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# A REMARK ON SIDON SETS

R. Kaufman

*Dedicado al Profesor Alberto González Domínguez*

Let $\Lambda = (\lambda_k)_1^\infty$ be an increasing sequence of positive numbers and E a compact set of real numbers. Then $\Lambda$ is a *Sidon set for* E provided an inequality

$$\max_{t\in E} |\sum a_k e^{i\lambda_k t}| \geqslant \delta \sum |a_k| \quad , \quad \delta > 0 \text{ constant}$$

holds for all polynomials $\sum a_k e^{i\lambda_k t}$ with frequencies in $\Lambda$. It is natural to study sets E with the property that there is a Sidon set $\Lambda$ for E subject to a growth condition; the most familiar condition is $\log \lambda_k = O(k)$ [4, p. 223]. Following the method of Helson and Kahane [1], it is proved in [5] that if $T > 1$ and Hausdorff dimension $E > 0$ [3,II], there exists a Sidon set $\Lambda$ for E fulfilling $\lambda_{k+1} < T\lambda_k$.

It seems very difficult to decide whether this condition on a Sidon set for E forces E to have positive dimension; concerning a related problem a final answer is obtained by Ivašev-Musatov [2]. In this note we prove that the theorem stated above is best-possible in a certain direction.

Let h(u) be a continuous increasing function on $[0,+\infty)$ and $h(0)=0$, and let us write $E \in (h)$ provided there is a Borel probability measure $\mu$ concentrated in E such that $\mu(I) = O(h(|I|))$ for all intervals I. A theorem of Frostman [3, p. 27] shows that $\dim E > 0$ if and only if $E \in (u^c)$ for a $c > 0$.

THEOREM 1. *Suppose that for every* $\alpha > 0$, $u^\alpha = o(h(u))$ $(u \to 0)$ . *Then we can construct a compact set* $E \in (h)$ *so that no Sidon set* $\Lambda$ *for* E *can fulfill* $\log \lambda_k = O(k)$.

THEOREM 2. *Let there exist, for each* $R > 1$, *and integer* $N > R$, *and a system* $(I_m)_{m=1}^N$ *of* N *intervals of length* $N^{-R}$ *whose union contains* E.

*Then no Sidon set* $\Lambda$ *for* E *can fulfill* $\log \lambda_k = O(k)$

Theorem 1 will be derived afterwards from Theorem 2. To prove Theorem 2 we suppose on the contrary that for any integer $M \geqslant 1$ and any choice of signs $\pm$ the polynomial $p(t) = \sum \pm e^{i\lambda_k t}$ satisfies $\max_E |p(t)| \geqslant \delta M$, while $\delta \log \lambda_k \leqslant k$ for each k. Let then $R \geqslant 4\delta^{-2}$ and N be the integer specified in the hypotheses; next let M be defined by $\lambda_M \leqslant N^{\frac{1}{2}R} < \lambda_{M+1}$, whence $1+M > \frac{1}{2} R\delta \log N$. Choose any $a_m \in I_m$ $(1 \leqslant m \leqslant N)$ and observe that

$$\max_E |p(t)| \leqslant \max_m |p(a_m)| + N^{-R}\max |p'|$$

$$\leqslant \max_m |p(a_m)| + M\lambda_M N^{-R} .$$

Thus $\max_m |p(a_m)| \geqslant \delta M - N^{-R}M\lambda_M \geqslant \frac{1}{2}\delta M$.

To complete the proof we choose the signs $\pm$ as the Rademacher functions $\phi_1(x),\ldots,\phi_M(x)$ on $(0,1)$; we write P instead of dx for Lebesgue measure, and $p(t;x)$ to indicate the dependence on x. We have only to prove that for large N

$$\sum_{m=1}^{M} P\{|p(a_m;x)| \geqslant \frac{1}{2}\delta M\} < 1 ,$$

and this is a consequence of

$$P\{|p(t;x)| \geqslant \frac{1}{2}\delta M\} < N^{-1} , \quad -\infty < t < \infty .$$

For any $y > 0$

$$\int_o^1 \exp y|\mathrm{Re}\ p(t;x)|dx \leqslant 2(\cos hy)^M \leqslant 2e^{\frac{1}{2}My^2} ,$$

and similarly for the imaginary part. Therefore for any $b > 0$ we obtain

$$P\{|p(t;x)| > b\} \leqslant 4e^{\frac{1}{2}My^2} e^{-\frac{1}{2}by}$$

$$= 4 \exp -\frac{1}{4} b^2M^{-1} \text{ (for the best value of } y > 0)$$

$$= 4 \exp -\frac{1}{16}\delta^2 M \quad \text{when} \quad b = \frac{1}{2}\delta M.$$

Using the inequality $M+1 > \frac{1}{2} R\delta \log N$ we obtain

$$P\{|p(t;x)| > b\} \leqslant CN^{-\frac{1}{32} R\delta^3}$$

This proves Theorem 2.

The deduction of Theorem 1 from Theorem 2 is an easy consequence of the facts in [3, I,II]. Let $r = (r_j)_1^\infty$ be a sequence of positive numbers decreasing to 0 and $E_r$ the set of all sums $\sum_{j=1}^\infty \pm r_1 \ldots r_j$ . Then $E_r$ has the property specified in Theorem 2. Moreover, if h is the function defined in Theorem 1, there is a sequence r such that $2^{-j} = o(h(r_1 \ldots r_j))$ and now $E_r \in (h)$.

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