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UNIFORM APPROXIMATION TO BOUNDED ANALYTIC FUNCTIONS

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Let Δ denote the open unit disc in the complex plane C, and let $H^\infty(\Delta)$ denote the algebra of bounded analytic functions on Δ. We wish to prove the following theorem, which was proved in the case that E is open by A. Stray [5].

THEOREM 1. *Let* $f \in H^\infty(\Delta)$ *, and let* E *be a subset of* $b\Delta$ *such that* f *extends continuously to each point of* E*. Then there is a sequence* $f_n \in H^\infty(\Delta)$ *such that each* f_n *extends to be analytic on some neighborhood of* E*, and* f_n *converges uniformly to* f *on* Δ.

For E a subset of $b\Delta$, let H^∞_E denote the subalgebra of $H^\infty(\Delta)$ of functions which extend continuously to each point of E. The theorem asserts that the functions in $H^\infty(\Delta)$ which extend analytically to a neighborhood of E are dense in H^∞_E. Combining the theorem with Carleson's corona theorem, we obtain the following corollary, which is due to Détraz [2].

COROLLARY. *The open unit disc* Δ *is dense in the maximal ideal space of* H^∞_E .

Proof of the main theorem. We proceed now directly to the proofs. The symbols C_0, C_1,.... will all denote universal constants. All norms will be supremum norms.

LEMMA 1. *Let* Q *be a closed subset of* $b\Delta$*, let* W *be an open subset of* C *at a positive distance from* Q*, and let* $\varepsilon > 0$*. Let* f *be a bounded Borel function on* C*, such that* f *is analytic on* Δ*. Suppose there is a continuous function* u *in a neighborhood of* Q *such that*

$$|f(z) - u(z)| < d$$

for all $z \in \Delta$ *which are near* Q*. Then there is a bounded Borel function* h *such that*

(i) h *is analytic on an open set containing* $\Delta \cup Q$.

(ii) h *extends analytically across any arc on* $b\Delta$ *across which* f *extends analytically.*

(iii) f-h *is analytic on* W *and satisfies* $|f-h| < \varepsilon$ *there.*

(iv) $|f(z)-h(z)| < C_1 d$ *for all* $z \in \Delta$.

Proof. For $\delta > 0$, the open δ-neighborhood of Q will be denoted by $Q(\delta)$. By hypothesis, we can choose $\delta_0 > 0$ so small that $Q(\delta_0)$ does not meet W, that u is defined on $Q(\delta_0)$, and that $|f(z)-u(z)| < d$ for $z \in Q(\delta_0) \cap \Delta$. Since u is uniformly continuous in a neighborhood of Q, we can shrink δ_0 so that also $|u(z)-u(\zeta)| < d$ for all $z,\zeta \in Q(\delta_0)$ satisfying $|z-\zeta| < 2\delta_0$.

Let Γ be the union of the arcs on $b\Delta$ across which f extends analytically. There is then an open set U containing Γ such that $|f(z)-u(z)| < d$ for all $z \in Q(\delta_0) \cap U$. Let F be the function which coincides with u on $Q(\delta_0)\setminus(\Delta \cup U)$, and which coincides with f elsewhere. Then F is a bounded Borel function which satisfies

(*) $\qquad |F(z)-F(\zeta)| < 3d$ whenever $z,\zeta \in Q(\delta_0)$, $|z-\zeta| < 2\delta_0$.

Since F coincides with f on Δ, on W, and in a neighborhood of Γ, it will suffice to obtain the conclusions of the lemma, with f replaced by F.

Now we are in a position to use Vitushkin's scheme for approximation, as developed for instance in Chapter VIII of [3], or in [6]. Because we are working on the unit circle, we can employ the version of this technique matching only one coefficient of the appropriate Laurent expansions (cf. [6], V.4). The details are as follows.

For a fixed δ satisfying $0 < \delta < \delta_0$, choose discs $\Delta_k = \{|z-z_k| < \delta\}$, $z_k \in Q$, which cover Q, and choose functions g_k supported on Δ_k such that $0 \leqslant g_k \leqslant 1$, $\sum g_k = 1$ in a neighborhood of Q, $\left|\dfrac{\partial g_k}{\partial \bar z}\right| \leqslant 4/\delta$, and no point z is contained in more than C_2 of the discs Δ_k. If

$$G_k(\zeta) = \frac{1}{\pi} \iint \frac{F(\zeta) - F(z)}{\zeta - z} \frac{\partial g_k}{\partial \bar z}\, dx\, dy, \quad \zeta \in \mathbb{C},$$

then G_k is a bounded Borel function, G_k is analytic wherever F is analytic, G_k is analytic off Δ_k, and $G_k(\infty) = 0$. Moreover, $F - \sum G_k$ is analytic on the interior of the set on which $\sum g_k$ assumes the value 1. In particular, $F - \sum G_k$ is analytic in a neighborhood of Q. The condition (*) can be used to estimate G_k, yielding the bound

$$\|G_k\| \leqslant C_3 d .$$

Suppose the expansion of G_k near ∞ is given by

$$G_k(z) = \frac{a_1}{z - z_k} + \ldots .$$

By Schwarz's lemma we have

$$|a_1| \leqslant \|G_k\| \delta .$$

Now the analytic capacity of the connected open set $\Delta_k \setminus \bar{\Delta}$ is at least one fourth its diameter. Hence we can find a continuous function H_k on C such that H_k is analytic off a compact subset of $\Delta_k \setminus \bar{\Delta}$,

$$\|H_k\| \leqslant 5 \|G_k\|$$

$$H_k(z) = \frac{a_1}{z - z_k} + \ldots .$$

Now $\|G_k - H_k\| \leqslant C_4 d$ so that

(**) $$|G_k(z) - H_k(z)| \leqslant C_4 d\, \delta^2 / |z - z_k|^2$$

for $z \in \Delta_k$. As H_k has been defined so that $G_k - H_k$ has a double zero at ∞, the estimate (**) persists for all $z \in C$.

Now we define

$$h = F - \sum (G_k - H_k) .$$

Since $F - \sum G_k$ is analytic wherever F is analytic, and each H_k is analytic in a neighborhood of $\bar{\Delta}$, the function h is analytic on Δ and extends analytically across Γ. Moreover, h is analytic in a neighborhood of Q, so that (i) and (ii) are valid. Since F - h is analytic off $Q(\delta)$, F - h is analytic on W. To complete the proof, it suffices now to obtain the estimates in (iii) and (iv).

To verify (iii), fix $z \in W$ and consider $F(z)-h(z) = \sum[G_k(z)-H_k(z)]$. Since no point lies in more than C_2 discs Δ_k and each Δ_k meets $b\Delta$, there is a grand total of at most $2\pi C_2/\delta$ discs Δ_k. Thus by (**)

$$(***)\qquad \sum|G_k(z) - H_k(z)| \leqslant 2\pi C_2C_4 d\ \delta/[\mathrm{dist}(W, \cup\Delta_k)]^2 \quad .$$

Taking δ much smaller than $\mathrm{dist}(W,Q(\delta_0))$, we get $|F - h| < \varepsilon$ on W.

To verify (iv), we first observe that $F - h = \sum[G_k - H_k]$ is analytic off $Q(\delta)$, so that it suffices to obtain the estimate

$$\sum|G_k(z) - H_k(z)| \leqslant C_1 d \qquad .$$

for $z \in Q(\delta)$. So fix a point $z \in Q(\delta)$. Let M(m) be the number of discs Δ_k whose centers satisfy $m\delta \leqslant |z - z_k| < (m+1)\delta$. Since no point z is contained in more than C_2 discs, there will be a constant C_5 such that

$$M(m) \leqslant C_5 \quad \text{if} \quad 0 \leqslant m \leqslant 1/\delta \quad ,$$

providing δ is sufficiently small. (Here we use the geometry of the unit circle, and the fact that z is close to the unit circle) Using the estimate $|G_k(z) - H_k(z)| \leqslant C_4 d$ for the at most C_2 indices k for which $|z - z_k| < \delta$, the estimate (**) for those k for which $m\delta \leqslant |z - z_k| < (m+1)\delta$ and $1 \leqslant k \leqslant 1/\delta$, and the same estimate used to obtain (***) for those k for which $|z - z_k| \geqslant 1$, we find that

$$\sum|G_k(z) - H_k(z)| \leqslant C_2C_4 d + \sum_{k=1}^{1/\delta} M(k)C_4 d/k^2 + 2\pi C_2C_4 d\delta \leqslant C_1 d \ .$$

That completes the proof.

LEMMA 2. *Let* $f \in H^\infty(\Delta)$ *, and let* E *be a subset of* $b\Delta$*. Suppose there is an open set* U *containing* E*, and a function* u *defined and continuous on* U*, such that* $|f(z)\cdot u(z)| < d$ *for all* $z \in U \cap \Delta$*. Then there is* $h \in H^\infty(\Delta)$ *such that* h *extends to be analytic in a neighborhood of* E*, and*

$$\sup_{z\in\Delta} |f(z) - h(z)| \leqslant C_0 d \quad .$$

Proof. By replacing E by $U \cap b\Delta$, we can assume that E is relatively open in $b\Delta$. Then we can write $E = (\cup Q_n) \cup (\cup R_n)$, where $Q_1, Q_2, \ldots$ are pairwise disjoint closed intervals, $R_1, R_2, \ldots$ are pairwise disjoint closed intervals, each Q_n joins the endpoints of two of the R_k's, and each R_n joins the endpoints of two of the Q_k's. Then we can choose $\delta_n > 0$ so that the δ_n-neighborhoods of the Q_n's are pairwise disjoint.

Starting with $\phi_0 = f$, we construct by induction a sequence of Borel functions ϕ_n such that

(i) ϕ_n is analytic on Δ, and ϕ_n is analytic on a neighborhood of Q_n.

(ii) $\phi_n - \phi_{n-1}$ is analytic off the δ_n-neighborhood of Q_n and satisfies $|\phi_n - \phi_{n-1}| < d/2^n$ there.

(iii) $\|\phi_n - \phi_{n-1}\| < 2C_1 d$.

Indeed, having chosen ϕ_{n-1}, we note that on the part of Δ near Q_n we have

$$|\phi_{n-1} - u| \leqslant |\phi_{n-1} - \phi_{n-2}| + \ldots + |\phi_1 - f| + |f - u|$$

$$< d/2^{n-1} + \ldots + d/2 + d < 2d \quad ,$$

so that Lemma 1 will provide the desired function ϕ_n.

For each z, $|\phi_j(z) - \phi_{j-1}(z)| < d/2^j$ for all but at most one index j, while always $|\phi_j - \phi_{j-1}| < 2C_1 d$. Hence the ϕ_j converge pointwise to a function ϕ satisfying

$$|\phi(z) - f(z)| \leqslant \sum |\phi_j(z) - \phi_{j-1}(z)| \leqslant (2C_1+1)d \ .$$

The convergence is uniform on any compact set at a positive distance from $\lim Q_n = bE$, so that ϕ is analytic on Δ. Since $\phi_j - \phi_{j-1}$ is analytic on the δ_n-neighborhood of Q_n for $j \neq n$, while $\phi_n - \phi_{n-1}$ is analytic in a neighborhood of Q_n, $\phi - f$ will also be analytic in a neighborhood of each Q_n.

Now we perform essentially the same construction on the R_n's, being careful to retain analyticity across the Q_n's. Choose $\varepsilon_n > 0$ so that the ε_n-neighborhoods of the R_n's are disjoint. Starting with $\psi_0 = \phi$, construct by induction a sequence ψ_n such that

(i) ψ_n is analytic on a neighborhood of $\Delta \cup R_n$.

(ii) ψ_n is analytic across the arcs of $b\Delta$ across which ψ_{n-1} is analytic.

(iii) $\psi_n - \psi_{n-1}$ is analytic off the ε_n-neighborhood of R_n and satisfies $|\psi_n - \psi_{n-1}| < d/2^n$ there.

(iv) $\|\psi_n - \psi_{n-1}\| < C_7 d$.

This is again possible by Lemma 1. As before we see that the ψ_n converge to a function h, uniformly on sets at a positive distance from bE, such that $h \in H^\infty(\Delta)$, h extends analytically across each Q_n and across each R_n, and $|h - \psi| < (C_7 + 1)d$. Then h is analytic across E, and $|h - f| < (C_7 + 2C_1 + 2)d$, so that h is the required function.

COROLLARY. *Let* $f \in H^\infty(\Delta)$, *let* E *be a subset of* $b\Delta$, *and let* $d > 0$. *Suppose that for each* $z \in E$, *the diameter of the cluster set of* f *at* z *is less than* d. *Then there is* $h \in H^\infty(\Delta)$ *such that* h *extends to be analytic in a neighborhood of* E, *and*

$$\sup_{z\in\Delta} |f(z) - h(z)| < C_0 d \ .$$

Proof. As the diameter of the cluster set of f at $z \in b\Delta$ is an upper semicontinuous function of z we can replace E by a larger open set. It is now easy to construct a continuous function satisfying the hypotheses of Lemma 2.

Proof of Theorem 1. If f extends continuously to each point of E, then we can take the d of the preceding corollary to be arbitrarily small. The resulting h's will approximate f uniformly on Δ, and they will be analytic on E.

Proof of the Corollary to Theorem 1. To show that Δ is dense in the maximal ideal space of H_E^∞, one must show that if $f_1,\ldots,f_n \in H_E^\infty$ satisfy $|f_1| +\ldots+ |f_n| \geq \delta > 0$ on Δ, then there are $g_1,\ldots,g_n \in H_E^\infty$ satisfying $\sum f_j g_j = 1$. In fact, it suffices to show this for $f_1,\ldots,f_n$ lying in any dense subalgebra of H_E^∞, so that by Theorem 1 we can assume that $f_1,\ldots,f_n$ extend analytically to a neighborhood of E. Then there is a simply connected open set $U \supseteq \Delta \cup E$ such that $f_1,\ldots,f_n$ are bounded on U and satisfy $|f_1| +\ldots+ |f_n| > \delta/2$ there. By Carleson's theorem, applied to U, there are bounded analytic functions $g_1,\ldots,g_n$ on U satisfying $\sum f_j g_j = 1$. Since the g_j's belong to H_E^∞, they are the required functions.

CONCLUDING REMARKS. For a subset E of $b\Delta$, let L_E^∞ denote the uniform closure of the functions in $L^\infty(d\theta)$ which extend continuously to an open set containing E. Then L_E^∞ consists of the functions in L^∞ which are constant on each "fiber" of the maximal ideal space of L^∞ lying over points of E. If we identify functions in $H^\infty(\Delta)$ with their radial boundary values, we can regard $H^\infty(\Delta)$ as a subalgebra of $L^\infty(d\theta)$. Under this identification, H_E^∞ becomes a subalgebra of L_E^∞. In fact, $H_E^\infty = H^\infty \cap L_E^\infty$, and H_E^∞ is a logmodular subalgebra of L_E^∞ (cf. Détraz [2]).

For $f \in L^\infty(d\theta)$, we define as usual the distance from f to L_E^∞ by

$$d(f,L_E^\infty) = \inf\{\|f - g\|_\infty : g \in L_E^\infty\} \quad ,$$

and we define $d(f,H_E^\infty)$ similarly. Lemma 2 can be restated as follows.

THEOREM 2. *There is a universal constant* C_0 *such that for all* $E \subset b\Delta$ *and all* $f \in H^\infty(\Delta)$,

$$d(f,L_E^\infty) \leqslant d(f,H_E^\infty) \leqslant C_0 d(f,L_E^\infty) \quad .$$

We hope to study the smallest possible constant C_0 in another paper.

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