t) + 2(n-1)^{-1}\,\frac{\partial}{\partial\nu}\,t^{-n}\eta(|x-y|/t)]\,d\sigma_z$$

Since, as readily seen, if $|x| = 1$ then

$$\frac{\partial}{\partial\nu}\,|x-y| = |x-y|^{-1}[1 - (x.y)]$$

and thus we have

$$\frac{\partial}{\partial\nu}\,t^{-n}\eta(|x-y|/t) = t^{-n-1}\eta'(|x-y|/t)\;|x-y|^{-1}\,[\,1 - (x.y)\,]\;.$$

Setting $|x-y| = s = 2\sin\theta/2$ we obtain

$$1 - (x.y) = 1 - \cos\theta = s^2/2$$

$$d\sigma = c(\sin\theta)^{n-2}d\theta = c\;s^{n-2}(1-s^2/4)^{(n-3)/2}ds = c\;s^{n-2}\;\Phi(s)ds$$

and substituting in (11) we get

$$(f_*\phi_t)(x) = \int_0^2 t^{-n}\eta(s/t)s^{n-2}\Phi(s)ds + (n-1)^{-1}\int_0^2 t^{-n-1}\eta'(s/t)s^{n-1}\Phi(s)ds$$

Since $t^{-n}\eta(s/t)$ and $t^{-n-1}\eta'(s/t)$ tend to zero as $t \to 0$, uniformly in $s \geqslant 1$, replacing the upper limit of integration by 1 and integrating by parts the second integral we obtain

$$(f_*\phi_t)(x) = -(n-1)^{-1} \int_0^1 t^{-n} s^{n-1} \eta(s/t)\, \Phi'(s)\, ds + o(1)$$

But, evidently, $|\Phi'(s)| \leq c\, s$ in $0 \leq s \leq 1$ and consequently the last integral is dominated by

$$c \int_0^1 t^{-n} s^n \eta(s/t)\, ds \leq c\, t \int_0^{1/t} s^n\, \eta(s)\, ds \leq c\, t^{\varepsilon}$$

as $t \to 0$, and we find that $(f_*\phi_t)(x) \to 0$ everywhere as $t \to 0$.

REFERENCES

[1] CALDERON A.P. and ZYGMUND A., *Local properties of solutions of elliptic partial differential equations*, Studia Math. 20 (1961), 171-225.

[2] SAKS S., *Theory of the integral*, Monografie Matematyczne, vol. VII, Warsaw, 1937.

[3] SHAPIRO V.L., *Cantor-type uniqueness of multiple trigonometric integrals*, Pacific J. Math. 5 (1955), 607-622.

[4] ----------- , *Removable sets for pointwise solutions of the generalized Cauchy-Riemann equations*, Ann. of Math. 92 (1970), 82-101.

[5] VERBLUNSKY S., *On trigonometric integrals and harmonic functions*, Proc. London Math. Soc. 38 (1935), 1-48.

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Recibido en setiembre de 1970.