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ON UNIQUENESS CONDITIONS FOR THE INITIAL VALUE PROBLEM FOR THE DIFFERENTIAL EQUATION $y' = f(x,y)$

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Consider the initial value problem

$$(1)\qquad y' = f(x,y) \quad , \qquad x \in (0,a)$$
$$y(0) = y_0 \qquad ,$$

with f a real valued function defined on the (open) strip S: $(0,a) \times (-\infty,\infty)$, where "a" is a positive constant. Notice that no continuity of any sort is assumed about f. By "a solution to problem (1) on the interval [0,a)" is meant, throughout the present paper, a real valued function y, defined and continuous on the "half-closed" interval $0 \leqslant x < a$, possessing a finite derivative $y'(x)$ on the open interval $0 < x < a$, and which satisfies the ordinary differential equation $y'(x) = f(x,y(x))$ on the open interval $0 < x < a$, and also obeys the initial condition $y(0)=y_0$. In this exposition, we are concerned with the uniqueness of the solution of (1) on [0,a), with f satisfying "one sided" versions of some of the various well known "two sided" conditions, such as Lipschitz, Nagumo [1], Osgood [2], Krasnosel'skii-Krein [3], and Moyer [4] - the last one is a general condition which, in a sense, includes the previous ones, as well as others. Furthermore, we wish to present a unified, direct, and, at the same time, elementary approach to the proofs of these uniqueness theorems. The only tool we will use here is the elementary mean value theorem of the differential calculus. Since we wish to stress the method, rather than the generality, of the results, we will, for simplicity, focus our attention on the case of a single equation. The method presented can, however, be easily adapted to the more general case of systems of ordinary differential equations, for example (compare the discussion in the book of W. Walter [5]).

Use of this elementary method was made in the paper of Diaz and Walter [6]; however, there it was applied only to the cases of the two sides Lipschitz and Nagumo conditions, and in a way which does not appear to be immediately applicable to other conditions on f.

In section 2 of the present paper, we will redo these two cases, in a yet simpler form, in one sided versions of the Lipschitz and Nagumo conditions (see Theorems 1 and 2); and then extend this method to prove theorems involving one sided versions of the conditions of Osgood, Krasnosel'skii-Krein, and Moyer.

Theorem 5 contains Theorem 1 (one sided Lipschitz) and Theorem 4 (one sided Osgood)as special cases, while Theorem 6 contains Theorem 2 (one sided Nagumo)as a special case. Thus, Theorem 5 appears as that one may call "a prototype of the one sided Lipschitz-Osgood uniqueness theorem", while Theorem 6 appears as what one may call "a prototype of the one sided Nagumo uniqueness theorem".

It may appear, at first glance, (see, for example, J.H. George [7], who states this opinion relative to the work of Moyer [4]) that all the uniqueness theorems in question may be obtained as "special cases of Okamura's theorem". But this first impression is not correct, because Okamura [8] always assumes explicitly that the function $f(x,y)$ is continuous in (x,y), while no explicit assumption about the continuity of $f(x,y)$ is made here.

a. LIPSCHITZ CONDITION.

THEOREM 1. *Let* f *satisfy a one sided Lipschitz condition, with Lipschitz constant* $L \geq 0$, *on the strip* S, *that is*

$$f(x,y) - f(x,z) \leq L(y-z) \quad ,$$

for $0 < x < a$, $-\infty < z < y < +\infty$. *Then the problem* (1) *has at most one solution on the interval* $[0,a)$.

Proof. Suppose not; then there exist *two* solutions $y(x)$ and $z(x)$, and two numbers x_0 , x_1 , with the property that $0 \leq x_0 < x_1 < a$, $y(x_0) = z(x_0)$, while $y(x) \neq z(x)$ for $x_0 < x \leq x_1$. (This is readily seen: Since $y \neq z$, there is a number x_1, with $0 < x_1 < a$ such that $y(x_1) \neq z(x_1)$; further, since the difference function $y-z$ is continuous on $[0,x_1)$, just take x_0 to be that zero of $y-z$ on $[0,x_1)$ which is closest to x_1).

Without loss, merely by interchanging the rôles of y and z, if need be, we can assume that $y(x) > z(x)$ for $x_0 < x \leq x_1$.

Hence, by the mean value theorem, applied to the product function $e^{-Lx}[y(x)-z(x)]$ on the closed interval $[x_0,x_1]$, there exists a number $\bar{x}$, with $x_0 < \bar{x} < x_1$, such that

$$\begin{aligned} 0 < e^{-Lx_1}[y(x_1)-z(x_1)] &= e^{-Lx_1}[y(x_1)-z(x_1)] - e^{-Lx_0}[y(x_0)-z(x_0)] \\ &= (x_1-x_0)e^{-L\bar{x}}\{-L[y(\bar{x})-z(\bar{x})]+[y'(\bar{x})-z'(\bar{x})]\} \\ &= (x_1-x_0)e^{-L\bar{x}}\{-L[y(\bar{x})-z(\bar{x})] + \\ &\quad + f(\bar{x},y(\bar{x}))- f(\bar{x},z(\bar{x}))\} \leq 0 \quad , \end{aligned}$$

where the very last inequality follows from the one sided Lipschitz condition. This contradiction completes the proof.

REMARK 1. It is to be noticed that the special case of the last theorem, when the Lipschitz constant $L = 0$, is of independent interest (here, the one sided Lipschitz condition reduces to the requirement that the function f is nondecreasing in its second argument, when its first argument is fixed). This uniqueness result, for "monotone" f, is *not* a special case, for $L = 0$, of the usual uniqueness theorem for the two sided Lipschitz condition:

$$|f(x,y) - f(x,z)| \leq L|y-z|$$

for $0 < x < a$, $-\infty < y < +\infty$, $-\infty < z < +\infty$.

WARNING. It is very tempting to say that "f satisfies a one sided Lipschitz condition on the strip S" means that:

$$f(x,y) - f(x,z) \leq L|y-z|$$

for $0<x<a, -\infty<y<+\infty, -\infty<z<+\infty$, a condition which differs only slightly, but in a very essential way, from the one sided condition formulated above in Theorem 1. There is a pitfall here, which should be appreciated, for the condition just written can be easily shown to be precisely equivalent to the "two sided" Lipschitz condition written previously, as can be realized merely by interchanging the roles of y and z. A similar word of caution applies when seeking to obtain one sided versions of the other known two sided conditions on $f(x,y)$.

b. NAGUMO CONDITION.

THEOREM 2. *Suppose that*

$$\lim_{\substack{x\to 0+\\ y\to y_0}} f(x,y)$$

exists. Furthermore, let f *satisfy a one sided Nagumo condition on the strip* S, *that is*

$$f(x,y)-f(x,z) \leqslant \frac{1}{x}(y-z) \quad ,$$

for $0 < x < a$, $-\infty < z < y < +\infty$. *Then the problem* (1) *possesses at most one solution on the interval* $[0,a)$.

Proof. Suppose not; then there exist *two* solutions, $y(x)$ and $z(x)$, of (1), on $[0,a)$. Define the auxiliary function g, on $[0,a)$, by

$$g(x) = \begin{cases} \dfrac{y(x)-z(x)}{x} & , \quad \text{for} \quad x > 0 \\ 0 & , \quad \text{for} \quad x = 0 \quad . \end{cases}$$

We first show that g is continuous at $x = 0$ (notice that g has a finite derivative, and hence is continuous, on $0 < x < a$). For $x > 0$, we have, by the mean value theorem, applied to the difference function $y-z$ on the interval $[0,x]$,

$$\begin{aligned} g(x) &= \frac{y(x)-z(x)-[y(0)-z(0)]}{x} \\ &= y'(\bar{x})-z'(\bar{x}) \\ &= f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x})) \quad , \end{aligned}$$

where $0 < \bar{x} < x$. Hence

$$\lim_{x\to 0+} g(x) = 0 \quad ,$$

since the limit

$$\lim_{\substack{x\to 0+\\ y\to y_0}} f(x,y)$$

exists.

Since $y \neq z$ on $[0,a)$, there exist two numbers x_0, x_1, with the property that $0 \leq x_0 < x_1 < a$, that $y(x_0) = z(x_0)$, while $y(x) \neq z(x)$ for $x_0 < x \leq x_1$; and, without loss, as in the proof of Theorem 1, we can assume that $y(x) > z(x)$ for $x_0 < x \leq x_1$.

Hence, by the mean value theorem, applied to the auxiliary function g on the interval $[x_0,x_1]$, there exists a number $\bar{x}$, with $x_0 < \bar{x} < x_1$, such that

$$
\begin{aligned}
0 < \frac{y(x_1)-z(x_1)}{x_1} &= g(x_1)-g(x_0) \\
&= (x_1-x_0)\,\frac{\bar{x}[y'(\bar{x})-z'(\bar{x})]-[y(\bar{x})-z(\bar{x})]}{\bar{x}^2} \\
&= \frac{x_1-x_0}{\bar{x}}\,[f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))] - \frac{1}{\bar{x}}\,[y(\bar{x})-z(\bar{x})] \\
&\leq 0
\end{aligned}
$$

where the very last inequality follows from the one sided Nagumo condition. This contradiction completes the proof.

REMARK 2. The results of Theorems 1 and 2 are similar to those of Diaz and Walter [6], except for the one sidedness of the conditions on f.

c. KRASNOSEL'SKII-KREIN CONDITION.

THEOREM 3. *Let* f *satisfy, simultaneously, a one sided Hölder condition, and a one sided Nagumo type condition, on the strip* S *; that is, suppose that there are constants* α, H, N, *with*

$$0 < \alpha < 1 \quad , \quad 0 < H \quad , \quad 0 < N$$

such that

$$f(x,y)-f(x,z) \leq H(y-z)^{\alpha}$$

and

$$f(x,y)-f(x,z) \leq \frac{N}{x}(y-z)$$

for $0 < x < a$, $-\infty < z < y < +\infty$ *; and suppose, further, that*

$$N < \frac{1}{1-\alpha}$$

(this is no restriction at all, when $N \leq 1$*). Then the problem* (1) *has at most one solution on the interval* $[0,a)$.

Proof. Suppose not; then there exist *two* solutions, $y(x)$ and $z(x)$, of (1), on $[0,a)$. Without loss, suppose that $z(x) \leq y(x)$ for $0 \leq x < a$, since the functions $\min_x (y(x),z(x))$ and $\max_x (y(x),z(x))$ are also solutions of (1). For each number β, with $0 < \beta < \frac{1}{1-\alpha}$, define the auxiliary function g_β, on the interval $[0,a)$, by

$$g_\beta(x) = \begin{cases} \dfrac{y(x)-z(x)}{x^\beta} & , \quad \text{for} \quad x > 0 \ , \\ 0 & , \quad \text{for} \quad x = 0 \ . \end{cases}$$

We first show, by mathematical induction, that the function $g_\beta(x)$ is continuous at $x = 0$, for the particular sequence of values $\beta_m = \sum_{i=0}^m \alpha^i$, where $m = 0,1,2,\ldots$; this implies the desired continuity for $0 < \beta < \frac{1}{1-\alpha}$. As in the proof of Theorem 2, we have, for $x > 0$, by the mean value theorem:

$$\frac{y(x)-z(x)}{x} = y'(\bar{x})-z'(\bar{x}) = f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x})) \ ,$$

for some $\bar{x}$, with $0 < \bar{x} < x$. Now, by the one sided Hölder condition, we have that

$$f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x})) \leq H(y(\bar{x})-z(\bar{x}))^\alpha \ .$$

Since $y(0) = z(0)$, this completes the proof of the continuity of the function g_β, for $\beta = \beta_0 = 1$. Suppose now that the desired continuity has been proved for g_β, where $\beta = \beta_m = \sum_{i=0}^m \alpha^i$, and m is a non-negative integer. Consider the function g_β, where $\beta = \sum_{i=0}^{m+1} \alpha^i$. For $x > 0$, we have, by the mean value theorem, that there is a number $\bar{x}$, with $0 < \bar{x} < x$, such that

$$g_{1+\alpha+\alpha^2+\ldots+\alpha^{m+1}}(x) = \frac{1}{x} \cdot \frac{y(x)-z(x)}{x^{\alpha+\ldots+\alpha^{m+1}}}$$

$$= \frac{y'(\bar{x})-z'(\bar{x})}{x^{\alpha+\ldots+\alpha^{m+1}}}$$

$$= \frac{f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))}{x^{\alpha+\ldots+\alpha^{m+1}}}$$

$$\leqslant \frac{H(y(\bar{x})-z(\bar{x}))^{\alpha}}{x^{\alpha+\ldots+\alpha^{m+1}}}$$

$$= H\left(\frac{\bar{x}}{x}\right)^{\alpha+\ldots+\alpha^{m+1}} \cdot \left(\frac{y(\bar{x})-z(\bar{x})}{\bar{x}^{1+\alpha+\ldots+\alpha^{m}}}\right)^{\alpha}$$

$$= H\left(\frac{\bar{x}}{x}\right)^{\alpha+\ldots+\alpha^{m+1}} \cdot g_{1+\alpha+\ldots+\alpha^{m}}(\bar{x}) \; .$$

The desired continuity of $g_\beta(x)$, at $x = 0$, for $\beta = \sum_{i=0}^{m+1} \alpha^i$, now follows from the induction hypothesis on m. This implies that $g_\beta(x)$ is continuous at $x = 0$, for all $0 < \beta < \frac{1}{1-\alpha}$.

Since $y \neq z$, and $y \geqslant z$, on $[0,a)$, there exist two numbers x_0, x_1, with the property that $0 \leqslant x_0 < x_1 < a$, and $y(x_0) = z(x_0)$, while $y(x) > z(x)$ for $x_0 < x \leqslant x_1$. For any number β such that $0 < \beta < \frac{1}{1-\alpha}$, one also has that $g_\beta(x_0) = \dfrac{y(x_0)-z(x_0)}{x_0^\beta} = 0$, while $g_\beta(x) = \dfrac{y(x)-z(x)}{x^\beta} > 0$ for $x_0 < x \leqslant x_1$ (here, if $x_0 = 0$, by $g_\beta(x_0)$ is meant simply zero).

Recall that, by hypothesis, the constant N, of the one sided Nagumo condition, is required to satisfy $0 < N < \frac{1}{1-\alpha}$, and consider the auxiliary function g_N. By the mean value theorem, applied to the function g_N on the interval $[x_0,x_1]$, there exists a number $\bar{x}$, with $x_0 < \bar{x} < x_1$, such that

$$0 < \frac{y(x_1)-z(x_1)}{x_1^N} = \frac{y(x_1)-z(x_1)}{x_1^N} - \frac{y(x_0)-z(x_0)}{x_0^N}$$

$$= (x_1-x_0)\,\frac{\bar{x}^N[y'(\bar{x})-z'(\bar{x})]-N\bar{x}^{N-1}[y(\bar{x})-z(\bar{x})]}{\bar{x}^{2N}}$$

$$= (x_1-x_0)(\bar{x})^{-N}\{[f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))]-\frac{N}{\bar{x}}[y(\bar{x})-z(\bar{x})$$

$$\leqslant 0 \quad ,$$

where the very last inequality follows from the one sided Nagumo condition. This contradiction completes the proof.

REMARK 3. As can be seen from the proof, the Hölder condition on f was used only to assure that the function $\frac{y(x)-z(x)}{x^N}$ tends to zero as x tends to zero; this is the essential fact used in the proof. This same fact could be obtained if, instead of the hypothesis of the Hölder condition for f, we substituted the hypothesis that the limit of $\frac{f(x,y)}{x^{N-1}}$ exists as (x,y) tends to $(0+,y_0)$. For in that case, we have, by the mean value theorem, for $x > 0$,

$$\frac{y(x)-z(x)}{x^N} = \frac{y'(\bar{x})-z'(\bar{x})}{x^{N-1}}$$

$$= \frac{f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))}{x^{N-1}}$$

Indeed, this alternative hypothesis was used, in conjuntion with the generalized Nagumo condition, in the results of Bownds and Metcalf [9].

d. OSGOOD CONDITION.

THEOREM 4. *Let* ω *be continuous on* $[0,\infty)$, $\omega(0) = 0$, $\omega(u) > 0$ *for* $u > 0$, *and* $\int_{0+} \frac{1}{\omega(u)}\,du = +\infty$. *Suppose that* f *satisfies the one sided Osgood condition on the strip* S, *that is*

$$f(x,y)-f(x,z) \leqslant \omega(y-z)$$

for $0 < x < a$, $-\infty < z < y < +\infty$. *Then the problem* (1) *possesses at most one solution on* $[0,a)$.

Proof. Let W be a function such that W has a continuous first derivative on $(0,\infty)$, $W(0+) = -\infty$, and $W'(x) = \frac{1}{\omega(x)}$ for $x > 0$ (hence $W(x) > -\infty$ for $x > 0$). For example, one may simply choose

$$W(x) = \int_1^x \frac{1}{\omega(u)}\,du \qquad \text{for} \quad x > 0 .$$

Suppose that the conclusion of the theorem is false. Then there exist *two* solutions, $y(x)$ and $z(x)$, of (1), on $[0,a)$. Without loss, as was done in the proof of Theorem 3, it may be assumed that $z(x) \leqslant y(x)$ for $0 \leqslant x < a$. Since $y \neq z$, and $y \geqslant z$, on $[0,a)$, there exist two numbers x_0, x_1 with the property that $0 \leqslant x_0 < x_1 < a$, and $y(x_0) = z(x_0)$, while $y(x) > z(x)$ for $x_0 < x \leqslant x_1$. This means that one also has that

$$e^{W(y(x_0)-z(x_0))} = 0 \text{ , while } e^{W(y(x)-z(x))} > 0$$

for $x_0 < x \leqslant x_1$. Then, by the mean value theorem, applied to the (product) function $e^{-x}e^{W(y(x)-z(x))}$ on the interval $[x_0,x_1]$, there exists a number $\bar{x}$, with $x_0 < \bar{x} < x_1$, such that

$$\begin{aligned}
0 < e^{-x_1}e^{W(y(x_1)-z(x_1))} &= e^{-x_1}e^{W(y(x_1)-z(x_1))} - e^{-x_0}e^{W(y(x_0)-z(x_0))} \\
&= (x_1-x_0)e^{-\bar{x}}e^{W(y(\bar{x})-z(\bar{x}))}\left\{\frac{y'(\bar{x})-z'(\bar{x})}{\omega(y(\bar{x})-z(\bar{x}))} - 1\right\} \\
&= (x_1-x_0)e^{-\bar{x}}e^{W(y(\bar{x})-z(\bar{x}))}\left\{\frac{f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))}{\omega(y(\bar{x})-z(\bar{x}))} - 1\right\} \\
&\leqslant 0 \quad ,
\end{aligned}$$

where the very last inequality follows from the one sided Osgood condition. This contradiction proves the theorem.

e. MOYER CONDITION.

In each of the preceding four proofs we examined an expression, call it $E(x,y(x)-z(x))$, involving the independent variable, and the difference of two solutions. (They were, respectively, $(y(x)-z(x))e^{-Lx}$, $\frac{y(x)-z(x)}{x}$, $\frac{y(x)-z(x)}{x^N}$ and $e^{-x}e^{W(y(x)-z(x))}$).

In each case, E is positive if y-z is positive, and E is equal to zero if $y = z$. Then, by the mean value theorem, we showed that $E(x,y(x)-z(x))$, under the preceding assumptions, must, indeed, always be zero.

Pursuing this simple approach further, we are now able to place the preceding results in a more unified setting, and in the process, obtain theorems which would include these results, as well as others. We will list the assumptions required, as their need arises.

We begin by considering a function $E(x,r)$, defined on $(0,a)\times(0,\infty)$, such that

(i) E is continuously differentiable on $(0,a) \times (0,\infty)$,

(ii) $E(x,r) > 0$ on $(0,a) \times (0,\infty)$, and $E(x,0+) = 0$ on $(0,a)$.

Suppose that problem (1) possesses two solutions y,z; then there are numbers x_0, x_1 such that $0 \leqslant x_0 < x_1 < a$ and (without loss) $y(x)-z(x) > 0$ for all $x_0 < x \leqslant x_1$ and $y(x_0)-z(x_0) = 0$. Therefore, $E(x_1,y(x_1)-z(x_1)) > 0$, and $E(x_0,y(x_0)-z(x_0)) = 0$, provided that $x_0 \neq 0$. If $x_0 = 0$, then $E(x_0,y(x_0)-z(x_0))$ is so far, not defined. Thus, we assume, further, that

(iii) E is such that $E(x,y(x)-z(x)) \to 0$ as $x \to 0+$. (We will understand $E(0,y(0)-z(0))$ to mean $\lim_{x\to 0+} E(x,y(x)-z(x))$).

Then, by the mean value theorem, there exists a number $\bar{x}$, with $x_0 < \bar{x} < x_1$, such that

$$0 < E(x_1,y(x_1)-z(x_1)) = E(x_1,y(x_1)-z(x_1))-E(x_0,y(x_0)-z(x_0))$$

$$= (x_1-x_0)\{E_x(\bar{x},y(\bar{x})-z(\bar{x})) +$$

$$+ E_r(\bar{x},y(\bar{x})-z(\bar{x}))(y'(\bar{x})-z'(\bar{x}))\}$$

$$= (x_1-x_0)\{E_x(\bar{x},y(\bar{x})-z(\bar{x})) +$$

$$+ E_r(\bar{x},y(\bar{x})-z(\bar{x}))(f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))\} .$$

If f is such that there exists a function E satisfying (i), (ii), and (iii), and such that the expression inside the brackets above is non-positive, then we would have the contradiction

$$0 < E(x_1,y(x_1)-z(x_1)) \leqslant 0 \quad ;$$

that is, problem (1) can have at most one solution. We therefore assume that f satisfies the one sided condition:

(iv) f is such that, for all $0 < x < a$, $-\infty < z < y < \infty$,

$$E_x(x,y-z) + E_r(x,y-z)[f(x,y)-f(x,z)] \leqslant 0 \quad .$$

This is the "one sided" condition of Moyer [4].

It now remains only to reformulate (iii) in a more suitable form, which does not involve, explicitly, a knowledge of the "two" solutions y(x) and z(x), so to speak. One condition sufficient for (iii) to hold, independently of the knowledge of the two solutions y(x) and z(x), is:

(iiia) $\lim\limits_{\substack{x\to 0+\\ r\to 0+}} E(x,r) = 0$

This is actually the case, in particular, in the Lipschitz and Osgood conditions, where $E(x,r) = re^{-Lx}$ and $E(x,r) = e^{W(r)}e^{-x}$, respectively. And we have, already, in the preceding discussion, proved:

THEOREM 5. *Suppose* f *is such that there exists a real valued function* E(x,r) *, defined on* $(0,a) \times (0,\infty)$*, satisfying the following*

(i) E *is continuously differentiable on* $(0,a) \times (0,\infty)$,

(ii) $E(x,r) > 0$ *on* $(0,a) \times (0,\infty)$ *and* $\lim\limits_{r\to 0+} E(x,r) = E(x,0+) = 0$ *on* $(0,a)$, (iiia) $\lim\limits_{\substack{x\to 0+\\ r\to 0+}} E(x,r) = 0$,

(iv) $E_x(x,y-z) + E_r(x,y-z)\,[f(x,y)-f(x,z)] \leqslant 0$ *for all*

$0 < x < a$, $-\infty < z < y < \infty$.

Then the initial value problem (1) possesses at most one solution on $[0,a)$.

If y and z are solutions of (1) sucht that $y(x) \geqslant z(x)$ for $0 \leqslant x \leqslant x_1$, for some x_1 , then

$$\begin{aligned} E(x,y(x)-z(x)) &= E(x,y(x)-z(x)-y(0)+z(0)) \\ &= E(x,x(y'(\bar{x})-z'(\bar{x}))) \qquad 0 < \bar{x} < x \\ &= E(x,x\,[f(\bar{x},y(\bar{x}))-f(\bar{x},z(\bar{x}))]) \ . \end{aligned}$$

Hence, another condition sufficient for (iii) to hold, independently of the knowledge of the two solutions $y(x)$ and $z(x)$, is:

(iiib) $\lim\limits_{\substack{x\to 0+\\ y\to y_0}} f(x,y)$ exists; and $\lim\limits_{x\to 0+} E(x,xh(x)) = 0$, where $h(x)$ is any non-negative function, defined on $(0,a)$, such that $\lim\limits_{x\to 0+} h(x) = 0$.

This is actually the case in the Nagumo condition, where $E(x,r) = \frac{r}{x}$. And we have already, in the preceding discussion, proved:

THEOREM 6. *Suppose* f *is such that there exists a real valued function* $E(x,r)$, *defined on* $(0,a) \times (0,\infty)$, *satisfying hypotheses* (i), (ii) *and* (iv) *of Theorem 5, and*

(iiib) $\lim\limits_{\substack{x\to 0+\\ y\to y_0}} f(x,y)$ *exists; and*
$\lim\limits_{x\to 0+} E(x,xh(x)) = 0$, *where* $h(x)$ *is any non-negative function, defined on* $(0,a)$, *such that* $\lim\limits_{x\to 0+} h(x) = 0$.

Then problem (1) possesses at most one solution on $[0,a)$.

To obtain "one sided" generalizations of most other "two sided" uniqueness theorems (such as that of Krasnosel'skii and Krein [3]), one need merely retain hypotheses (i), (ii), (iv) of Theorem 5, and derive other conditions sufficient for (iii) to hold. For the sake of brevity, they will not be listed here; the reader is referred to Moyer [4] for a compilation of some of these "two sided" uniqueness conditions just mentioned.

REFERENCES

[1] M. NAGUMO, *Eine Hinreichende Bedindung für die Unität der Lösung von Differentialgleichungen*, Japan J. Math., 3 (1926), 107-112.

[2] W. OSGOOD, *Beweise der Existenz einer Lösung der Differential gleichungen* $\frac{dy}{dx} = f(x,y)$ *ohne Hinzunahme der Cauchy-Lipschitz Bedindung*, Monatsh. Math. Physik, 9 (1898), 331-345.

[3] M.A. KRASNOSEL'SKII and S.G. KREIN, *On a Class of Uniqueness Theorems for the Equation* $y'=f(x,y)$, Uspehi Mat. Nauk, (67), 11 (1956), 209-213. (Russian).

[4] R.D. MOYER, *A General Uniqueness Theorem*, Proc. Amer. Math. Soc. 17 (1966) , 602-607.

[5] W. WALTER, *Differential-und Integral-Ungleichungen*, Springer-Verlag, 1964.

[6] J.B. DIAZ and W.L. WALTER, *On Uniqueness Theorems for Ordinary Differential Equations and for Partial Differential Equations of Hyperbolic Type*, Trans. Amer. Math. Soc., 96 (1960), 90-100.

[7] John H. GEORGE, *On Okamura's Uniqueness Theorem*, Proc. Amer. Math. Soc., 18 (1967), 764-765.

[8] H. OKAMURA, *Condition nécessaire et suffisante remplie par les équations différentielles ordinaires sans points de Peano*, Mem. Coll. Sci. Kyoto Univ. A24, (1942) 21-28.

[9] J.M. BOWNDS and F.T. METCALF, *An Extension of the Nagumo Uniqueness Theorem*, to appear in Proc. Amer. Math. Soc.

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