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ON THE BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS

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To Professor Alberto González Domínguez with affection and gratitude

I. INTRODUCTION.

The purpose of this note is to present a characterization of the Hardy space H^2 for Siegel domains of type II. For domains of type I, i.e., for tube domains over regular convex cones Bochner's theorem [1] characterizes H^2: a function belongs to H^2 if and only if the Fourier transform of its boundary function vanishes outside a certain cone. For domains of type II only part of this result carries over: it follows from Gindikin's representation theorem [2] , [5] that a certain *partial* Fourier transform (i.e., one not involving all variables) of an H^2 function must be zero outside a cone attached to the domain, however, this condition is not sufficient. Therefore, one has to search for additional conditions which together with the one just stated will yield a sufficient one.

Among the Siegel domains of type II those which are hermitian symmetric spaces form an important subclass. Every such domain is holomorphically equivalent to a bounded domain in some C^n to which it bears the same relation as the classical upper half-plane bears to the unit disc. In a recent paper [8] W. Schmid proved that for the bounded realization of a hermitian symmetric space, provided it contains no irreducible factor of tube type, the boundary values of H^2 functions are characterized by the fact that they satisfy the tangential Cauchy-Riemann equations on the distinguished boundary of the domain. This circumstance and the unit disc--half-plane analogy naturally suggest an additional condition for the membership in H^2, viz., that of satisfying the tangential Cauchy-Riemann equations on the distinguished boundary.

It will be shown here that the two conditions outlined above do suffice to characterize H^2 of an arbitrary Siegel domain of type II. Moreover, it turns out that if the domain is hermitian symmetric and contains no irreducible factor of type I, then, in analogy with Schmid's result, the second condition alone is sufficient.

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2. STATEMENT OF RESULTS.

For all definitions and basic facts about Siegel domains of type II and their Hardy spaces, we refer to [2], [3], and [4]. We shall follow the notation of [3], and make one additional definition; if D is a Siegel domain and B its distinguished boundary, we shall denote by $H^2(B)$ the closed subspace of $L^2(B)$ which consists of the boundary functions of elements of $H^2(D)$. $H^2(B)$ is isomorphic, as a Hilbert space, to $H^2(D)$.

We now state our results in detail. D is a Siegel domain of type II, B its Bergman-Silov boundary, V_1, V_2, Ω and Φ have their usual meaning [3].

THEOREM. *If* f *is a function in* $L^2(B)$*, then it belongs to* $H^2(B)$ *if and only if the following two conditions are satisfied:*

(α) *For almost every* $\zeta \in V_2$ *the Fourier transform of* $f(\cdot,\zeta)$ *vanishes almost everywhere outside* Ω'*, the dual cone of* Ω.

(β) f *satisfies the tangential Cauchy-Riemann equations on* B *in the sense of distributions.*

LEMMA. *Hypothesis* (α) *of the theorem is implied by* (β) *if and only if*: (γ) *For every* $\lambda \notin \overline{\Omega}'$ *there exists a* $\zeta \in V_2$ *for which* $\langle \lambda, \Phi(\zeta,\zeta) \rangle < 0$.

The condition (γ) can be analyzed in terms of the geometry of the domain; we shall say that a subset of $\overline{\Omega}$ generates Ω if the set of non-negative linear combinations formed with elements of that subset contains Ω. We have the following

PROPOSITION. (i) (γ) *holds if and only if* $\Sigma = \{x \in \mathrm{Re}V_1 \,|\, x = \Phi(\zeta,\zeta),\ \zeta \in V_2\}$ *generates* Ω.

(ii) *If* D *is homogeneous,* Σ *generates* Ω *if and only* *if it spans* ReV_1.

(iii) *If* D *is hermitian symmetric, then* Σ *spans* ReV_1 *if and only if* D *contains no irreducible factor of type I.*

Assertion (iii) of the proposition is essentially an unpublished result of Korányi, the proof given here is ours. The above statements clearly imply the following

COROLLARY. *If* D *is hermitian symmetric, then the following two facts are equivalent:*

(i) D *has no irreducible factor of type I.*

(ii) $H^2(B)$ *consists of exactly those functions of* $L^2(B)$ *which, in the sense of distributions, satisfy the tangential Cauchy-Riemann equations on* B.

3. PROOF OF THE THEOREM AND THE LEMMA.

We begin by determining explicitly the tangential Cauchy-Riemann equations on B. To do this we first introduce suitable coordinates in $V_1 \times V_2$. The usual way of putting coordinates on $V_1 \times V_2$ is to choose a euclidean coordinate system in V_1 which is adapted to ReV_1, and any euclidean coordinate system in V_2. Thus, if

$$D = \{(z_1,z_2) \in V_1 \times V_2 | \mathrm{Im} z_1 - \Phi(z_2,z_2) \in \Omega\}$$

and

$$B = \{(z_1,z_2) \in V_1 \times V_2 | \mathrm{Im} z_1 - \Phi(z_2,z_2) = 0\} ,$$

then

$$z_1=x_1+iy_1=(z_1{}^1,z_1{}^2,\ldots,z_1{}^{n_1}),x_1,y_1 \in \mathrm{ReV}_1, z_1{}^j=x_1{}^j+iy_1{}^j, n_1=\dim V_1$$

$$z_2=x_2+iy_2=(z_2{}^1,z_2{}^2,\ldots,z_2{}^{n_2}),\ z_2{}^j = x_2{}^j+iy_2{}^j ,\ n_2=\dim V_2 .$$

$\Phi(z_2,z_2')$ in this system has coordinates $\Phi_k(z_2,z_2')$, $k=1,2,\ldots,n$, where each Φ_k is a (numerical) hermitian form. We introduce a slight modification as follows: Set

$$x=x_1 \in \mathrm{Re}V_1 \qquad x=(x_1,\ldots,x_{n_1}) \ , \ x_j=x_1^{\,j}$$

$$t=y_1-\Phi(z_2,z_2) \in \mathrm{Re}V_1 \qquad t=(t_1,\ldots,t_{n_1}) \ , \ t_j=y_1^{\,j}-\Phi_j(z_2,z_2)$$

$$\zeta=z_2 \in V_2 \qquad \zeta=(\zeta_1,\ldots,\zeta_{n_2}) \ , \ \zeta_j=\xi_j+i\eta_j=z_2^{\,j}=x_2^{\,j}+iy_2^{\,j}$$

In this coordinate system we have $D = \{(x,t,\zeta)\,|\,t \in \Omega\}$ and $B = \{(x,t,\zeta)\,|\,t = 0\}$.

We shall denote points of B by (x,ζ), $x \in \mathrm{Re}V_1$, $\zeta \in V_2$. All functions defined on B will be given, unless specifically stated to the contrary, in terms of the (x,t,ζ) coordinate system. The measure $d\beta$ on B [3] is just euclidean measure on $\mathrm{Re}V_1 \times V_2$ and will be written whenever convenient as $dx dV_\zeta$. Clearly we have

$$\frac{\partial}{\partial x_1^{\,j}} = \frac{\partial}{\partial x_j} \qquad \frac{\partial}{\partial y_1^{\,j}} = \frac{\partial}{\partial t_j} \qquad\qquad j = 1,2,\ldots,n_1$$

$$\left.\begin{aligned} \frac{\partial}{\partial x_2^{\,j}} &= \frac{\partial}{\partial \xi_j} - \sum_{k=1}^{n_1} \frac{\partial}{\partial \xi_j}\Phi_k(\zeta,\zeta)\frac{\partial}{\partial t_k} \\ \frac{\partial}{\partial y_2^{\,j}} &= \frac{\partial}{\partial \eta_j} - \sum_{k=1}^{n_1} \frac{\partial}{\partial \eta_j}\Phi_k(\zeta,\zeta)\frac{\partial}{\partial t_k} \end{aligned}\right. \qquad j = 1,2,\ldots,n_2$$

$$\frac{\partial}{\partial \bar z_1^{\,j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial t_j}\right) \qquad\qquad j = 1,2,\ldots,n_1$$

$$\frac{\partial}{\partial \bar z_2^{\,j}} = \frac{\partial}{\partial \bar\zeta_j} - \sum_{k=1}^{n_1} \frac{\partial}{\partial \bar\zeta_j}\Phi_k(\zeta,\zeta)\frac{\partial}{\partial t_k} \qquad j = 1,2,\ldots,n_2$$

If we denote the basis elements of V_2 by $\zeta_1^{\,o}\ldots\zeta_{n_2}^{\,o}$, it is easy to see that

$$\frac{\partial}{\partial \bar\zeta_j}\Phi_k(\zeta,\zeta) = \Phi_k(\zeta,\zeta_j^{o}) \ .$$

Now let $\qquad X = \sum_{j=1}^{n_1} X_1^{\,j}\dfrac{\partial}{\partial \bar z_1^{\,j}} + \sum_{j=1}^{n_2} X_2^{\,j}\dfrac{\partial}{\partial \bar z_2^{\,j}}$

be a C^∞ vector field of type (0,1) defined in a neighborhood of B. The condition that it be tangent to B is that expressed in (x,t,ζ) coordinates the coefficients of the $\partial/\partial t_j$'s vanish on B, i.e.,

$$X_1{}^j = -2i \sum_{k=1}^{n_2} X_2{}^k \, \Phi_k(\zeta,\zeta_j{}^o) .$$

A function $f \in C^1(B)$ is said to satisfy the tangential Cauchy-Riemann equations on B if it is annihilated by the restriction to B of all such vector fields, i.e., if

$$\sum_{j=1}^{n_2} X_2{}^j (-i \sum_{k=1}^{n_1} \Phi_k(\zeta,\zeta_j{}^o) \frac{\partial f}{\partial x_k} + \frac{\partial f}{\partial \bar{\zeta}_j}) = 0$$

Since we can find for every point of B C^∞ functions, $X_2{}^j$ defined in a neighborhood of B,and such that at the point $X_2{}^{j_o} = 1$ and $X_2{}^j = 0$ for $j \neq j_o$, it follows that f satisfies the tangential Cauchy-Riemann equations on B if and only if

$$(-i \sum_{k=1}^{n_1} \Phi_k(\zeta,\zeta_j{}^o) \frac{\partial}{\partial x_k} + \frac{\partial}{\partial \bar{\zeta}_j})f = 0 \qquad j=1,2,\ldots,n_2 \tag{3.1}$$

We denote the operator on the lefthand side of (3.1) by Z_j. If $f \in C^1(B)$ and $\phi \in C_o^\infty(B)$, one verifies that

$$\int_B f \, Z_j \phi d\beta = - \int \phi Z_j \, f \, d\beta \quad ,$$

and, therefore, a function $f \in L^1_{loc}(B)$ is said to satisfy the tangential Cauchy-Riemann equations, in the sense of distributions, on B if for every $\phi \in C_o^\infty(B)$

$$\int_B f \, Z_j \phi d\beta = 0 \quad . \tag{3.2}$$

We now proceed to the proof of the theorem; in what follows we carefully distinguish between "functions in L^2", i.e., measurable and square integrable functions, and "elements of L^2", i.e., equivalence classes of such functions. Let f be a function in $L^2(B) = L^2(\mathrm{Re}V_1 \times V_2)$. Let S be a subset of full measure in V_2 such that for every $\zeta \in S$ the function $f(\cdot,\zeta)$ is in $L^2(\mathrm{Re}V_1)$.

For fixed $\zeta \in S$ and $\varepsilon > 0$ consider the function

$$F_\zeta^{\varepsilon}(\lambda) = \int_{ReV_1} e^{-2\pi i<\lambda,x>-2\pi|x|\varepsilon} f(x,\zeta)dx \quad .$$

There exists a function F_ζ in $L^2((ReV_1)')$ such that $F_\zeta^{\varepsilon} \to F_\zeta$ in quadratic mean and pointwise for $\lambda \in (ReV_1)'$, except when λ is in a set $E_\zeta \subset (ReV_1)'$ of measure zero. F_ζ is (a representative of) the L^2 Fourier transform of $f(\cdot,\zeta)$. Observe that the function

$$(\lambda,\zeta) \mapsto F_\zeta^{\varepsilon}(\lambda)$$

is measurable on $(ReV_1)' \times S$. By Plancharel's theorem and known properties of the Poisson integral,

$$\int_{(ReV_1)'} |F_\zeta^{\varepsilon}(\lambda)|^2 \, d\lambda \leqslant \int_{ReV_1} |f(x,\zeta)|^2 \, dx \quad .$$

The righthand side of this inequality is an integrable function of ζ by Fubini's theorem. Therefore, by the Lebesgue dominated convergence theorem,

$$\int_S dV_\zeta \int_{(ReV_1)'} | F_\zeta^{\varepsilon}(\lambda) - F_\zeta^{\eta}(x)|^2 \, d\lambda \to 0 \quad \text{if} \quad \varepsilon,\eta \to 0 \ .$$

Consequently, there exists a function $\hat{f}$ in $L^2((ReV_1)' \times S)$ to which the function $(\lambda,\zeta) \mapsto F_\zeta^{\varepsilon}(\lambda)$ converges in quadratic mean . The set $E = \cup_{\zeta \in S} E_\zeta \times \{\zeta\}$ is of measure zero, therefore, since $F_\zeta^{\varepsilon}(\lambda) \to F_\zeta(\lambda)$ on $(ReV_1)' \times S - E$ we have that almost everywhere on this set

$$\hat{f}(\lambda,\zeta) = F_\zeta(\lambda) \quad .$$

We can, of course, extend the definition of $\hat{f}$ to all of V_2 by setting it equal to 0 outside $(ReV_1)' \times S$. To sum up we have shown that for $\zeta \in S$ the function $\hat{f}(\cdot,\zeta)$ is almost everywhere in $(ReV_1)'$ equal to (a representative of) the L^2 Fourier transform of $f(\cdot,\zeta)$.

Assume now f satisfies the hypotheses (α) and (β) of the theorem. We take S to be the set of ζ's for which $f(\cdot,\zeta) \in L^2$ and $F_\zeta(\lambda) = 0$ for almost every $\lambda \notin \overline{\Omega}'$. Consequently, $\hat{f}(\lambda,\zeta) = 0$ for almost every $(\lambda,\zeta) \in (\overline{\Omega}')^{c} \times S$. Let now $\phi_1 \in C_o^{\infty}(ReV_1)$, $\phi_2 \in C_o^{\infty}(V_2)$, and set

$\phi = \phi_1 \otimes \phi_2$. By assumption (3.2) holds for $j = 1,2,\ldots,n_2$:

(3.3) $$0 = \int_B fZ_j\phi d\beta = \int_{V_2} dV_\zeta \int_{\mathrm{Re}V_1} Z_j\phi f dx \quad .$$

Since $Z_j\phi \in C_o^\infty$ there is no difficulty in taking its Fourier transform in x, the resulting function will be in $C^\infty((\mathrm{Re}V_1)'\times V_2)$, and no problems of measurability arise. Denoting by F the Fourier transform in x, the inner integral by Plancherel's theorem is equal, for $\zeta \in S$, to

(3.4) $$\int_{(\mathrm{Re}V_1)'} \overline{FZ_j\phi}\; F_\zeta(\lambda)d\lambda = \int_{(\mathrm{Re}V_1)'} \overline{FZ_j\phi}\; \hat{f}(\lambda,\zeta)d\lambda \quad .$$

The integrand of the righthand side of (3.4) is in $L^1((\mathrm{Re}V_1)'\times V_2)$, so after substituting back into (3.3) we can use Fubini's theorem to obtain

(3.5) $$\int_{(\mathrm{Re}V_1)'} d\lambda \int_{V_2} \overline{FZ_j\phi}\; \hat{f}(\lambda,\zeta)dV_\zeta = 0 \quad .$$

Now

$$\overline{FZ_j\phi} = [-2\pi \sum_{k=1}^{n_1} \phi_k(\zeta,\zeta_j{}^o)\lambda_k\phi_2(\zeta) + \frac{\partial}{\partial\bar{\zeta}_j}\phi_2(\zeta)]\,\overline{(F\bar{\phi}_1)(\lambda)}$$

$$= \frac{\partial}{\partial\bar{\zeta}_j}[e^{-2\pi<\lambda,\Phi(\zeta,\zeta)>}\phi_2(\zeta)]\,e^{2\pi<\lambda,\Phi(\zeta,\zeta)>}\,\overline{(F\bar{\phi}_1)(\lambda)} \quad .$$

Substituting this into (3.5) we have:

(3.6)

$$\int_{(\mathrm{Re}V_1)'} d\lambda\,\overline{(F\bar{\phi}_1)(\lambda)} \int_{V_2} \frac{\partial}{\partial\bar{\zeta}_j}[e^{-2\pi<\lambda,\Phi(\zeta,\zeta)>}\phi_2(\zeta)]\,e^{2\pi<\lambda,\Phi(\zeta,\zeta)>}\hat{f}(\lambda,\zeta)dV_\zeta = 0$$

Since the functions $F\bar{\phi}_1$ are dense in $L^2((\mathrm{Re}V_1)')$, the inner integral vanishes for almost all $\lambda \in (\mathrm{Re}V_1)'$. Note that any C_o^∞ function ψ on V_2 can be written as $\exp(-<\lambda,\Phi(\zeta,\zeta)>)\phi_2(\zeta)$ with suitable ϕ_2, therefore, we obtain from (3.6), that

(3.7) $$\int_{V_2} \frac{\partial}{\partial\bar{\zeta}_j}\psi(\zeta)e^{2\pi<\lambda,\Phi(\zeta,\zeta)>}\hat{f}(\lambda,\zeta)dV_\zeta = 0 \quad j=1,2,\ldots,n_2$$

for all $\psi \in C_o^\infty(V_2)$ and almost all $\lambda \in (\mathrm{Re}V_1)'$. Note, however, that the set of λ's where (3.7) holds may depend on ψ. Let us set

$$g(\lambda,\zeta) = e^{2\pi<\lambda,\Phi(\zeta,\zeta)>}\hat{f}(\lambda,\zeta) \quad .$$

Let K be a compact subset of Ω', and $h \in L^2(K)$. Then by (3.7) for every $\psi \in C_o^\infty(V_2)$ and $j = 1,2,\ldots,n_2$

$$\int_K \overline{h(\lambda)}\,[\int_{V_2} \frac{\partial\psi}{\partial\bar{\zeta}_j}\, g(\lambda,\zeta)dV_\zeta]\,d\lambda = \int_{V_2} \frac{\partial\psi}{\partial\bar{\zeta}_j}[\int_K \overline{h(\lambda)g(\lambda,\zeta)d\lambda}]\,dV_\zeta = 0.$$

Therefore, the inner integral is almost everywhere equal to a holomorphic function of ζ on V_2, and, consequently, $\zeta \mapsto g(\cdot,\zeta)$ is a weakly, and hence also strongly, holomorphic map of V_2 into $L^2(K)$. Since $g(\lambda,\zeta) = 0$ almost everywhere outside $\bar{\Omega}' \times V_2$ we have

$$\int_{\Omega'\times V_2} e^{-4\pi<\lambda,\Phi(\zeta,\zeta)>}|g(\lambda,\zeta)|^2 d\lambda\, dV_\zeta = \int_{\Omega'\times V_2} |\hat{f}(\lambda,\zeta)|^2 d\lambda\, dV_\zeta < \infty \ ,$$

and thus can conclude that g is contained in the space $\hat{L}^2$ of [5]. Therefore, we now can use the theorem of Gindikin [5, Theorem 4.1], and switching to the (z_1,z_2) coordinates, obtain finally that the function F is defined on D by

$$F(z_1,z_2) = \int_{\Omega'} e^{2\pi i<\lambda,z_1>}g(\lambda,z_2)d\lambda$$

belongs to $H^2(D)$.

We still have to show that the boundary function of F coincides with f almost everywhere on B. If $z_1 = x + it + i\Phi(z_2,z_2)$, $t \in \Omega$, $z_2 = \zeta$, define F_t on B by

$$F_t(x,\zeta) = F(z_1,z_2) \ .$$

$F_t \in L^2(B)$ and we know [2], [5] that F_t tends in $L^2(B)$ to a function $\tilde{F} \in H^2(B)$. Now

$$F_t(x,\zeta) = \int_{\Omega'} e^{2\pi i<\lambda,x>}e^{-2\pi<\lambda,t>}\hat{f}(\lambda,\zeta)d\lambda \ .$$

For $\zeta \in S$, $e^{-2\pi<\lambda,t>}\hat{f}(\lambda,\zeta) \to \hat{f}(\lambda,\zeta) = F_\zeta(\lambda)$ as $t \to 0$. The convergence is dominated so we have also convergence in $L^2((\mathrm{Re}V_1)')$, and, therefore, by Plancherel's theorem

$$F_t(\cdot,\zeta) \to f(\cdot,\zeta)$$

in $L^2(\mathrm{ReV}_1)$. This together with the fact that $F_t \to \tilde{F}$ in $L^2(B)$ immediately yields $\tilde{F} = f$ almost everywhere on B. This finishes the proof of the sufficiency of the conditions (α) and (β).

The necessity of these conditions is a straightforward consequence of the existence of boundary values of H^2 functions, and (for (α)) of a lemma of Stein [9].

We now come to the proof of the lemma. Let S consist of all $\zeta \in V_2$ for which $f(\cdot,\zeta) \in L^2(\mathrm{ReV}_1)$. Construct g as before. Note that g has all the properties it had before, except that possibly it is not zero outside $\overline{\Omega}' \times V_2$. We want to conclude that if (γ) holds g, and, therefore, $\hat{f}$ is zero almost everywhere outside $\overline{\Omega}' \times V_2$. By Lemma 3.2 of [5] one can modify g on a set of measure zero so that for almost all $\lambda \in (\mathrm{ReV}_1)'$, $g(\lambda,\cdot)$ is a holomorphic function on V_2. Now it follows from

$$\int_{(\mathrm{ReV}_1)'\times V_2} e^{-4\pi<\lambda,\Phi(\zeta,\zeta)>}|g(\zeta,\zeta)|^2 d\lambda dV_\zeta = \int_{(\mathrm{ReV}_1)'\times V_2} |\hat{f}(\lambda,\zeta)|^2 d\lambda dV_\zeta < \infty$$

that for almost all $\lambda \in (\mathrm{ReV}_1)'$

$$\int_{V_2} e^{-4\pi<\lambda,\Phi(\zeta,\zeta)>}|g(\lambda,\zeta)|^2 dV_\zeta < \infty \tag{3.8}$$

Using the fact that $V_2 - \{0\}$ can be considered as a complex line bundle over the complex projective space of dimension n_2-1, P^{n_2-1}, we can parametrize $V_2-\{0\}$ by a non-zero complex number, and a point in P^{n_2-1}. Introducing these new coordinates in (3.8) we check that for almost all $\zeta_o \in V_2$

$$\int_{C(\zeta_o)} e^{-4\pi<\lambda,\Phi(z\zeta_o,z\zeta_o)>}|g(\lambda,z\zeta_o)|^2 dV_z < \infty \tag{3.9}$$

where $C(\zeta_o)$ is the complex line determined by ζ_o, $z \in C$, and dV_z the euclidean area element on $C(\zeta_o)$. We omit the routine but somewhat cumbersome details. Now let $\lambda \notin \overline{\Omega}'$ such that $g(\lambda,\cdot)$ is holomorphic and (3.8) holds. Then there exists $\zeta \in V_2$ such that $<\lambda,\Phi(\zeta,\zeta)> < 0$. By continuity this inequality holds in a whole neighborhood N of ζ. Let $\zeta_o \in N$ such that (3.9) holds; then since

$$e^{-4\pi<\lambda,\Phi(z\zeta_o,z\zeta_o)>}|g(\lambda,z\zeta_o)|^2 \geqslant |g(\lambda,z\zeta_o)|^2$$

we have

$$\int_{C(\zeta_o)} |g(\lambda,z\zeta_o)|^2 dV_z < \infty \quad .$$

This implies that the holomorphic function

$$z \mapsto g(\lambda,z\zeta_o)$$

is identically zero, in particular, $g(\lambda,\zeta_o) = 0$. We can repeat this argument for almost all $\zeta_o \in N$, and conclude that $g(\lambda,\cdot)$ vanishes on a set of positive measure, and hence identically in V_2 .

To prove the converse statement we use the fact, proved in Section 4, that if (γ) does not hold, then there exists an open set of λ's in $(\mathrm{Re}V_1)' - \overline{\Omega}'$ such that for λ in this set, $<\lambda,\Phi(\zeta,\zeta)> > 0$, for all $\zeta \neq 0$. Let N be a closed ball contained in this set. Let g be an entire function on V_2 such that

$$\int_{V_2} e^{-4\pi<\lambda,\Phi(\zeta,\zeta)>}|g(\zeta)|^2 dV_\zeta < 0 \qquad \lambda \in N \ ,$$

e.g., $g = g_1 \otimes g_2 \otimes \ldots \otimes g_{n_2}$ with each g_j entire of exponential type will do, because $|<\lambda,\Phi(\zeta,\zeta)>| \geqslant c|\zeta|^2$ with $c > 0$ since N is compact and bounded away from the origin (it cannot intersect $\overline{\Omega}'$). Let $\phi \in C^\infty((\mathrm{Re}V_1)')$ with support contained in N. Set

$$h(\lambda,\zeta) = e^{-2\pi<\lambda,\Phi(\zeta,\zeta)>}g(\zeta)\phi(\lambda) \quad ,$$

$h \in L^2((\mathrm{Re}V_1)' \times V_2) \cap C^\infty$ and satisfies

$$(3.10) \qquad (-2\pi \sum_{k=1}^{n_1} \lambda_k \Phi_k(\zeta,\zeta_j{}^o) + \frac{\partial}{\partial\overline{\zeta}_j}) \, h = 0$$

Let

$$f(x,\zeta) = \int_{(\mathrm{Re}V_1)'} e^{2\pi i<\lambda,x>}h(\lambda,\zeta)d\lambda \quad ,$$

$f \in L^2(B) \cap C^\infty$, and by (3.10) satisfies the tangential Cauchy-Rie-mann equations on B; moreover, for every $\zeta \in V_2$, $f(\cdot,\zeta) \in L^1(\mathrm{Re}V_1)$ and, therefore, for every (λ,ζ), $F_\zeta(\lambda) = h(\lambda,\zeta)$, but the support of h is contained in $N \subset \overline{\Omega}'^{\underline{c}} \times V_2$. This concludes the proof of the lemma.

4. PROOF OF THE PROPOSITION.

Let ω denote the set of linear combinations formed with elements of Σ and non-negative coefficients, i.e., the convex cone spanned by Σ. If Σ generates Ω, then $\Omega \subset \omega \subset \overline{\Omega}$, and, therefore, ω', the dual cone of ω, is equal to Ω'. If $\langle\lambda,\Phi(\zeta,\zeta)\rangle \geqslant 0$ for all ζ in V_2, then $\langle\lambda,x\rangle \geqslant 0$ for all x in Ω and, therefore, by continuity, also for all x in $\overline{\Omega}$, i.e., $\lambda \in \overline{\Omega}'$. This proves statement (i) in one direction. To prove the converse, let us note first that a convex dense subset of a convex open set U must equal U, and second that the boundary of a convex set is nowhere dense. Suppose now that Ω is not contained in ω, then by the above remark the interior of $\Omega - \omega$ is nonempty, and, consequently, ω' contains Ω' properly and the interior of $\omega' - \Omega'$ is nonempty. It follows that the interior of $\omega' - \overline{\Omega}'$ is nonempty as well. Let λ be a point in the interior of $\omega' - \overline{\Omega}'$. Then $\langle\lambda,x\rangle > 0$ for all x in $\overline{\omega}$, in particular for all $\Phi(\zeta,\zeta)$.

To prove assertion (ii) let us recall that if D is homogeneous, then there exists a group of linear transformations of $V_1 \times V_2$ which carries D into itself, which acts *transitively* on Ω and whose elements g have the property that

$$g\Phi(\zeta,\zeta) = \Phi(g\zeta,g\zeta) \quad . \tag{4.1}$$

If Σ spans $\mathrm{Re}V_1$, then pick a basis in Σ, and consider the convex cone generated by this basis. This cone is contained in ω and its interior is nonempty, and, hence, intersects the interior of Ω because $\partial\Omega$ is nowhere dense. Now by (4.1) and the transitivity on Ω of the group whose elements are g, non-negative linear combinations of the $\Phi(\zeta,\zeta)$'s fill up Ω if $\omega \cap \Omega \neq \phi$ and, therefore, $\Omega \subset \omega$. The converse is clear from the last remark.

We finally come to (iii). One half of the statement is true with out assuming D to be hermitian symmetric or homogeneous. Let D be the product of two domains D_1 and D_2, D_1 being of type I. Let, with obvious notation, $\lambda_1 \notin \bar{\Omega}'_1$ and $\lambda_2 \in \Omega'_2$. Then $\lambda = (\lambda_1,\lambda_2) \notin \bar{\Omega}'_1 \times \bar{\Omega}'_2 = \overline{(\Omega_1 \times \Omega_2)}'$. Now $\Phi = (0,\Phi_2)$, therefore, for all $\zeta \in (V_2)_2$ $((V_2)_1 = \{0\})$ we have

$$\langle\lambda,\Phi(\zeta,\zeta)\rangle = \langle\lambda_1,0\rangle + \langle\lambda_2,\Phi_2(\zeta,\zeta)\rangle > 0 \quad ,$$

In other words (γ) does not hold and, hence, by (i) Σ does not generate Ω.

Now let D be hermitian symmetric. We need some of the more detailed knowledge about the structure of D, as given in [6], and begin by stating the relevant information. There is a hermitian inner product (,) on $V_1 \times V_2$ which, restricted to $\mathrm{Re}V_1$, is a real inner product. Ω is selfdual with respect to (,). A Lie group of linear transformations Γ acts on $V_1 \times V_2$. Γ leaves V_1, $\mathrm{Re}V_1$, and V_2 invariant, and is transitive on Ω.[1] The elements of Γ are holomorphic automorphisms of $V_1 \times V_2$, and their restrictions to D carry D into itself. Moreover, we again have

$$(4.1) \qquad g\Phi(\zeta,\zeta) = \Phi(g\zeta,g\zeta) \qquad g \in \Gamma .$$

If A^* denotes the adjoint relative to (,) of the linear transformation A of $V_1 \times V_2$, then for $g \in \Gamma$ we also have $g^* \in \Gamma$.[2]

Suppose now that the subspace of $\mathrm{Re}V_1$ spanned by Σ is proper; we have to show that D as a hermitian symmetric space is a product one of whose factors is a Siegel domain of type I. By (4.1) M is invariant under Γ. Since $g \in \Gamma$ implies that $g^* \in \Gamma$, it follows that $M^{\perp}$, the orthocomplement of M in $\mathrm{Re}V_1$, is also invariant. Therefore, denoting by π the orthogonal projection of $\mathrm{Re}V_1$ onto M, we have

$$(4.2) \qquad \pi g = g\pi \qquad g \in \Gamma .$$

Let us note first that since π is linear and open, $\pi\Omega$ and $(I-\pi)\Omega$ are open convex cones in M and $M^{\perp}$ respectively. (4,2) implies

1. In the notation of [6]: $V_1=\underline{p}_1^-$, $\mathrm{Re}V_1=\underline{n}_1^-$, $V_2=\underline{p}_2$, $\Omega=\underline{c}$, $(,)=\langle , \rangle_\nu$, $\Gamma=\mathrm{ad}(K_1^*)$.
2. This follows easily from the fact that for every $V \in \underline{g}^c$ one has $\mathrm{ad}(V)^*=-\mathrm{ad}(\nu V)$, [6,p.282], and that $\underline{k}_1^* = \underline{\ell}_1+i\underline{q}_1$, the Lie algebra of K_1^* is invariant under ν [6,p.284].

that Γ acts transitively on both. Let ω be as above. Clearly, $\omega \subset M$, but ω is contained in $\partial\Omega$ too; if it were not, $\omega \cap \Omega$ would be nonempty, and then by the final remark in the proof of (ii) we should have $\Omega \subset \omega$, which contradicts the fact that M is proper. Note further that by (4.1) $\Gamma\omega \subset \omega$, and that

$$\omega = \pi\omega \subset \pi\overline{\Omega} \subset \overline{\pi\Omega} \tag{4.3}$$

where the last inclusion follows from the continuity of π. Finally, note that the interior of ω is nonempty. Therefore, by (4.3) $\omega \cap \pi\Omega \neq \phi$. Let $x \in \omega \cap \pi\Omega$. Then $\pi\Omega = \Gamma x \subset \omega \subset \partial\Omega$, hence, $\overline{\pi\Omega} \subset \overline{\omega}$, and so by (4.3) $\overline{\pi\Omega} = \overline{\omega}$. Now let $x \in \Omega$, $y \in \Omega$, then the three numbers $(x,\pi x)$, (x,y), $(y,\pi x)$ are positive. In the three dimensional subspace of M spanned by πx, x, y, orthonormalize these three vectors in this order. In this coordinate system $x = (x_1,x_2,0)$, $y = (y_1,y_2,y_3)$, and $\pi x = (x_1,0,0)$ where all the nonzero coordinates are positive. Consequently, we have

$$(x-\pi x,y) = (x,y)-(\pi x,y) = \sum_{i=1}^{3} x_i y_i - x_1 y_1 = x_2 y_2 > 0 \quad .$$

Since for fixed x this holds for arbitrary y in Ω, by the self-duality of Ω it follows that $(x-\pi x,\phi(\zeta,\zeta)) > 0$ for all ζ in V_2. Consequently, we have $(I-\pi)\Omega \subset \partial\Omega$.

Now let $\tilde{\Omega} = \pi\Omega + (I-\pi)\Omega$. This is an open convex cone in $\mathrm{Re}V_1$. Clearly, $\Omega \subset \tilde{\Omega}$, but on the other hand, since $\pi\Omega$ and $(I-\pi)\Omega$ are contained in $\overline{\Omega}$, we also have $\tilde{\Omega} \subset \overline{\Omega}$. Since $\tilde{\Omega}$ is open, and (an open convex set being the interior of its closure) Ω is the interior of $\overline{\Omega}$, we have $\Omega = \tilde{\Omega}$. It now follows that $\pi\Omega$ and $(I-\pi)\Omega$ are regular: let $x \in \overline{\pi\Omega}$, then $x \in \overline{\Omega}$. Therefore, if $-x$ were contained in $\overline{\pi\Omega}$, then it also would be in $\overline{\Omega}$, which contradicts the regularity of Ω. Same argument for $(I-\pi)\Omega$. The cone Ω is, therefore, the product of two regular convex cones. Note next that the splitting of $\mathrm{Re}V_1$ into $M + M^\perp$ also induces a splitting of $V_1 \times V_2$ (we identify $V_1 + V_2$ with $V_1 \times V_2$ to unify the notation):

$$V_1 + V_2 = (M + iM + V_2) + (M^\perp + iM^\perp) \quad .$$

By expanding $\Phi(\zeta+\zeta',\zeta+\zeta')$ and $\Phi(\zeta+i\zeta',\zeta+i\zeta')$ and using the fact that $\phi(\zeta,\zeta) \in \overline{\pi\Omega}$, we find that $\Phi(\zeta,\zeta') \in M + iM$. So $M^\perp + iM^\perp$ and $(I-\pi)\Omega$ determine a Siegel domain of type I, D'. Similarly, $M + iM$, $\pi\Omega$, V_2, Φ determine a domain of type II, D". Clearly, D is the product of D' and D".

To conclude the proof we have to check that D as a hermitian symmetric space is also the product of D' and D". To do this let $g \in \Gamma$, and define a map $f: V_1+V_2 \to V_1+V_2$ as follows:

$$f = \begin{cases} g & \text{on} \quad M + iM + V_2 \\ 1 & \text{on} \quad M^{\perp} + iM^{\perp} . \end{cases}$$

f clearly is a linear automorphism of $V_1 + V_2$ and, hence, holomorphic; its restriction to D is a holomorphic automorphism of D which leaves D' pointwise fixed. Therefore, using Lemma 3 on page 134 of [7], we infer that D' is a hermitian symmetric space. A similar argument works for D". Since D can be a hermitian symmetric space in only one way, the truth of the assertion (iii) and of the proposition now follow.

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